International Applied Mechanics, Vol. 44, No. 11, 2008

DETERMINING THE CONSTRAINT REACTIONS OF A WHEELED ROBOTIC VEHICLE WITH ONE STEERABLE WHEEL

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The reactions of the nonholonomic constraints of a wheeled robotic vehicle with one steerable wheel are determined. Simplified (asymptotic) relations are derived in addition to the exact ones. They are used to estimate the reactions. The efficiency of the approximate formulas is demonstrated by an example

Keywords: nonholonomic constraints, wheeled transport robot, one steerable wheel, reactions of constraints

Introduction. One of the tasks in the development of manipulators is to choose the velocity to move a workpiece along a set trajectory with allowance for the constraints imposed on the dynamic parameters of the actuating mechanisms (see, e.g., [12, 18, 19]). A similar task arises in relation to wheeled robotic vehicles [10–17]. However, constraints here are associated with the admissible reaction of nonholonomic constraints rather than with the capabilities of the actuating mechanisms [2, 4, 5, 8, 9]. This reaction should not generally exceed the robot's weight multiplied by the dynamic coefficient of friction. Note that it may appear of primary necessity to determine constraint forces (for example, holonomic) in various applied problems (see, e.g., [7]). Generally, analytic expressions for the reaction forces of nonholonomic constraints are rather complex and, therefore, there is a need to derive simpler expressions that would permit relatively easy determination of the constraint forces.

The present paper uses a model of a wheeled robotic vehicle with one steerable wheel (Lineikin model [4], kinematic car [17], see also [10, 11, 13]) to analyze the problem of determining constraint forces (obtain exact equations) and derive approximate (asymptotic) relations to estimate the constraint forces. The efficiency of the approximate formulas is demonstrated by way of examples [11–13, 17].

1. General Equations [9]. The motion of a mechanical system with nonholonomic constraints is described by the equations

$$M(q)\ddot{q}+F(q,\dot{q})=R+B(q)u, \tag{1.1}$$

$$J(q)\dot{q}=0, \qquad R=J^{\mathrm{T}}(q)\lambda, \tag{1.2}$$

where q, λ, and u are the vectors of generalized coordinates, Lagrangian multipliers, and controls; R is the vector of constraint forces; and $F(q,\dot{q})$ is a vector function. It is assumed that the matrix $M(q)=M^{\mathrm{T}}(q)$ is reversible, and the matrices $J(q)$ and $B(q)$ are of full rank. The superscript "T" denotes transposition. Let there be a matrix $C(q)$ such that the vector $\dot{q}$ can be expressed in terms of $\dot{q}_1$ of lower dimension:

$$\dot{q}=C(q)\dot{q}_1. \tag{1.3}$$

Note that

$$J(q)C(q)=0 \tag{1.4}$$

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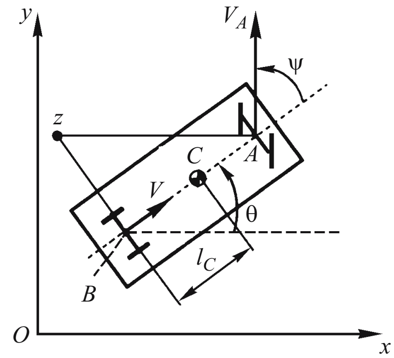

Fig. 1

according to (1.1). Hereafter the arguments q and $\dot{q}$ can be omitted. Substituting (1.3) into (1.1), we get

$$MC\ddot{q}_1 + M\dot{C}\dot{q}_1 + F = J^{\mathrm{T}}\lambda + Bu. \tag{1.5}$$

Left-multiplying (1.5) by JM^{-1} and considering (1.2) and (1.4), we derive an expression for the R-vector of constraint forces:

$$R = J^{\mathrm{T}}(JM^{-1}J^{\mathrm{T}})^{-1}J(\dot{C}\dot{q}_1 + M^{-1}F - M^{-1}Bu). \tag{1.6}$$

2. Description of the Model. Kinematic Approximation [11, 13]. Let us derive the equations of motion (in kinematic approximation) for a mobile robot with one steerable wheel [11, 13]. Let the robot undergo plane-parallel motion on the plane XOY. Its position is characterized by the segment AB (Fig. 1). It is assumed that the velocity of the point B is directed along AB, and the velocity of the point A makes an angle ψ with AB (ψ can be interpreted as the angle of turn of the steerable wheel). The position of this system is described by the coordinates (x, y) of the point B and the angle θ between AB and the OX-axis. If Z is the instantaneous center of velocities, V_A and V are the velocities of the points A and B, $L = |AB|$ ($|\cdot|$ is the length of the segment), then the equation of motion can be written as

$$\dot{x} = V\cos\theta, \quad \dot{y} = V\sin\theta, \quad \dot{\theta} = \frac{V}{L}\tan\psi. \tag{2.1}$$

It is assumed that the steerable wheel is turned according to the formula

$$\dot{\psi} = vV\cos\theta, \tag{2.2}$$

where v is the control.

Thus, (2.1) and (2.2) describe, in kinematic approximation, the motion of a mobile robot with one steerable wheel (analog of Eqs. (11) [17]).

Next, we assume that $|\theta|, |\psi| < \pi/2$, and $V > 0$. This allows us to use x as an independent variable in Eqs. (2.1) and (2.2) and, thus, to reduce the order of the system. In this case, analogs of system (2.1), (2.2) are the following equations:

$$y' = \frac{dy}{dx} = \tan\theta, \quad \theta' = \frac{d\theta}{dx} = \frac{\tan\psi}{L\cos\theta}, \quad \psi' = \frac{d\psi}{dx} = v. \tag{2.3}$$

The prime denotes differentiation with respect to x.

Considering that

$$y' = \tan\theta, \quad y'' = \frac{\tan\psi}{L(\cos\theta)^3}, \tag{2.4}$$

we replace system (2.3) by one differential equation of the third order:

$$y''' = v_1, \tag{2.5}$$

$$v_1 = \frac{v}{L(\cos\theta)^3(\cos\psi)^2} + \frac{3\sin\theta(\sin\psi)^2}{L^2(\cos\theta)^5(\cos\psi)^2}. \tag{2.6}$$

3. Dynamic Effects. Let us detail Eqs. (1.1) with reference to the model in Fig. 1. Let the center of gravity lie on the segment AB at a distance l_C from the point B. The vector of generalized coordinates is $q = \begin{bmatrix} \theta \\ x_C \\ y_C \end{bmatrix}$, where x_C and y_C are the coordinates of the center of gravity. Denoting the central moment of inertia and the mass of the system by I and m, we find the form of the matrices appearing in (1.1) and (1.2):

$$M(q) = \operatorname{diag}\{I, \quad m, \quad m\}, \quad F = 0, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix},$$

$$u = \begin{bmatrix} \Omega \\ P \end{bmatrix}, \quad J = \begin{bmatrix} l\cos\psi & -\sin(\theta+\psi) & \cos(\theta+\psi) \\ -l_C & -\sin\theta & \cos\theta \end{bmatrix},$$

where $l = L - l_C$, Ω and P are the moment and force (acting along AB).

Considering (2.1), we concretize (1.3) by finding how $\dot{q}$ depends on V:

$$\dot{q} = CV, \quad C = \frac{1}{L\cos\psi}\begin{bmatrix} \sin\psi \\ l\cos\psi\cos\theta + l_C\cos(\theta+\psi) \\ l\cos\psi\sin\theta + l_C\sin(\theta+\psi) \end{bmatrix}. \tag{3.1}$$

Note that, as shown in [14], the kinetic energy T of the robot is expressed in terms of V and other parameters as follows:

$$T = \frac{mV^2}{2}\cdot\frac{W}{L^2(\cos\psi)^2}, \quad W = \mu^2(\sin\psi)^2 + L^2(\cos\psi)^2, \quad \mu^2 = \frac{I + ml_C^2}{m}. \tag{3.2}$$

Given kinematic parameters θ, ψ, $\dot{\psi}$, and V, the above expressions for the matrices M, J, and, C allow us, according to (1.6), to determine the constraint forces acting on a moving robot. However, as already mentioned, since expression (1.6) is complicated, it can be used only in numerical calculations, even in the case of a relatively simple system with nonholonomic constraints. Therefore, there is a need for simple approximate formulas to estimate R. One possible approach is to use an asymptotic formula for R defined by (1.6) assuming that θ and ψ are small, while $\dot{\psi}$ is not, during the motion of the robot. Here is such a formula that estimates R up to terms of the second order of smallness:

$$R = \frac{mV}{L}\begin{bmatrix} \dfrac{I}{m}\dot{\psi} \\ -l_C\dot{\psi}\theta - \left(\dfrac{I}{m} + l_C^2\right)\dfrac{\psi\dot{\psi}}{L} \\ l_C\dot{\psi} + V\psi \end{bmatrix} + \begin{bmatrix} -1 & \dfrac{I}{mL}\psi \\ \dfrac{\psi}{L} & 0 \\ 0 & \dfrac{l_C}{L}\psi \end{bmatrix} u. \tag{3.3}$$

According to the chosen vector of generalized coordinates q, the components of the vector $R = \begin{bmatrix} R_0 \\ R_x \\ R_y \end{bmatrix}$ have the following physical meaning: R_0 is the moment of constraint forces; R_x and R_y are the projections of the resultant of constraint forces onto the axes OX and OY, i.e., the absolute magnitude of the reaction applied to the robot is defined by $|R| = \sqrt{R_x^2 + R_y^2}$.

Being simple, the asymptotic formula (3.3) allows easy estimation of the effect of some parameters of a wheeled robot on the components of the vector R.

For example, if $\psi = 0$ and $\theta = 0$ (rectilinear motion with velocity V), the formula $|R| = \frac{ml_C}{L} V\dot{\psi}$ following from (3.3) may be considered as a constraint on the admissible rotation rate of the steerable wheel ($\dot{\psi}$), given limiting value of $|R|$. In what follows, we will assess the quality of approximation of (1.6) by (3.3) in motion planning (choosing a maneuver [17]) for coasting ($u = 0$). In this case, it is possible to find the exact analytic solution of the problem. We need to know the values of $\theta(x)$, $\psi(x)$, $\dot{\psi}(x)$, and $\dot{\theta}(x)$ to determine the vector of constraint forces, according to (1.6) or (3.3). The functions $\psi(x)$ can be obtained in kinematic approximation by solving a two-point boundary-value problem [13]. Having this solution, we can find $V(x)$ from (3.2) (T = const):

$$V = L\cos\psi\sqrt{\frac{2T}{mW}}, \tag{3.4}$$

and then $\dot{\theta}(x)$ and $\dot{\psi}(x)$, according to (2.1) and (2.2).

4. Motion Planning [13]. Let boundary conditions for y, θ, and ψ be prescribed at the initial ($x = 0$) and final ($x = x_f$) points.

Rearranging (2.5) into

$$w' = Aw + Bv_1, \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.2}$$

and considering (2.4), we obtain boundary conditions for the phase vector $w^{\mathrm{T}} = \begin{bmatrix} y & y' & y'' \end{bmatrix}$ at the initial and final points rather than the values of y, θ, and ψ.

Since the control v is uniquely determined by v_1, according to (2.6), we formulate a motion-planning problem as follows. Let the dynamics of the vector w be described by (4.2). Given $w(0)$ and $w(x_f)$, it is necessary to find a function v_1 that would minimize the quadratic functional

$$J = \int_0^{x_f} ((y')^2 + rv_1^2)dx, \tag{4.3}$$

where $r > 0$ is a weight coefficient.

After solving this problem (determining $v_1(x)$), the value of v can be found using (2.6):

$$v = v_1 L(\cos\theta)^3 (\cos\psi)^2 - \frac{3\sin\theta(\sin\psi)^2}{L(\cos\theta)^2}. \tag{4.4}$$

Thus, we have reduced the original nonlinear two-point boundary-value problem to a linear two-point problem with control defined by (2.6).

The solution of problem (4.2), (4.3) is known (see, e.g., [1, 3, 6]) and found in the following steps.

A vector of conjugate variables $\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^{\mathrm{T}}$ is introduced and the Hamiltonian matrix H of the variational problem (4.2), (4.3) is formed:

$$H = \begin{bmatrix} A & -S \\ -Q & -A^{\mathrm{T}} \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad S = \frac{1}{r} BB^{\mathrm{T}}. \tag{4.5}$$

The expression for $v_1(x)$ is as follows:

$$v_1(x) = -\frac{1}{r} B^{\mathrm{T}} \lambda(x), \tag{4.6}$$

w and λ satisfying the conditions

$$z(x) = \Phi(x) z(0), \qquad z = \begin{bmatrix} w \\ \lambda \end{bmatrix}, \qquad \Phi(x) = e^{Hx}. \tag{4.7}$$

The value $\lambda(0)$ is determined by the boundary conditions $w(0)$ and $w(x_f)$ as

$$\lambda(0) = (\Phi_{12}(x_f))^{-1}(w(x_f) - \Phi_{11}(x_f) w(0)),$$

$$\Phi(x_f) = \begin{bmatrix} \Phi_{11}(x_f) & \Phi_{12}(x_f) \\ \Phi_{21}(x_f) & \Phi_{22}(x_f) \end{bmatrix}. \tag{4.8}$$

Thus, relations (4.6)–(4.8) define solutions of the problem because the functions $w(x)$ and $v_1(x)$ define the parameters $y(x)$, $\theta(x)$, and $\psi(x)$:

$$\theta = \arctan(y'), \qquad \psi = \arctan(y'' L (\cos\theta)^3), \tag{4.9}$$

$\psi'(x) = v$ being defined by (4.4).

Next, as already mentioned, we find V using (3.4) and determine $\dot{\theta}(x)$ and $\dot{\psi}(x)$ using (2.1) and (2.2).

5. Examples. *Example 1.* Let us consider an example [13, 14, 17]: It is necessary to steer the wheeled robot shown in Fig. 1 from the point $x = -5$ m with kinematic parameters $y = 1$ m, $\theta = 0.05$, $\psi = 1$ to the point $x = 0$ with parameters $y = 0.5$ m, $\theta = 0$, $\psi = 0$. As in [13], changing the origin of coordinates, we suppose that $x = 0$ and $x_f = 5$. These parameters can be used to determine, using (2.4) and setting $L = 2$ m, the vectors $w(0)$ and $w(x_f)$ appearing in (4.8). Now we have $w(0) = [1 \quad 0.05 \quad 0.7816]^{\mathrm{T}}$ and $w(5) = [0.5 \quad 0 \quad 0]^{\mathrm{T}}$. Let $r = 0.1$ in (4.3). Since, unlike [13], we are solving a motion-planning problem (determining the constraint forces upon a coasting robot ($u = 0$ in (1.1), (1.6), and (3.3)), the above kinematic parameters should be supplemented with the inertial characteristics of the robot. Let such characteristics, which appear in, e.g., (3.2) be the following: $l_C = L/2$, $I = m\rho^2$, $\rho = 0.7$ m. The robot's mass (m) is of no interest because we are interested in not constraint forces but rather in the dynamic coefficient of friction at which the constraints remain. Therefore, the constraint forces are normalized, i.e., divided by the robot's weight mg, where $g = 9.81$ m/sec^2.

If $V = 1.5$ m/sec is the initial velocity, then we can determine, according to (3.2), the kinetic energy, which remains constant during the maneuver under consideration. The results of modeling this maneuver are presented in Figs. 2–8. Figures 2 and 3 show the functions $\theta(x)$ and $\psi(x)$. Figure 4 demonstrates how the velocity V changes during the maneuver. Figure 5 depicts the function $\dot{\psi}(x)$, which makes it possible to formulate requirements to the speed of the actuating mechanism of the steerable wheel. Figures 6–8 show R_x / mg, R_y / mg, and $\sqrt{R_x^2 + R_y^2} / mg$ found using (1.6) (solid line) and (3.3) (dashed line). Noteworthy is the high quality of the approximation provided by (3.3) despite the fact that $\theta(x)$ and $\psi(x)$ are not small (see Figs. 2 and 3). Figure 8 indicates that the dynamic coefficient of friction necessary to execute the maneuver should be greater than 0.6, which is, apparently, difficult to ensure. To reduce the coefficient of friction, it is possible either to decrease the kinetic energy of the robot or to alter the maneuver.

Now we will consider another example.

Example 2. Let us show that it is possible to reduce the constraint force by changing the maneuver. To this end, we keep all the initial data of Example 1 and change the factor r in (4.3). Namely, we set $r = 1$. The results of modeling are presented in Figs. 9 and 10. Figure 9 shows the function $\dot{\psi}(x)$ and Fig. 10 shows $\sqrt{R_x^2 + R_y^2} / mg$. In Fig. 10, as in Fig. 8, the solid and dashed lines represent the results obtained using (1.6) and (3.3), respectively. Note that, according to Fig. 10, the quality of the approximation provided by (3.3) is high too. Comparing Figs. 8 and Fig. 10, we conclude that changing the maneuver has substantially reduced (by a factor of 1.5) the constraint forces (and, hence, the dynamic coefficient of friction). Comparing Figs. 5 and 9 reveals that the maximum rate of turn of the steerable wheel has decreased, i.e., the requirements to the actuating mechanism have been reduced.

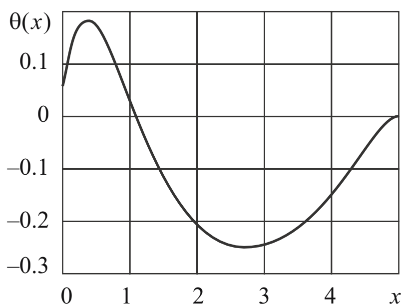

Fig. 2

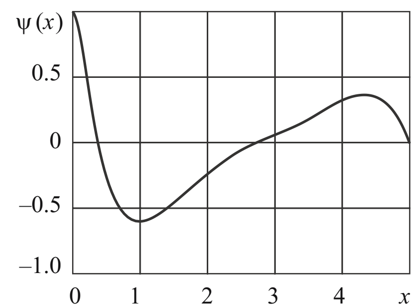

Fig. 3

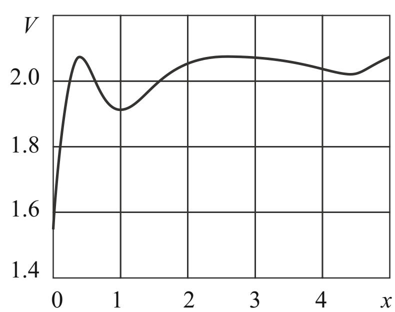

Fig. 4

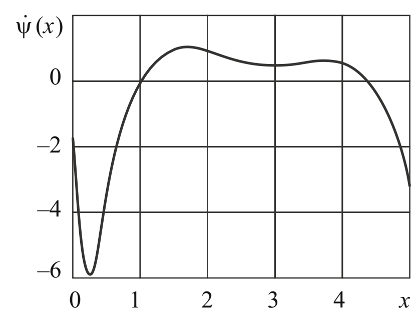

Fig. 5

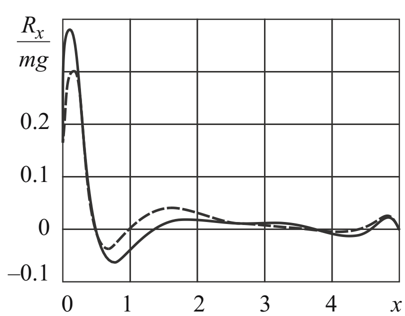

Fig. 6

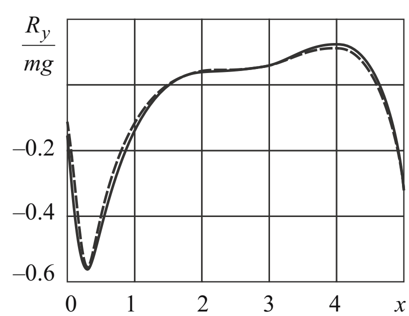

Fig. 7

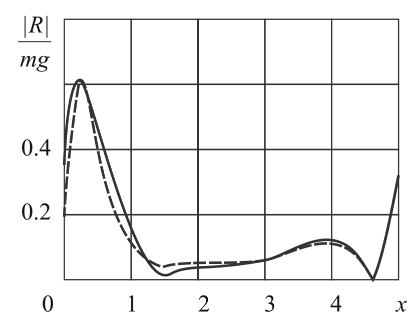

Fig. 8

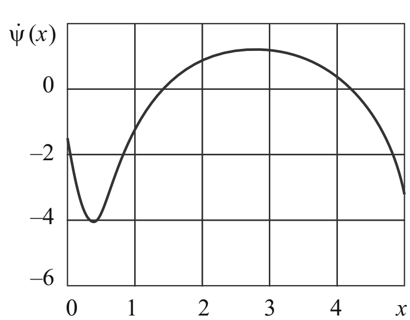

Fig. 9

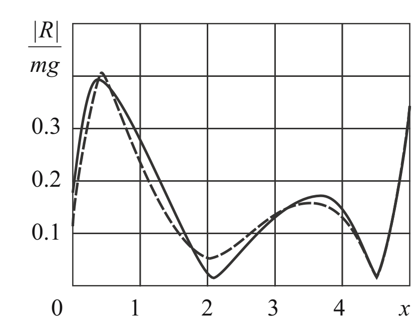

Fig. 10

Conclusions. We have analyzed the problem of determining the constraint forces on a wheeled robotic vehicle with one steerable wheel. In addition to the exact equations, we have derived simplified (asymptotic) equations that allow estimating the constraint forces. The efficiency of the approximate formulas has been demonstrated by way of examples.

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