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FRACTURE OF A BODY WITH A PERIODIC SET OF COAXIAL CRACKS UNDER FORCES DIRECTED ALONG THEM: AN AXISYMMETRIC PROBLEM

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Linearized solid mechanics is used to solve an axisymmetric problem for an infinite body with a periodic set of coaxial cracks. Two nonclassical fracture mechanisms are considered: fracture of a body with initial stresses acting in parallel to crack planes and fracture of materials compressed along cracks. Numerical results are obtained for highly elastic materials described by the Bartenev–Khazanovich, Treloar, and harmonic elastic potentials. The dependence of the fracture parameters on the loading conditions, the physical and mechanical characteristics of the material, and the geometrical parameters is analyzed

Keywords: fracture, prestresses, stress intensity factors, circular cracks, highly elastic materials

Introduction. Fracture mechanics is still one of the most intensively developing branches of solid mechanics, which is primarily due to its great practical importance for the quantitative assessment of critical loads for materials and structures with cracks. However, some problems in this area are yet to be completely resolved. Among them are the effect of initial (residual) stresses on the stress–strain state of materials and compression of cracked bodies. That researchers are interested in such problems is evidenced by the great many recent publications on the subject (see, e.g., [11, 16–18, 23, 26]).

Of special interest are cases where prestresses (or compressive forces) act along crack surfaces. According to [5, 7, 8, 16, 17], such are nonclassical problems of fracture mechanics because they cannot be solved within the framework of classical linear fracture mechanics. The reason is that according to the linear theory of elasticity, the load components parallel to the crack planes do not appear in the expressions for the stress intensity factors and crack opening displacements and, hence, cannot be allowed for in classical failure criteria such as Griffiths–Irwin or critical crack opening.

To solve problems of fracture mechanics for bodies with initial stresses acting along cracks, the papers [2, 4, 5] proposed an approach based on the three-dimensional linearized theory of elasticity and a brittle-failure criterion analogous to the Griffiths–Irwin criterion. This approach made it possible to solve some static and dynamic problems (mainly for single cracks in an unbounded material) and reveal new mechanical effects of prestresses (see [5, 6, 13, 16, 20, 24] for reviews of relevant studies).

In [3, 7], local buckling near cracks was considered a failure mechanism in materials compressed by forces parallel to crack planes. Relevant problems were formulated and solved within the framework of the three-dimensional linearized theory of stability of deformable bodies [14, 15] (see [7, 12, 18] for a review of studies based on the approach for homogeneous and composite materials in a continuum formulation). There are also a few recent results obtained using a piecewise-homogeneous medium to model a material containing and compressed along parallel interfacial plane cracks [19].

What these two approaches to nonclassical problems of fracture mechanics (fracture of prestressed materials and bodies compressed along cracks) have in common is that they use linearized relations (equilibrium equations, boundary conditions for stresses, and constitutive equations). Moreover, in solving problems of fracture of prestressed bodies, the following mechanical effect was discovered: the fracture parameters change resonantly as the prestresses tend to the levels corresponding to local

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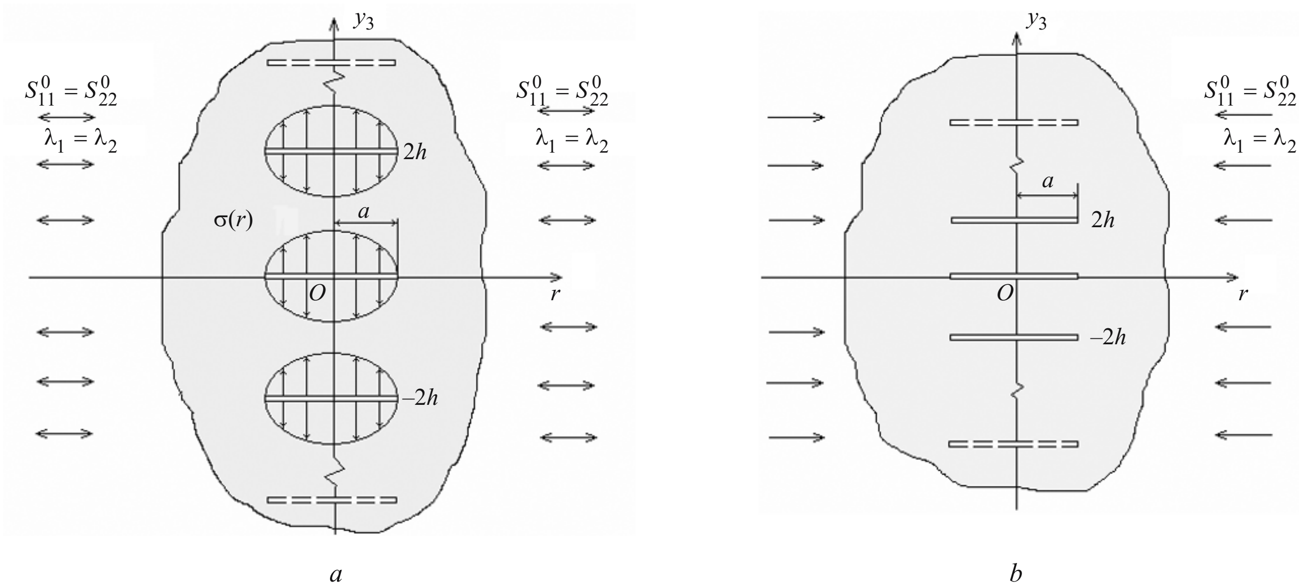

Fig. 1

buckling of cracked bodies under compression [4, 5]. It is natural that in fracture problems for prestressed materials, compressive prestresses cannot exceed local-buckling stresses near cracks. Therefore, it appears reasonable (with respect both to the mathematical tools used and to the need to take fully into account and interpret correctly all the mechanical effects) to solve problems and simultaneously analyze two nonclassical failure mechanisms (fracture of prestressed bodies and compressive failure due to local buckling).

Taking the foregoing into account, we will use the equations of the three-dimensional linearized mechanics of deformable bodies to simultaneously solve two groups of three-dimensional axisymmetric problems for a body with a periodic set of parallel coaxial circular cracks, namely, fracture of a material with prestresses parallel to mode I crack planes and compression of a body along cracks. The analysis applies to both compressible and incompressible elastic bodies. The Hankel transform will be used to reduce the original equations to dual integral equations and then to Fredholm equations of the second kind. Expressions for the stress intensity factors near cracks will be derived. For some types of elastic potentials, we will calculate stress intensity factors as functions of prestresses and geometrical parameters and the critical compressive loads corresponding to local buckling near cracks. According to the approach outlined above, the critical compressive loads are determined as the initial compressive stresses at which the stress intensity factors for a prestressed material with a periodic set of parallel coaxial cracks change abruptly (resonantly).

1. Problem Formulation. We introduce a Lagrangian coordinate system $y_j\ (j = 1, 2, 3)$ that coincides with the Cartesian one in the initial (precritical) stress–strain state. We also use the following notation: S_{ij}^0 are the components of the symmetric stress tensor per unit area of the nondeformed (natural state) body; λ_j are the tensile (compressive) strains induced by the prestresses along the coordinate axes y_j; Q'_{ij} are the components of the asymmetrical stress tensor per unit area of the body with prestresses S_{ij}^0 (initial state); and u_j are the components of the corresponding displacement vector.

Let the body have an infinite row of parallel coaxial circular cracks of radius a located in parallel planes $y_3 = \text{const}$: $\{r < a, 0 \le \theta < 2\pi, y_3 = 2hn, n = 0, \pm 1, \pm 2, \ldots\}$, where (r, θ, y_3) are the circular cylindrical coordinates obtained from the Cartesian coordinates y_j (Fig. 1*a*).

Assume that the prestresses acting along the Oy_1- and Oy_2-axes are equal and induce a homogeneous initial stress–strain state:

$$S_{33}^0 = 0, \quad S_{11}^0 = S_{22}^0 = \text{const} \ne 0, \quad \lambda_j = \text{const}, \quad \lambda_1 = \lambda_2 \ne \lambda_3, \tag{1.1}$$

in which the displacements are defined by

$$u_j^0 = \lambda_j^{-1}(\lambda_j^{-1} - 1)y_j, \quad j = \overline{1,3}. \tag{1.2}$$

The linearized equilibrium equations can be written for the displacement components [4, 5]:

$$\omega'_{im\alpha\beta} \frac{\partial^2 u_\alpha}{\partial y_i \partial y_\beta} = 0 \tag{1.3}$$

for compressible bodies and

$$\kappa'_{im\alpha\beta} \frac{\partial^2 u_\alpha}{\partial y_i \partial y_\beta} + q'_{im} \frac{\partial p}{\partial y_i} = 0, \qquad q'_{im} \frac{\partial u_m}{\partial y_i} = 0 \tag{1.4}$$

for incompressible bodies (with incompressibility condition).

The linearized elastic equations are

$$Q'_{ij} = \omega'_{ij\alpha\beta} \frac{\partial u_\alpha}{\partial y_\beta} \tag{1.5}$$

for compressible bodies and

$$Q'_{ij} = \kappa'_{ij\alpha\beta} \frac{\partial u_\alpha}{\partial y_\beta} + q'_{ij} p \tag{1.6}$$

for incompressible bodies. Here the components of the tensors $\omega'_{im\alpha\beta}$ and $\kappa'_{im\alpha\beta}$ depend on the material model and prestresses [4, 5]; p is the scalar hydrostatic pressure.

As in classical fracture mechanics (no prestresses [22]), mode I cracks are meant cracks acted upon by normal loads symmetric about the crack planes. The following boundary conditions (axisymmetric case) are prescribed at the crack edges:

$$Q'_{33} = -\sigma(r), \quad Q'_{3r} = 0 \quad (y_3 = (2hn)_\pm, \; r < a), \tag{1.7}$$

where $n = 0, \pm 1, \pm 2, \ldots$, and the subscripts "+" and "–" refer to the crack edges. It is assumed that the stress–strain state of the body is perturbed by additional stresses $\sigma(r)$ that are much less than the prestresses S_{ij}^0, which makes it possible to apply linearized equations.

For a material compressed along a periodic set of parallel coaxial cracks (Fig. 1*b*), the boundary conditions at the cracks edges are

$$Q'_{33} = 0, \quad Q'_{3r} = 0 \quad (y_3 = (2hn)_\pm, r < a). \tag{1.8}$$

Since the geometry and forces are symmetric about the plane $y_3 = 0$ and the components of the stress tensor and the displacement vector are periodic (with period 2*h*) in the variable y_3, we reduce the original problem to the problem for a layer $0 \le y_3 \le h$ with the following conditions at its edges:

$$u_3 = 0 \quad (y_3 = 0, r > a), \quad Q'_{33} = -\sigma(r) \quad (y_3 = 0, r < a),$$

$$Q'_{3r} = 0 \quad (y_3 = 0, 0 \le r < \infty), \quad u_3 = 0, \quad Q'_{3r} = 0 \quad (y_3 = h, \; 0 \le r < \infty). \tag{1.9}$$

The general solutions of the linearized equilibrium equations were expressed in [4, 5] in terms of harmonic potential functions in the case of homogeneous initial state (1.1). The form of the expressions depends on the ratio of the roots of the characteristic equation. For example, if the roots are unequal ($n_1 \ne n_2$), we have the following solutions expressed in circular cylindrical coordinates:

$$u_r = \frac{\partial}{\partial r}(\varphi_1 + \varphi_2), \quad u_3 = \frac{m_1}{\sqrt{n_1}} \frac{\partial \varphi_1}{\partial z_1} + \frac{m_2}{\sqrt{n_2}} \frac{\partial \varphi_2}{\partial z_2},$$

$$Q'_{33} = C_{44}\left(d_1 l_1 \frac{\partial^2 \varphi_1}{\partial z_1^2} + d_2 l_2 \frac{\partial^2 \varphi_2}{\partial z_2^2} \right), \quad Q'_{3r} = C_{44}\left(\frac{d_1}{\sqrt{n_1}} \frac{\partial^2 \varphi_1}{\partial r \partial z_1} + \frac{d_2}{\sqrt{n_2}} \frac{\partial^2 \varphi_2}{\partial r \partial z_2} \right), \tag{1.10}$$

where $d_i \equiv 1 + m_i$, $z_i \equiv n_i^{-1/2} y_3$ $(i = 1, 2)$, and φ_j $(j = \overline{1, 3})$ are harmonic functions satisfying the Laplace equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z_i^2} \right) \varphi_j (r, z_i) = 0, \qquad i = \overline{1, 3},$$

where C_{44}, n_i, m_i, l_i $(i = 1, 2)$ are determined by the choice of a material model and dependent on the prestresses [5].

If the roots are equal ($n_1 = n_2$), we have

$$u_r = -\frac{\partial \varphi}{\partial r} - z_1 \frac{\partial F}{\partial r}, \qquad u_3 = \frac{m_1 - m_2 + 1}{\sqrt{n_1}} F - \frac{m_1}{\sqrt{n_1}} \Phi - \frac{m_1}{\sqrt{n_1}} z_1 \frac{\partial F}{\partial z_1},$$

$$Q'_{33} = C_{44}\left[(d_1 l_1 - d_2 l_2) \frac{\partial F}{\partial z_1} - d_1 l_1 \frac{\partial \Phi}{\partial z_1} - d_1 l_1 z_1 \frac{\partial^2 F}{\partial z_1^2} \right],$$

$$Q'_{3r} = C_{44}\left\{ \frac{1}{\sqrt{n_1}} \frac{\partial}{\partial r} \big[(d_1 - d_2) F - d_1 \Phi \big] - \frac{d_1}{\sqrt{n_1}} z_1 \frac{\partial^2 F}{\partial r \partial z_1} \right\}, \qquad \Phi \equiv \frac{\partial \varphi}{\partial z_1}, \tag{1.11}$$

where φ, F, Φ, and φ_3 are harmonic functions. Note that solutions (1.11) somewhat differ from those obtained in [5] due to the new potential functions:

$$\varphi \equiv -(\varphi_1 + \varphi_2), \qquad F \equiv -\frac{d\varphi_2}{dz_1}.$$

Substituting (1.10) and (1.11) into conditions (1.9), we obtain the boundary conditions for harmonic (in the layer $0 \le y_3 \le h$) potential functions at the layer edges. For example, if the roots are unequal ($n_1 \ne n_2$), then

$$\frac{m_1}{\sqrt{n_1}} \frac{\partial \varphi_1}{\partial z_1} + \frac{m_2}{\sqrt{n_2}} \frac{\partial \varphi_2}{\partial z_2} = 0 \qquad (z_1 = z_2 = 0, r > a),$$

$$C_{44}\left(d_1 l_1 \frac{\partial^2 \varphi_1}{\partial z_1^2} + d_2 l_2 \frac{\partial^2 \varphi_2}{\partial z_2^2} \right) = -\sigma(r) \qquad (z_1 = z_2 = 0, r < a),$$

$$C_{44}\left(\frac{d_1}{\sqrt{n_1}} \frac{\partial^2 \varphi_1}{\partial r \partial z_1} + \frac{d_2}{\sqrt{n_2}} \frac{\partial^2 \varphi_2}{\partial r \partial z_2} \right) = 0 \qquad (z_1 = z_2 = 0, 0 \le r < \infty),$$

$$\frac{m_1}{\sqrt{n_1}} \frac{\partial \varphi_1}{\partial z_1} + \frac{m_2}{\sqrt{n_2}} \frac{\partial \varphi_2}{\partial z_2} = 0 \qquad (z_1 = h_1, z_2 = h_2, 0 \le r < \infty),$$

$$C_{44}\left(\frac{d_1}{\sqrt{n_1}} \frac{\partial^2 \varphi_1}{\partial r \partial z_1} + \frac{d_2}{\sqrt{n_2}} \frac{\partial^2 \varphi_2}{\partial r \partial z_2} \right) = 0 \qquad (z_1 = h_1, z_2 = h_2, 0 \le r < \infty),$$

$$h_i \equiv h / \sqrt{n_i}, \qquad i = 1, 2.$$

If the roots are equal ($n_1 = n_2$), then

$$\frac{m_1 - m_2 + 1}{\sqrt{n_1}} F - \frac{m_1}{\sqrt{n_1}} \Phi = 0 \qquad (z_1 = 0, r > a),$$

$$C_{44} \left[(d_1 l_1 - d_2 l_2) \frac{\partial F}{\partial z_1} - d_1 l_1 \frac{\partial \Phi}{\partial z_1} \right] = -\sigma(r) \qquad (z_1 = 0, r < a),$$

$$C_{44} \frac{1}{\sqrt{n_1}} \frac{\partial}{\partial r} \left[(d_1 - d_2) F - d_1 \Phi \right] = 0 \qquad (z_1 = 0, 0 \le r < \infty),$$

$$\frac{m_1 - m_2 + 1}{\sqrt{n_1}} F - \frac{m_1}{\sqrt{n_1}} \Phi - \frac{m_1}{\sqrt{n_1}} h_1 \frac{\partial F}{\partial z_1} = 0 \qquad (z_1 = h_1, 0 \le r < \infty),$$

$$C_{44} \frac{1}{\sqrt{n_1}} \frac{\partial}{\partial r} \left[(d_1 - d_2) F - d_1 \Phi - d_1 h_1 \frac{\partial F}{\partial z_1} \right] = 0 \qquad (z_1 = h_1, 0 \le r < \infty). \tag{1.13}$$

2. Deriving Fredholm Equations of the Second Kind. We will now reduce the problem to a system of dual integral equations and then to the Fredholm equation of the second kind. To this end, we apply the Hankel transform with respect to the coordinate r to the harmonic potential functions appearing in (1.10) and (1.11):

$$\varphi_1(r, z_1) = \int_0^\infty \left[A_1(\lambda) \cosh \lambda (h_1 - z_1) + A_2(\lambda) \sinh \lambda (h_1 - z_1) \right] J_0(\lambda r) \frac{\partial \lambda}{\lambda \sinh \lambda h_1},$$

$$\varphi_2(r, z_2) = \int_0^\infty \left[B_1(\lambda) \cosh \lambda (h_2 - z_2) + B_2(\lambda) \sinh \lambda (h_2 - z_2) \right] J_0(\lambda r) \frac{\partial \lambda}{\lambda \sinh \lambda h_2} \tag{2.1}$$

for unequal roots ($n_1 \ne n_2$) and

$$\varphi(r, z_1) = -\int_0^\infty \left[B_1(\lambda) \sinh \lambda (h_1 - z_1) + B_2(\lambda) \cosh \lambda (h_1 - z_1) \right] J_0(\lambda r) \frac{\partial \lambda}{\lambda \sinh \lambda h_1},$$

$$F(r, z_1) = \int_0^\infty \left[A_1(\lambda) \cosh \lambda (h_1 - z_1) + A_2(\lambda) \sinh \lambda (h_1 - z_1) \right] J_0(\lambda r) \frac{\partial \lambda}{\sinh \lambda h_1},$$

$$\Phi(r, z_1) = \int_0^\infty \left[B_1(\lambda) \cosh \lambda (h_1 - z_1) + B_2(\lambda) \sinh \lambda (h_1 - z_1) \right] J_0(\lambda r) \frac{\partial \lambda}{\sinh \lambda h_1} \tag{2.2}$$

for equal roots ($n_1 = n_2$).

Satisfying the boundary conditions on the entire domain $y_3 =$ const (last three relations in (1.12) and (1.13)), we obtain the following relations between the unknown functions in (2.1) and (2.2):

$$A_2(\lambda) = 0, \quad B_1(\lambda) = -\frac{d_1 \sqrt{n_2}}{d_2 \sqrt{n_1}} A_1(\lambda), \quad B_2(\lambda) = 0 \tag{2.3}$$

for unequal roots ($n_1 \ne n_2$) and

$$A_1(\lambda) = 0, \quad B_1(\lambda) = h_1 \lambda A_2(\lambda), \quad B_2(\lambda) = \left(1 - \frac{d_2}{d_1} - h_1 \lambda \coth \lambda h \right) A_2(\lambda) \tag{2.4}$$

for equal roots ($n_1 = n_2$).

The remaining boundary conditions lead to the dual integral equations

$$\int_0^\infty [1-g(\lambda)]\lambda A(\lambda)J_0(\lambda r)d\lambda = \Sigma(r), \qquad r<a,$$

$$\int_0^\infty A(\lambda)J_0(\lambda r)d\lambda = 0, \qquad r>a, \tag{2.5}$$

where

$$A(\lambda)\equiv A_1(\lambda), \quad g(\lambda)=\frac{1}{k}\left(k_2\frac{e^{-\mu_2}}{\sinh\mu_2}-k_1\frac{e^{-\mu_1}}{\sinh\mu_1}\right), \quad \mu_i\equiv\lambda h_i, \quad i=1,2,$$

$$k_1=\frac{l_1}{\sqrt{n_2}}, \quad k_2=\frac{l_2}{\sqrt{n_1}}, \quad k=k_1-k_2, \quad \Sigma(r)=-\frac{k_1\sigma(r)}{kC_{44}d_1l_1} \tag{2.6}$$

for unequal roots and

$$A(\lambda)\equiv A_2(\lambda), \quad g(\lambda)=-\frac{e^{-\mu_1}}{\sinh\mu_1}-\frac{\mu_1}{k\sinh^2\mu_1},$$

$$\mu_1\equiv\lambda h_1, \quad k\equiv\frac{d_2(l_1-l_2)}{d_1l_1}, \quad \Sigma(r)=\frac{\sigma(r)}{kC_{44}d_1l_1} \tag{2.7}$$

for equal roots.

The system of dual integral equations (2.5) is solved by the method of substitution [9], selecting the solution of the dual equations in such a form that the second equation in (2.5) holds identically:

$$A(\lambda)=\int_0^a \omega(t)\sin\lambda t\,dt, \tag{2.8}$$

where $\omega(t)$ is an unknown function that is continuous together with its first derivative in $[0,a]$.

Substituting (2.8) into the first equation in (2.5), we obtain

$$\int_0^\infty\left[\int_0^a \omega(t)\sin\lambda t\,dt\right]J_0(\lambda r)\lambda d\lambda=\int_0^\infty\left[\int_0^a \omega(t)\sin\lambda t\,dt\right]g(\lambda)\lambda J_0(\lambda r)d\lambda+\Sigma(r). \tag{2.9}$$

Considering the relation

$$r^{-1}\frac{d}{dr}\left[rJ_1(\lambda r)\right]=\lambda J_0(\lambda r) \tag{2.10}$$

and the Weber–Schafheitlin integral [9]

$$\int_0^\infty \sin\lambda t J_1(\lambda r)\,d\lambda=\begin{cases}0, & 0\le r<t,\\ \dfrac{t}{r\sqrt{r^2-t^2}}, & 0<t<r,\end{cases} \tag{2.11}$$

we reduce the integral on the left-hand side of (2.9) to the form

$$r^{-1}\frac{d}{dr}\int_0^r \frac{t\omega(t)}{\sqrt{r^2-t^2}}dt, \qquad r<a. \tag{2.12}$$

Multiplying both sides of Eq. (2.9) by r and integrating $\int_0^r \ldots(\rho)d\rho$, we obtain the equation

$$\int_0^r \frac{t\omega(t)}{\sqrt{r^2-t^2}}dt = r\int_0^a \omega(t)\left[\int_0^\infty \sin(\lambda t)g(\lambda)J_1(\lambda r)d\lambda\right]dt - \int_0^r \rho\Sigma(\rho)d\rho.$$

Substituting $t = r\sin\theta$ into the left-hand side of this equation, we obtain Schlomilch's integral equation

$$\int_0^{\pi/2} r\sin\theta\,\omega(r\sin\theta)d\theta = N(r), \qquad r\le a, \tag{2.13}$$

where

$$N(r) \equiv r\int_0^a \omega(t)\left[\int_0^\infty \sin(\lambda t)g(\lambda)J_1(\lambda r)d\lambda\right]dt - \int_0^r \rho\Sigma(\rho)d\rho.$$

Considering that Schlomilch's equation $\int_0^{\pi/2} f(r\sin\theta)d\theta = N(r)$ has the following solution [9]:

$$f(x) = \frac{2}{\pi}\left[N(0) + x\int_0^{\pi/2} N'(x\sin\theta)d\theta\right]$$

and applying Sonine's integral

$$\int_0^{\pi/2} \sin\theta J_0(\lambda x\sin\theta)d\theta = \sqrt{\frac{\pi}{2\lambda x}}J_{1/2}(\lambda x)$$

to (2.13), we obtain the inhomogeneous Fredholm equation of the second kind

$$\omega(x) - \frac{1}{\pi}\int_0^a \omega(t)\widetilde{K}(x,t)dt = \frac{1}{\pi}x\int_0^{\pi/2} \sin\theta\Sigma(x\sin\theta)d\theta, \qquad 0\le x\le a, \tag{2.14}$$

where

$$\widetilde{K}(x,t) = 2\int_0^\infty \left(\frac{k_2}{k}\frac{e^{-\mu_2}}{\sinh\mu_2} - \frac{k_1}{k}\frac{e^{-\mu_1}}{\sinh\mu_1}\right)\sin\lambda t\sin\lambda x d\lambda \tag{2.15}$$

for unequal roots ($n_1 \ne n_2$) and

$$\widetilde{K}(x,t) = 2\int_0^\infty \left(-\frac{e^{-\mu_1}}{\sinh\mu_1} - \frac{\mu_1}{k\sinh^2\mu_1}\right)\sin\lambda t\sin\lambda x d\lambda \tag{2.16}$$

for equal roots ($n_1 = n_2$).

It was shown in [7] that the improper integrals in (2.15) and (2.16) can be represented in terms of the psi-function:

$$\int_0^\infty \frac{e^{-\lambda h}\sin\lambda t\sin\lambda x}{\sinh\lambda h}d\lambda = -\frac{1}{2h}\left\{\operatorname{Re}\psi\left[1+\frac{i(x-t)}{2h}\right]-\operatorname{Re}\psi\left[1+\frac{i(x+t)}{2h}\right]\right\},$$

$$\int_0^\infty \frac{\lambda\sin\lambda t\sin\lambda x}{\sinh^2\lambda h}d\lambda = -\frac{1}{2h^2}\left\{\operatorname{Re}\psi\left[1+\frac{i(x-t)}{2h}\right]-\operatorname{Re}\psi\left[1+\frac{i(x+t)}{2h}\right]\right\}$$

$$+\frac{1}{4h^3}\left\{(x-t)\operatorname{Im}\psi_1\left[1+\frac{i(x-t)}{2h}\right]-(x+t)\operatorname{Im}\psi_1\left[1+\frac{i(x+t)}{2h}\right]\right\}, \tag{2.17}$$

where $\operatorname{Re}\psi\left(1+\frac{iz}{2h}\right)$ and $\operatorname{Im}\psi_1\left(1+\frac{iz}{2h}\right)$ are the real part of the psi-function $\psi(z)=\frac{d}{dz}\ln\Gamma(z)$ ($\Gamma(z)$ is the gamma function) and the imaginary part of its derivative $\psi_1(z)=\frac{d}{dz}\psi(z)$.

Then the kernel of Eq. (2.14) can be represented as

$$\widetilde{K}(x,t)=\widetilde{R}(x-t)-\widetilde{R}(x+t), \tag{2.18}$$

where

$$\widetilde{R}(z)=\frac{1}{k}\left[\frac{k_1}{h_1}\operatorname{Re}\psi\left(1+\frac{iz}{2h_1}\right)-\frac{k_2}{h_2}\operatorname{Re}\psi\left(1+\frac{iz}{2h_2}\right)\right]$$

for unequal roots ($n_1 \neq n_2$) and

$$\widetilde{R}(z)=\frac{1}{kh_1}\left[(k+1)\operatorname{Re}\psi\left(1+\frac{iz}{2h_1}\right)-\frac{z}{2h_1}\operatorname{Im}\psi_1\left(1+\frac{iz}{2h_1}\right)\right]$$

for equal roots ($n_1 = n_2$).

Introducing the dimensionless variables and functions

$$\xi\equiv a^{-1}x,\quad \eta\equiv a^{-1}t,\quad f(\xi)\equiv a^{-1}\omega(a\xi),\quad \mathrm{P}(\xi)\equiv\Sigma(a\xi), \tag{2.19}$$

we nondimensionalize the Fredholm equation (2.14)

$$f(\xi)-\frac{1}{\pi}\int_0^1 f(\eta)K(\xi,\eta)d\eta=\frac{2}{\pi}\xi\int_0^{\pi/2}\sin\theta\,\mathrm{P}(\xi\sin\theta)d\theta,\quad 0\le\xi,\eta\le 1. \tag{2.20}$$

The kernel has the form

$$K(\xi,\eta)=R(\xi-\eta)-R(\xi+\eta) \tag{2.21}$$

where

$$R(z)=\frac{1}{k}\left[\frac{k_1}{\beta_1}\operatorname{Re}\psi\left(1+\frac{iz}{2\beta_1}\right)-\frac{k_2}{\beta_2}\operatorname{Re}\psi\left(1+\frac{iz}{2\beta_2}\right)\right],$$

$$\beta_1=a^{-1}h_1,\quad \beta_2=a^{-1}h_2,$$

for unequal roots ($n_1 \neq n_2$) and

$$R(z)=\frac{1}{k\beta_1}\left[(k+1)\operatorname{Re}\psi\left(1+\frac{iz}{2\beta_1}\right)-\frac{z}{2\beta_1}\operatorname{Im}\psi_1\left(1+\frac{iz}{2\beta_1}\right)\right], \qquad \beta_1=a^{-1}h_1,$$

for equal roots ($n_1=n_2$).

3. Stress Intensity Factors. We can use the solution of Eq. (2.20) with (2.19), (2.1), (2.2), (1.10), and (1.11) to determine the distribution of stresses and displacements. Let us find the stress components in the crack plane $y_3=0$ beyond the crack, $r>a$. In the case of unequal roots ($n_1\neq n_2$), we use (1.10) with (2.1) and (2.8) and integrate by parts to get

$$Q'_{33}(r,0)=C_{44}d_1l_1\frac{k}{k_1}\left\{-\frac{\omega(a)}{\sqrt{r^2-a^2}}+\frac{\omega(0)}{r}+\int_0^a\frac{d}{dt}[\omega(t)]\frac{dt}{\sqrt{r^2-t^2}}\right.$$

$$\left.-\frac{1}{k}\int_0^a\omega(t)\left[\int_0^\infty\left(k_2\frac{e^{-\mu_2}}{\sinh\mu_2}-k_1\frac{e^{-\mu_1}}{\sinh\mu_1}\right)\sin\lambda tJ_0(\lambda r)\lambda d\lambda\right]dt\right\},$$

$$Q'_{3r}(r,0)=0,\qquad Q'_{3\theta}(r,0)=0. \tag{3.1}$$

In the case of equal roots ($n_1=n_2$), we use (1.12) and take (2.2) and (2.11) into account to obtain

$$Q'_{33}(r,0)=-C_{44}d_1l_1k\left\{-\frac{\omega(a)}{\sqrt{r^2-a^2}}+\frac{\omega(0)}{r}+\int_0^a\frac{d}{dt}[\omega(t)]\frac{dt}{\sqrt{r^2-t^2}}\right.$$

$$\left.+\int_0^a\omega(t)\left[\int_0^\infty\left(\frac{e^{-\mu_1}}{\sinh\mu_1}+\frac{\mu_1}{k}\frac{1}{\sinh^2\mu_1}\right)\sin\lambda tJ_0(\lambda r)\lambda d\lambda\right]dt\right\},$$

$$Q'_{3r}(r,0)=0,\qquad Q'_{3\theta}(r,0)=0. \tag{3.2}$$

As in the linear fracture mechanics of materials without prestresses [22], we define stress intensity factors as follows:

$$K_{\mathrm{I}}=\lim_{r\to+a}[2\pi(r-a)]^{1/2}Q'_{33}(r,0),$$

$$K_{\mathrm{II}}=\lim_{r\to+a}[2\pi(r-a)]^{1/2}Q'_{3r}(r,0),$$

$$K_{\mathrm{III}}=\lim_{r\to+a}[2\pi(r-a)]^{1/2}Q'_{3\theta}(r,0). \tag{3.3}$$

Then, with (3.1) and (3.2), we obtain the following expressions for the stress intensity factors:

$$K_{\mathrm{I}}=-C_{44}d_1l_1\frac{k}{k_1}\sqrt{\frac{\pi}{a}}\omega(a),\qquad K_{\mathrm{II}}=0,\qquad K_{\mathrm{III}}=0 \tag{3.4}$$

for unequal roots ($n_1\neq n_2$) and

$$K_{\mathrm{I}}=C_{44}d_1l_1k\sqrt{\frac{\pi}{a}}\omega(a),\qquad K_{\mathrm{II}}=0,\qquad K_{\mathrm{III}}=0 \tag{3.5}$$

for equal roots ($n_1=n_2$), where $\omega(a)$ is found by solving Eq. (2.14).

Changing over to dimensionless functions in (3.4) and (3.5), we obtain

$$K_{\mathrm{I}}=-C_{44}d_1l_1\frac{k}{k_1}\sqrt{\pi a}f(1),\qquad K_{\mathrm{II}}=0,\qquad K_{\mathrm{III}}=0 \tag{3.6}$$

for unequal roots ($n_1 \neq n_2$) and

$$K_{\mathrm{I}} = C_{44} d_1 l_1 k \sqrt{\pi a} f(1), \quad K_{\mathrm{II}} = 0, \quad K_{\mathrm{III}} = 0 \tag{3.7}$$

for equal roots ($n_1 = n_2$), where $f(1)$ is found by solving the Fredholm equation (2.20).

As is seen, the stress intensity factor K_{I} depends on the prestresses (because the parameters C_{44}, d_1, l_1, k and the function f depend on the initial tensile (compressive) strains along the coordinate axes $\lambda_j, j = \overline{1,3}$) and the geometrical parameters (radius of and distance between cracks).

Let us examine the limiting case of infinite distance between cracks. From the expressions for the kernels of the integral equations (2.18), it follows that the kernels vanish as $h \to \infty$:

$$\lim_{h \to \infty} \widetilde{K}(x,t) = 0. \tag{3.8}$$

Then, substituting $t = x \sin\theta$ into Eqs. (2.14), we obtain the limiting value of the function ω:

$$\omega^{\infty}(x) \equiv \lim_{h \to \infty} \omega(x) = -\frac{2}{\pi} \frac{1}{C_{44} d_1 l_1} \frac{k_1}{k} \int_0^x \frac{t\sigma(t)}{\sqrt{x^2 - t^2}} dt \tag{3.9}$$

for unequal roots and

$$\omega^{\infty}(x) \equiv \lim_{h \to \infty} \omega(x) = \frac{2}{\pi} \frac{1}{C_{44} d_1 l_1} \frac{1}{k} \int_0^x \frac{t\sigma(t)}{\sqrt{x^2 - t^2}} dt \tag{3.10}$$

for equal roots.

Substituting (3.9) and (3.10) into (3.4) and (3.5), we obtain the stress intensity factors in the limiting case ($h \to \infty$) for unequal and equal roots of the characteristic equation:

$$K_{\mathrm{I}}^{\infty} \equiv \lim_{h \to \infty} K_{\mathrm{I}} = \frac{2}{\sqrt{\pi a}} \int_0^a \frac{t\sigma(t)}{\sqrt{a^2 - t^2}} dt, \quad K_{\mathrm{II}}^{\infty} = 0, \quad K_{\mathrm{III}}^{\infty} = 0. \tag{3.11}$$

From (3.11) it is seen that in this limiting case, the stress intensity factors are independent of the prestresses and are equal (to within notation) to the stress intensity factors for a mode I crack in an infinite prestressed material [5] and for a mode I crack in an infinite body without prestresses [22].

For example, when the crack edges are under uniform pressure

$$\sigma(r) = \sigma_0 = \text{const}, \tag{3.12}$$

expression (3.11) yields

$$K_{\mathrm{I}}^{\infty} = 2\sigma_0 \sqrt{\frac{a}{\pi}}, \quad K_{\mathrm{II}}^{\infty} = 0, \quad K_{\mathrm{III}}^{\infty} = 0. \tag{3.13}$$

4. Numerical Results. We used the Bubnov–Galerkin method and power functions as a system of coordinate functions for the numerical analysis of the Fredholm equation of the second kind (2.20) and calculation of the stress intensity factors (3.6) and (3.7). Gaussian quadratures were used for numerical integration. The values of the factors for hyperelastic compressible and incompressible materials with different elastic potentials under a uniform normal load applied to crack edges (3.12) are presented below.

4.1. Bartenev–Khazanovich Potential [1] (Case of Equal Roots; Incompressible Body). The parameters appearing in (1.11) are defined as follows:

$$\lambda_3 = \lambda_1^{-2}, \quad n_1 = n_2 = \lambda_1^{-3}, \quad m_1 = \lambda_1^{-3}, \quad m_2 = 1, \quad l_1 = \lambda_1^3, \quad l_2 = \frac{1}{2}(1 - \lambda_1^3),$$

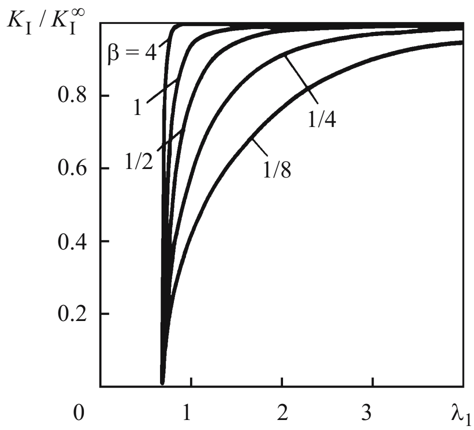

Fig. 2

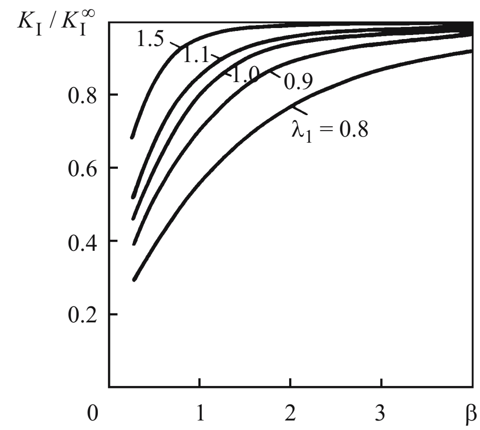

Fig. 3

$$d_1 = \lambda_1^{-3} + 1, \quad d_2 = 2, \quad k = \frac{3\lambda_1^3 - 1}{\lambda_1^3 + 1}, \quad C_{44} = 2\mu\lambda_1^{-1}(1+\lambda_1^3)^{-1}, \quad S_{11}^0 = 2\mu\lambda_1^{-1}(1-\lambda_1^{-3}). \tag{4.1}$$

Figure 2 shows the dependence of the ratio of stress intensity factors K_I / K_I^∞ (K_I^∞ is the stress intensity factor defined by (3.13) for a crack in an infinite material) on the initial compressive (tensile) strain λ_1 caused by the compressive (tensile) prestresses S_{11}^0 ($\lambda_1 < 1$ corresponds to initial compression, $\lambda_1 > 1$ to initial tension, $\lambda_1 = 1$ to zero prestresses) for different values of the dimensionless semidistance between cracks $\beta = ha^{-1}$. As is seen, the prestresses have a strong effect on the stress intensity factors. Note also that the curves (K_I / K_I^∞ versus λ_1) in Fig. 2 have vertical asymptotes corresponding to the abrupt (resonant) change in the stress intensity factors as the initial compressive forces tend to $\lambda_1^* = 0.6934$ at which local buckling (symmetric mode) occurs near cracks when the body is compressed along a periodic set of parallel coaxial cracks.

Figure 3 illustrates, for this material, the dependence of K_I / K_I^∞ on β for different values of λ_1. It can be seen that the interaction of cracks in the linearized problem leads to decrease (especially for small β) in the stress intensity factor compared with K_I^∞ for a single (noninteracting) crack in an infinite body, as for a periodic set of cracks in an infinite body without prestresses [10]. As the distance between cracks increases, the stress intensity factors K_I for a periodic set of coaxial mode I cracks tend to K_I^∞. If the distance between cracks exceeds eight crack radii, the interaction of cracks can be neglected for practical purposes because the difference between K_I and K_I^∞ is less than 5% for all values of λ_1.

4.2. Treloar Potential [25] (Case of Unequal Roots, Incompressible Body). The parameters in (1.10) are given by

$$\lambda_3 = \lambda_1^{-2}, \quad n_1 = \lambda_1^{-6}, \quad n_2 = 1, \quad m_1 = \lambda_1^{-6}, \quad m_2 = 1, \quad l_1 = \frac{2\lambda_1^6}{1+\lambda_1^6},$$

$$l_2 = \frac{1}{2}(1+\lambda_1^6), \quad d_1 = 1+\lambda_1^{-6}, \quad d_2 = 2, \quad k_1 = \frac{2\lambda_1^6}{1+\lambda_1^6}, \quad k_2 = \frac{1}{2}\lambda_1^3(1+\lambda_1^6),$$

$$k = k_1 - k_2, \quad C_{44} = 2C_{10}\lambda_1^{-4}, \quad S_{11}^0 = 2c_{10}\lambda_1^{-2}(\lambda_1^2 - \lambda_3^2). \tag{4.2}$$

Figure 4 illustrates the dependence of K_I / K_I^∞ on λ_1 for different values of $\beta = ha^{-1}$. Figure 5 shows K_I / K_I^∞ versus β. It is seen that the stress intensity factors are strongly dependent on both prestresses and distance between cracks. As this distance increases, the interaction between cracks weakens, and the stress intensity factors for a periodic set of coaxial mode I cracks tend to those for a single crack in an infinite material.

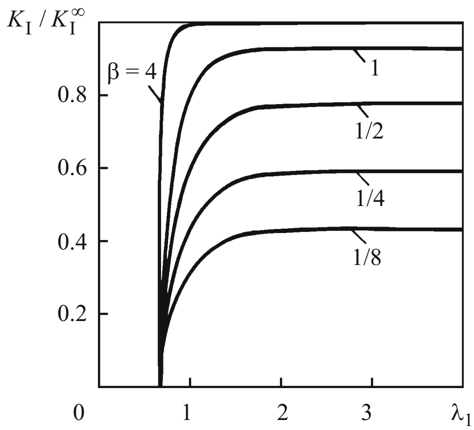

Fig. 4

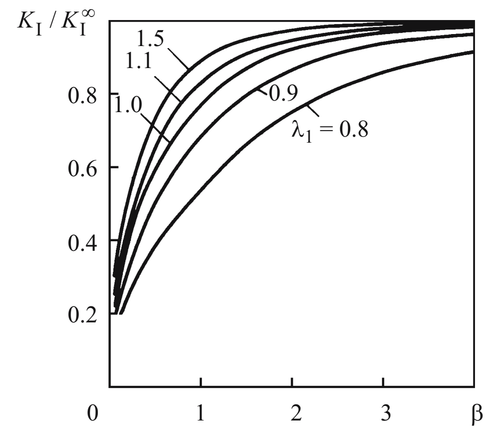

Fig. 5

The curves in Fig. 4 display resonant behavior as the initial compressive strain tends to $\lambda_1^* = 0.6661$, which corresponds to local buckling (symmetric mode) of a material described by the Treloar potential, containing a periodic set of cracks, and subjected to compression along crack planes.

4.3. Harmonic Potential [21] (Case of Equal Roots; Compressible Material). The parameters appearing in (1.11) are defined by

$$\lambda_3 = 1 - 2\nu(1-\nu)^{-1}, \quad n_1 = n_2 = \frac{\lambda_3^2}{\lambda_1^2}, \quad m_1 = \frac{\lambda_3}{\lambda_1}, \quad m_2 = \frac{\nu\lambda_1 + (3\nu - 1)\lambda_3}{\nu\lambda_1 + (1-\nu)\lambda_3},$$

$$l_1 = \frac{\lambda_1}{\lambda_3}, \quad l_2 = \frac{1 + \nu + (\nu - 2)\lambda_1}{2\nu\lambda_3}, \quad d_1 = 1 + \frac{\lambda_3}{\lambda_1}, \quad d_2 = \frac{2\nu(\lambda_1 + \lambda_3)}{\nu\lambda_1 + (1-\nu)\lambda_3},$$

$$k = \frac{\nu(4-\nu)\lambda_1 - 1}{\nu\lambda_1 + (1-\nu)\lambda_3}, \quad C_{44} = \frac{2\mu\lambda_3}{\lambda_1(\lambda_1 + \lambda_3)}, \quad S_{11}^0 = \frac{E}{(1+\nu)} \frac{\lambda_1 - \lambda_3}{\lambda_1}. \tag{4.3}$$

Figure 6 shows K_I / K_I^∞ versus λ_1 for Poisson's ratio $\nu = 0.1$ and different values of β. As is seen, the prestresses have a strong effect on the stress intensity factors. Figure 7 shows K_I / K_I^∞ versus λ_1 for $\beta = 0.5$ and $\nu = 0.1, 0.3, 0.5$ to illustrate how the compressibility of a prestrained material described by a harmonic potential affects the stress intensity factors. For example, when $\lambda_1 = 1.2$, the K_I for $\nu = 0.5$ exceeds that for $\nu = 0.1$ by 21%.

Conclusions. The results obtained here lead us to the following conclusions.

Only the stress intensity factor K_I is nonzero in a prestressed material with a periodic set of coaxial parallel mode I cracks.

In all the materials examined, the prestresses have a strong effect on the stress intensity factors near cracks. These factors change abruptly (resonantly) as the initial compressive forces tend to the level corresponding to local buckling (symmetric mode) near cracks. This makes it possible to determine the critical compressive loads directly, by solving the relevant problem of the fracture mechanics of prestressed materials.

Allowing for the interaction of periodic coaxial cracks in a prestressed body decreases (especially if the dimensionless distance between cracks β is small) the stress intensity factor K_I compared with K_I^∞ for a single crack in an infinite prestrained material.

As the distance between cracks increases, the interaction of cracks gradually weakens and the stress intensity factors near cracks tend to those for a single crack in an infinite material. For practical purposes, this interaction may be neglected if the distance between cracks exceeds eight crack radii.

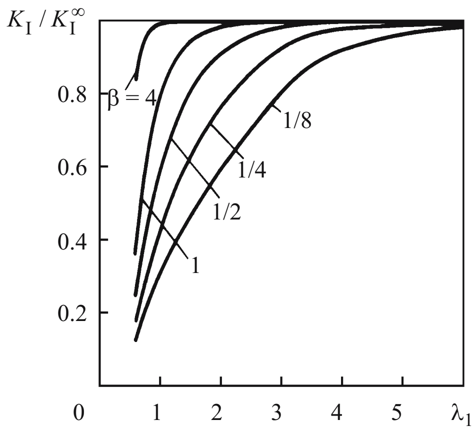

Fig. 6

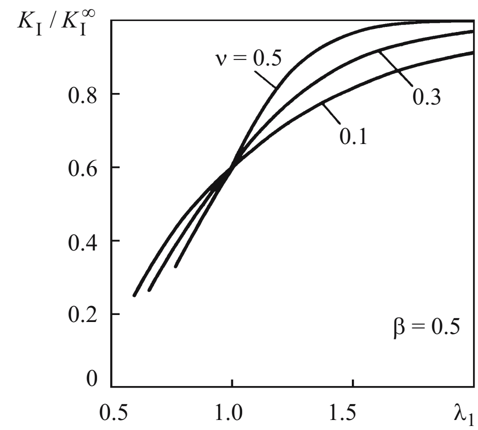

Fig. 7

The stress intensity factors of a compressible material with a harmonic elastic potential are greatly dependent on Poisson's ratio.

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