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ANALYSIS OF THE NONAXISYMMETRIC VIBRATIONS OF FLEXIBLE ELLIPSOIDAL SHELLS DISCRETELY REINFORCED WITH TRANSVERSE RIBS UNDER NONSTATIONARY LOADS

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The problem of forced nonaxisymmetric vibrations of reinforced ellipsoidal shells under nonstationary loads is formulated. A numerical algorithm of solving it is developed and the results obtained are analyzed

Keyword: reinforced ellipsoidal shells, geometrically nonlinear theory, numerical method, nonstationary vibrations

The problem of forced vibrations of reinforced shells is well understood now. According to reviews and monographs on the subject, the axisymmetric and nonaxisymmetric harmonic vibrations of reinforced shells of simple geometry (cylindrical, conical, and spherical) were mainly studied [1–3, 6, 18]. Results on the forced vibrations of reinforced shells under impulsive loads are presented in [7–11, 15–17]. Studies on the dynamic behavior of reinforced shells of more complex geometry are very few. Among them are the studies [8, 9, 11], which are concerned with the forced vibrations of shells of revolution, including reinforced ellipsoidal shells. It is of interest to study the nonaxisymmetric vibrations of shells reinforced with discrete ribs and subjected to nonstationary loads.

We will present equations describing the nonaxisymmetric vibrations of a discretely reinforced ellipsoidal shell. To describe the casing and ribs, we will use Timoshenko's refined model of shells and rods [9, 12]. To derive the vibration equations, the Hamilton–Ostrogradsky variational principle will be used. The dynamic equations will be solved numerically using the integro-interpolation method for differencing equations with discontinuous coefficients. The nonaxisymmetric vibrations of a transversely reinforced ellipsoidal shell under a distributed internal load will be considered as a numerical example.

1. Initial Assumptions for the Casing. Consider an inhomogeneous elastic structure that is an ellipsoidal shell discretely reinforced with transverse ribs. The geometry of the mid-surface of the casing is described as follows [4, 5]:

$$x = R\sin\alpha_1 \sin\alpha_2, \qquad y = R\sin\alpha_1\cos\alpha_2, \qquad z = kR\cos\alpha_1, \tag{1.1}$$

where the parameters α_1 and α_2 are the Gaussian curvilinear coordinates on the shell surface, α_1 representing the meridional direction and α_2 the circumferential direction; $k = b/a$ is the aspect ratio; and a and b are the ellipse semiaxes.

With (1.1), the components of the midsurface metric and shape are expressed as

$$a_{11} = R^2(\cos^2\alpha_1 + k^2\sin^2\alpha_1), \qquad a_{22} = R^2\sin^2\alpha_1,$$

$$b_{11} = kR(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{-1/2}, \qquad b_{22} = kR\sin^2\alpha_1(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{-1/2}. \tag{1.2}$$

According to (1.2), the coefficients of the first quadratic form and the curvature of the ellipsoidal midsurface are given by

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$$A_1 = a(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{1/2}, \qquad A_2 = a\sin\alpha_1,$$

$$k_1 = \frac{b}{a^2}(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{-3/2}, \qquad k_2 = \frac{b}{a^2}(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{-1/2}. \tag{1.3}$$

In modeling the dynamic deformation of the structure, we will use the geometrically nonlinear Timoshenko theory of shells, which is based on the following assumptions.

The displacements along the thickness of the shell are approximated as follows (in the coordinate system (s_1, s_2, z)):

$$u_1^z(s_1, s_2, z) = u_1(s_1, s_2) + z\varphi_1(s_1, s_2), \qquad u_2^z(s_1, s_2, z) = u_2(s_1, s_2) + z\varphi_2(s_1, s_2),$$

$$u_3^z(s_1, s_2, z) = u_3(s_1, s_2), \qquad z \in [-h/2, h/2], \tag{1.4}$$

where u_1, u_2, u_3, φ_1, and φ_2 are the components of the generalized displacement vector of the mid-surface; $s_1 = \alpha_1 A_1$, $s_2 = \alpha_2 A_2$, where A_1 and A_2 are the coefficients of the first quadratic form of the ellipsoidal shell.

The strains are approximated as follows [13]:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial s_1} + k_1 u_3 + \frac{1}{2}\theta_1^2, \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial s_2} + \frac{1}{A_2}\frac{\partial A_2}{\partial s_1}u_1 + k_2 u_3 + \frac{1}{2}\theta_1^2,$$

$$\varepsilon_{12} = \omega + \theta_1\theta_2, \qquad \varepsilon_{13} = \varphi_1 + \theta_1, \qquad \varepsilon_{23} = \varphi_2 + \theta_2, \qquad \omega = \omega_1 + \omega_2,$$

$$\omega_1 = \frac{\partial u_2}{\partial s_1}, \qquad \omega_2 = \frac{\partial u_1}{\partial s_2} - \frac{1}{A_2}\frac{\partial A_2}{\partial s_1}u_2, \qquad \theta_1 = \frac{\partial u_3}{\partial s_1} - k_1 u_1, \qquad \theta_2 = \frac{\partial u_3}{\partial s_2} - k_2 u_2,$$

$$\chi_{11} = \frac{\partial \varphi_1}{\partial s_1}, \qquad \chi_{22} = \frac{\partial \varphi_2}{\partial s_2} + \frac{1}{A_2}\frac{\partial A_2}{\partial s_1}\varphi_1, \qquad \chi_{12} = \tau_1 + \tau_2 + k_1\omega_1 + k_2\omega_2,$$

$$\tau_1 = \frac{\partial \varphi_2}{\partial s_1}, \qquad \tau_2 = \frac{\partial \varphi_1}{\partial s_2} - \frac{1}{A_2}\frac{\partial A_2}{\partial s_1}\varphi_2. \tag{1.5}$$

2. Initial Assumptions for the Ribs. In modeling the deformation of the jth rib aligned with the α_2-axis, we hypothesize that the cross-section of the rib is rigid according to Timoshenko's geometrically nonlinear theory of rods. As a special case, we will use the following approximation of the displacements in the cross-section of the jth rib:

$$U_{1j}^{xz}(x, s_2, z) = U_{1j}(s_2) + z\varphi_{1j}(s_2), \qquad U_{2j}^{xz}(x, s_2, z) = U_{2j}(s_2) + z\varphi_{2j}(s_2),$$

$$U_{3j}^{xz}(x, s_2, z) = U_{3j}(s_2), \tag{2.1}$$

where U_{1j}, U_{2j}, U_{3j}, φ_{1j}, and φ_{2j} are the components of the generalized displacement vector of the center of gravity of the cross-section of the jth rib.

The strains in the ribs are approximated as

$$\varepsilon_{22j} = \frac{\partial u_2}{\partial s_2} \pm h_{cj}\frac{\partial \varphi_2}{\partial s_2} + k_{2j}u_3 + \frac{1}{2}\theta_{1j}^2 + \frac{1}{2}\theta_{2j}^2, \qquad \varepsilon_{21j} = \theta_{2j}, \qquad \varepsilon_{23j} = \varphi_2 + \theta_{1j},$$

$$\theta_{1j} = \frac{\partial u_3}{\partial s_2} - k_{2j}(u_2 \pm h_{cj}\varphi_2), \qquad \theta_2 = \frac{\partial u_1}{\partial s_2} \pm h_{cj}\frac{\partial \varphi_1}{\partial s_2}, \qquad \chi_{21j} = \frac{\partial \varphi_1}{\partial s_2}, \qquad \chi_{22j} = \frac{\partial \varphi_2}{\partial s_2}. \tag{2.2}$$

3. Vibration Equations. The components of the displacement vector of the center of gravity of the cross-section of the jth rib aligned with the α_2-axis and the components of the generalized displacement vector of the initial mid-surface are related as follows [1, 3, 8, 9]:

$$U_{1j}(s_2) = U_1(s_{1j}, s_2) \pm h_{cj}\varphi_2(s_{1j}, s_2), \qquad U_{2j}(s_2) = U_2(s_{1j}, s_2) \pm h_{cj}\varphi_1(s_{1j}, s_2),$$

$$U_{3j}(s_2)=U_3(s_{1j},s_2),\qquad \varphi_{1j}(s_2)=\varphi_2(s_{1j},s_2),\qquad \varphi_{2j}(s_2)=\varphi_1(s_{1j},s_2), \tag{3.1}$$

where $h_{cj}=0.5(h+h_j)$ is the distance from the mid-surface to the line of the center of gravity of the *j*th rib cross-section; h_j is the height of the *j*th rib aligned with the α_2-axis; α_{1j} is the coordinate of the projection line of the center of gravity of the *j*th rib cross-section onto the coordinate mid-surface of the casing.

We will use the integral form of (3.1) to derive the equations of motion [1]:

$$U_{1j}(s_2)=\int_{s_1}\left[U_1(s_{1j},s_2)\pm h_{cj}\varphi_2(s_{1j},s_2)\right]\delta(s_1-s_{1j})A_1ds_1,$$

$$U_{2j}(s_2)=\int_{s_1}\left[U_2(s_{1j},s_2)\pm h_{cj}\varphi_1(s_{1j},s_2)\right]\delta(s_1-s_{1j})A_1ds_1,$$

$$U_{3j}(s_2)=\int_{s_1}U_3(s_{1j},s_2)\delta(s_1-s_{1j})A_1ds_1,$$

$$\varphi_{1j}(s_2)=\int_{s_1}\varphi_2(s_{1j},s_2)\delta(s_1-s_{1j})A_1ds_1,$$

$$\varphi_{2j}(s_2)=\int_{s_1}\varphi_1(s_{1j},s_2)\delta(s_1-s_{1j})A_1ds_1. \tag{3.2}$$

To derive the equations of motion of a discretely reinforced structure, we will use the Hamilton–Ostrogradsky variational principle

$$\int_{t_1}^{t_2}\left[\delta(\Pi-K)+\delta A\right]dt=0, \tag{3.3}$$

where $\Pi=\Pi_0+\sum_{j=1}^{n_2}\Pi_j$, $K=K_0+\sum_{j=1}^{n_2}K_j$, Π_0 and K_0 are the potential and kinetic energies of the casing; Π_j and K_j are the potential and kinetic energies of the *j*th rib; A is the work done by external forces; and δK and $\delta\Pi$ are expressed as

$$\delta\Pi=\delta\Pi_0+\sum_{j=1}^{n_2}\delta\Pi_j,\qquad \delta K=\delta K_0+\sum_{j=1}^{n_2}\delta K_j,$$

$$\delta\Pi_0=\iint_S\left[T_{11}\delta\varepsilon_{11}+T_{22}\delta\varepsilon_{22}+S\delta\varepsilon_{12}+T_{13}\delta\varepsilon_{13}+T_{23}\delta\varepsilon_{23}\right.$$

$$\left.+M_{11}\delta\kappa_{11}+M_{22}\delta\kappa_{22}+H\delta(\tau_1+\tau_2)\right]ds,$$

$$\delta\Pi_j=\int_{l_2}\left[T_{21j}\delta\varepsilon_{21j}+T_{22j}\delta\varepsilon_{22j}+T_{23j}\delta\varepsilon_{23j}+M_{21j}\delta\kappa_{21j}+M_{22j}\delta\kappa_{22j}\right]dl_2,$$

$$\delta K_0=\rho h\iint_S\left[\frac{\partial U_1}{\partial t}\delta\frac{\partial U_1}{\partial t}+\frac{\partial U_2}{\partial t}\delta\frac{\partial U_2}{\partial t}+\frac{\partial U_3}{\partial t}\delta\frac{\partial U_3}{\partial t}+\frac{h^2}{12}\left(\frac{\partial\varphi_1}{\partial t}\delta\frac{\partial\varphi_1}{\partial t}+\delta\frac{\partial\varphi_2}{\partial t}\right)dS\right],$$

$$\delta K_j=\rho_j h_j\int_{l_2}\left[\frac{\partial U_{1j}}{\partial t}\delta\frac{\partial U_{1j}}{\partial t}+\frac{\partial U_{2j}}{\partial t}\delta\frac{\partial U_{2j}}{\partial t}+\frac{\partial U_{3j}}{\partial t}\delta\frac{\partial U_{3j}}{\partial t}\right.$$

$$\left.+\frac{I_{crj}}{F_j}\frac{\partial\varphi_{1j}}{\partial t}\delta\frac{\partial\varphi_{1j}}{\partial t}+\frac{I_{2j}}{F_j}\frac{\partial\varphi_{2j}}{\partial t}\delta\frac{\partial\varphi_{2j}}{\partial t}\right]dl_2. \tag{3.4}$$

After variation and integration in view of conditions (3.1) and (3.2), functional (3.3) becomes

$$\int_{t_1}^{t_2}\left\{\iint_s \left\{\left\{\rho h\frac{\partial^2 U_1}{\partial t^2}-L_1(\overline{U})+\sum_{j=1}^{n_1}\left[\rho_j F_j\frac{\partial^2 U_{1j}}{\partial t^2}-L_{1j}(\overline{U}_j)\right]\delta\left(\alpha_2-\alpha_{2j}\right)\right\}\delta U_1\right.\right.$$

$$+\left\{\rho h\frac{\partial^2 U_2}{\partial t^2}-L_2(\overline{U})+\sum_{j=1}^{n_1}\left[\rho_j F_j\frac{\partial^2 U_{1j}}{\partial t^2}-L_{2j}(\overline{U}_j)\right]\delta(\alpha_2-\alpha_{2j})\right\}\delta U_2$$

$$+\left\{\rho h\frac{\partial^2 U_3}{\partial t^2}-L_3(\overline{U})+\sum_{j=1}^{n_1}\left[\rho_j F_j\frac{\partial^2 U_{1j}}{\partial t^2}-L_{3j}(\overline{U}_j)\right]\delta(\alpha_2-\alpha_{2j})\right\}\delta U_3$$

$$+\left\{\rho\frac{h^3}{12}\frac{\partial^2\varphi_1}{\partial t^2}-L_4(\overline{U})+\sum_{j=1}^{n_1}\left[\rho_j F_j\left(\frac{\partial^2 U_{1j}}{\partial t^2}+\frac{I_{crj}}{F_j}\frac{\partial^2\varphi_{1j}}{\partial t^2}\right)-L_{4j}(\overline{U}_j)\right]\delta(\alpha_1-\alpha_{1j})\right\}\delta\varphi_1$$

$$\left.+\left\{\rho\frac{h^3}{12}\frac{\partial^2\varphi_2}{\partial t^2}-L_5(\overline{U})+\sum_{j=1}^{n_1}\left[\rho_j F_j\left(\frac{\partial^2 U_{2j}}{\partial t^2}+\frac{I_{2j}}{F_j}\frac{\partial^2\varphi_{2j}}{\partial t^2}\right)-L_{5j}(\overline{U}_j)\right]\delta(\alpha_1-\alpha_{1j})\right\}\delta\varphi_2\right\}dS$$

$$\iint_s\left[\rho h\left(\frac{\partial U_1}{\partial t}\delta U_1+\frac{\partial U_2}{\partial t}\delta U_2+\frac{\partial U_3}{\partial t}\delta U_3\right)+\rho\frac{h^3}{12}\left(\frac{\partial\varphi_1}{\partial t}\delta\varphi_1+\frac{\partial\varphi_2}{\partial t}\delta\varphi_2\right)\right]\Bigg|_{t_1}^{t_2}dS$$

$$-\int_{l_1}\left\{\sum_{j=1}^{n_1}\left[\rho_j h_j\left(\frac{\partial U_{1j}}{\partial t}\delta U_{1j}+\frac{\partial U_{2j}}{\partial t}\delta U_{2j}+\frac{\partial U_{3j}}{\partial t}\delta U_{3j}\right)\right.\right.$$

$$\left.\left.+\rho_j\left(I_{crj}\frac{\partial\varphi_{1j}}{\partial t}\delta\varphi_{1j}+I_{2j}\frac{\partial\varphi_{2j}}{\partial t}\delta\varphi_{2j}\right)\right]\delta(\alpha_1-\alpha_{1j})\Bigg|_{t_1}^{t_2}\right\}dl_1=0, \tag{3.5}$$

where

$$L_1(\overline{U})=\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}(A_2T_{11})-\frac{\partial A_2}{\partial\alpha_1}T_{22}+\frac{\partial}{\partial\alpha_2}\left[A_1(S+k_1H)\right]+\frac{\partial A_1}{\partial\alpha_2}(S+k_2H)\right\}+k_1\overline{T}_{13},$$

$$L_2(\overline{U})=\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\left[A_2(S+k_2H)\right]-\frac{\partial A_2}{\partial\alpha_1}(S+k_1H)+\frac{\partial}{\partial\alpha_2}(A_1T_{22})-\frac{\partial A_1}{\partial\alpha_2}T_{11}\right\}+k_2\overline{T}_{23},$$

$$L_3(\overline{U})=\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}(A_2\overline{T}_{13})-\frac{\partial}{\partial\alpha_1}(A_1\overline{T}_{13})\right]-k_1T_{11}-k_2T_{22},$$

$$L_4(\overline{U})=\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}(A_2M_{11})-\frac{\partial A_2}{\partial\alpha_1}M_{22}+\frac{\partial}{\partial\alpha_1}(A_1H)+\frac{\partial A_1}{\partial\alpha_2}H\right]-T_{13},$$

$$L_5(\overline{U})=\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}(A_2H)-\frac{\partial A_2}{\partial\alpha_1}H+\frac{\partial}{\partial\alpha_2}(A_1M_{22})-\frac{\partial A_1}{\partial\alpha_2}M_{11}\right]-T_{23},$$

$$L_{1j}(\overline{U}_j)=\frac{1}{A_2}\frac{\partial\overline{T}_{21j}}{\partial\alpha_2},\qquad L_{2j}(\overline{U}_j)=\frac{1}{A_2}\frac{\partial T_{22j}}{\partial\alpha_2}+k_{2j}\overline{T}_{23j},$$

$$L_{3j}(\overline{U}_j)=\frac{1}{A_2}\frac{\partial \overline{T}_{23j}}{\partial \alpha_2}-k_{2j}T_{22j}, \quad L_{4j}(\overline{U}_j)=\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}(M_{21j}\pm h_{cj}\overline{T}_{21j}),$$

$$L_{5j}(\overline{U}_j)=\frac{1}{A_2}\frac{\partial M_{22j}}{\partial \alpha_2}-T_{23j}\pm h_{cj}\left(\frac{1}{A_2}\frac{\partial T_{22j}}{\partial \alpha_2}+k_{2j}\overline{T}_{23j}\right), \tag{3.6}$$

and $\overline{U}=(U_1,U_2,U_3,\varphi_1,\varphi_2)$, $\overline{U}_j=(U_{1j},U_{2j},U_{3j},\varphi_{1j},\varphi_{2j})$, $l_2=A_2 d\alpha_2$.

After standard transformations, we obtain two groups of equations:

the equations of vibration of the shell in the smooth region

$$\frac{1}{A_2}\left[\frac{\partial}{\partial s_1}(A_2T_{11})-\frac{\partial A_2}{\partial s_1}T_{22}\right]+k_1\overline{T}_{13}+\frac{1}{A_1}\frac{\partial}{\partial s_2}(A_1T_{21})=\rho h\frac{\partial^2 u_1}{\partial t^2},$$

$$\frac{1}{A_2}\left[\frac{\partial}{\partial s_1}(A_2T_{12})-\frac{\partial A_2}{\partial s_1}T_{21}\right]+k_2\overline{T}_{23}+\frac{1}{A_1}\frac{\partial}{\partial s_2}(A_1T_{22})=\rho h\frac{\partial^2 u_2}{\partial t^2},$$

$$\frac{1}{A_2}\frac{\partial}{\partial s_1}(A_2\overline{T}_{13})-k_1T_{11}-k_2T_{22}+P_3+\frac{1}{A_1}\frac{\partial}{\partial s_2}(A_1\overline{T}_{23})=\rho h\frac{\partial^2 u_3}{\partial t^2},$$

$$\frac{1}{A_2}\left[\frac{\partial}{\partial s_1}(A_2M_{11})-\frac{\partial A_2}{\partial s_1}M_{22}\right]-T_{13}+\frac{1}{A_1}\frac{\partial}{\partial s_2}(A_1M_{21})=\rho\frac{h^3}{12}\frac{\partial^2 \varphi_1}{\partial t^2},$$

$$\frac{1}{A_2}\left[\frac{\partial}{\partial s_1}(A_2M_{12})+\frac{\partial A_2}{\partial s_1}M_{21}\right]+\frac{1}{A_1}\frac{\partial}{\partial s_2}(A_1M_{22})-T_{23}=\rho\frac{h^3}{12}\frac{\partial^2 \varphi_2}{\partial t^2}, \tag{3.7}$$

$$T_{11}=B_{11}\varepsilon_{11}+B_{12}\varepsilon_{22}, \quad T_{22}=B_{21}\varepsilon_{11}+B_{22}\varepsilon_{22}, \quad T_{12}=S+k_2H, \quad T_{21}=S+k_1H,$$

$$T_{13}=B_{13}\varepsilon_{13}, \quad T_{23}=B_{23}\varepsilon_{23}, \quad \overline{T}_{13}=T_{13}+T_{11}\theta_1+S\theta_2, \quad \overline{T}_{23}=T_{23}+T_{22}\theta_2+S\theta_1,$$

$$S=B_s\varepsilon_{12}, \quad M_{11}=D_{11}\chi_{11}+D_{12}\chi_{22}, \quad M_{22}=D_{21}\chi_{11}+D_{22}\chi_{22}, \quad M_{12}=M_{21}=H, \quad H=D_s\chi_{12}, \tag{3.8}$$

the equations of vibration of the jth rib

$$\frac{\partial \overline{T}_{21j}}{\partial s_2}+[T_{11}]_j=\rho_jF_j\left(\frac{\partial^2 U_1}{\partial t^2}\pm h_{cj}\frac{\partial^2 \varphi_1}{\partial t^2}\right),$$

$$\frac{\partial T_{22j}}{\partial s_2}+k_{2j}\overline{T}_{23j}+[S]_j=\rho_jF_j\left(\frac{\partial^2 U_2}{\partial t^2}\pm h_{cj}\frac{\partial^2 \varphi_2}{\partial t^2}\right),$$

$$\frac{\partial \overline{T}_{23j}}{\partial s_2}-k_{2j}T_{22j}+[\overline{T}_{13}]_j=\rho_jF_j\frac{\partial^2 U_3}{\partial t^2},$$

$$\frac{\partial M_{21j}}{\partial s_2}\pm h_{cj}\frac{\partial \overline{T}_{21j}}{\partial s_2}+[M_{11}]_j=\rho_jF_j\left(\pm h_{cj}\frac{\partial^2 U_1}{\partial t^2}+\left(h_{cj}^2+\frac{I_{crj}}{F_j}\right)\frac{\partial^2 \varphi_1}{\partial t^2}\right),$$

$$\frac{\partial M_{22j}}{\partial s_2}-T_{23j}\pm h_{cj}\left(\frac{\partial T_{22j}}{\partial s_2}+k_{2j}\overline{T}_{23j}\right)+[H]_j=\rho_jF_j\left(\pm h_{cj}\frac{\partial^2 U_2}{\partial t^2}+\left(h_{cj}^2+\frac{I_{2j}}{F_j}\right)\frac{\partial^2 \varphi_2}{\partial t^2}\right), \tag{3.9}$$

$$\overline{T}_{21j}=T_{21j}+T_{22j}\theta_{1j}, \quad T_{21j}=G_jF_j\varepsilon_{21j}, \quad T_{22j}=E_jF_j\varepsilon_{22j}, \quad \overline{T}_{23j}=T_{23j}+T_{22j}\theta_{2j},$$

$$T_{23j}=G_jF_jk_j^2\varepsilon_{23j}, \quad M_{21j}=G_jI_{crj}\chi_{21j}, \quad M_{22j}=E_jI_{2j}\chi_{22j}. \tag{3.10}$$

Equations (3.7) and (3.10) are supplemented with natural boundary and initial conditions, which follow from (3.5).

4. Numerical Algorithm of Solving Nonlinear Problems. Equations (3.7)–(3.9) constitute a system of nonlinear partial differential equations with the variables s_1, s_2, and t and spatial discontinuities in s_2. The spatial discontinuities are the projection lines of the centers of gravity of the rib cross-sections onto the mid-surface of the ellipsoidal shell. Because of this, a numerical algorithm for solving the original problem is as follows: the solution is sought in the smooth region of the ellipsoidal shell (Eq. (3.7)) and on the spatial discontinuity lines (Eqs. (3.9)) [8, 9]. The equations in the smooth region and on the discontinuity lines are written and integrated. The difference algorithm is based on the integro-interpolation method of differencing in the space coordinates and an explicit finite-difference approximation in the time coordinate [14]. The components of the generalized displacement vector are approximated at integer points of the difference mesh, and the components of strains and forces at half-integer points. Such an approach allows maintaining the divergent form of the finite-difference equations and the validity of the law of conservation of total mechanical energy in difference form [3]. The transition from the continuous system to the finite-difference system involves two stages. The first stage is finite-difference approximation of the divergent vibration equations written for forces and moments.

Equations (3.7) can be approximated as

$$L_1(\bar{U}^n_{l,m}) + P^n_{1l,m} = \rho h (U^n_{1l,m})_{\bar{t}t}, \quad L_2(\bar{U}^n_{l,m}) + P^n_{2l,m} = \rho h (U^n_{2l,m})_{\bar{t}t},$$

$$L_3(\bar{U}^n_{l,m}) + P^n_{3l,m} = \rho h (U^n_{3l,m})_{\bar{t}t}, \quad L_4(\bar{U}^n_{l,m}) = \rho \frac{h^3}{12} (\varphi^n_{1l,m})_{\bar{t}t}, \quad L_5(\bar{U}^n_{l,m}) = \rho \frac{h^3}{12} (\varphi^n_{2l,m})_{\bar{t}t}, \tag{4.1}$$

where

$$L_1(\bar{U}^n_{l,m}) = \frac{1}{A_{2l}} \left(\frac{A_{2l+1/2,m} T^n_{11l+1/2,m} - A_{2l-1/2,m} T^n_{11l-1/2,m}}{\Delta s_1} \right)$$

$$- \frac{1}{A_{2l}} \frac{A_{2l+1/2} - A_{2l-1/2}}{\Delta s_1} T^n_{22l,m} + \frac{1}{A_{1l}} \frac{A_{1l} T^n_{21l,m+1/2} - A_{1l} T^n_{21l,m-1/2}}{\Delta s_2} + k_{1l} \bar{T}^n_{13l,m},$$

$$L_2(\bar{U}^n_{l,m}) = \frac{1}{A_{2l}} \left(\frac{A_{2l+1/2} T^n_{12l+1/2,m} - A_{2l-1/2} T^n_{12l-1/2,m}}{\Delta s_1} \right)$$

$$- \frac{1}{A_{2l}} \frac{A_{2l+1/2} - A_{2l-1/2}}{\Delta s_1} T^n_{21\,l,m} + \frac{1}{A_{1l}} \left(\frac{A_{1l} T^n_{22\,l,m+1/2} - A_{1l} T^n_{22\,l,m-1/2}}{\Delta s_2} \right) + k_{2l} \bar{T}^n_{23\,l,m},$$

$$L_3(\bar{U}^n_{l,m}) = \frac{1}{A_{2l}} \left(\frac{A_{2l+1/2} \bar{T}^n_{13l+1/2,m} - A_{2l-1/2} \bar{T}^n_{13l-1/2,m}}{\Delta s_1} \right) - k_{1l} T^n_{11l,m}$$

$$+ \frac{1}{A_{1l}} \left(\frac{A_{1l} \bar{T}^n_{23l,m+1/2} - A_{1l} \bar{T}^n_{23l,m-1/2}}{\Delta s_2} \right) - k_{2l} T^n_{22l,m},$$

$$L_4(\bar{U}^n_{l,m}) = \frac{1}{A_{2l}} \left(\frac{A_{2l+1/2} M^n_{11l+1/2,m} - A_{2l-1/2} M^n_{11l-1/2,m}}{\Delta s_1} \right) - \frac{1}{A_{2l}} \frac{A_{2l+1/2} - A_{2l-1/2}}{\Delta s_1} M^n_{22l,m}$$

$$+ \frac{1}{A_{1l}} \left(\frac{A_{1l} M^n_{21l,m+1/2} - A_{1l} M^n_{21l,m-1/2}}{\Delta s_2} \right) - T^n_{13l,m},$$

$$L_5(\overline{U}^n_{l,m}) = \frac{1}{A_{2l}}\left(\frac{A_{2l+1/2}M^n_{12l+1/2,m} - A_{2l-1/2}M^n_{12l-1/2,m}}{\Delta s_1}\right) - \frac{1}{A_{2l}}\frac{A_{2l+1/2} - A_{2l-1/2}}{\Delta s_1}M^n_{21l,m}$$

$$+\frac{1}{A_{1l}}\left(\frac{A_{1l}M^n_{22l,m+1/2} - A_{1l}M^n_{22l,m-1/2}}{\Delta s_2}\right) - T^n_{23l,m}. \tag{4.2}$$

Equations (3.9) are approximated as

$$L_{1j}(\overline{U}^n_{l,m}) = \rho_j F_j (U^n_{1l,m})_{\bar{t}t} \pm h_{cj}\rho_j F_j(\varphi^n_{1l,m})_{\bar{t}t},$$

$$L_{2j}(\overline{U}^n_{l,m}) = \rho_j F_j (U^n_{2l,m})_{\bar{t}t} \pm h_{cj}\rho_j F_j(\varphi^n_{2l,m})_{\bar{t}t}, \qquad L_{3j}(\overline{U}^n_{l,m}) = \rho_j F_j (U^n_{3l,m})_{\bar{t}t},$$

$$L_{4j}(\overline{U}^n_{l,m}) = \pm h_{cj}\rho_j F_j (U^n_{1l,m})_{\bar{t}t} + \rho_j F_j\left(h^2_{cj} + \frac{I_{crj}}{F_j}\right)(\varphi^n_{1l,m})_{\bar{t}t},$$

$$L_{5j}(\overline{U}^n_{l,m}) = \pm h_{cj}\rho_j F_j (U^n_{2l,m})_{\bar{t}t} + \rho_j F_j\left(h^2_{cj} + \frac{I_{2j}}{F_j}\right)(\varphi^n_{2l,m})_{\bar{t}t}, \tag{4.3}$$

where

$$L_{1j}(\overline{U}^n_{l,m}) = \frac{\overline{T}^n_{21j,m+1/2} - \overline{T}^n_{21j,m-1/2}}{\Delta s_2} + [T_{11}]^n_j,$$

$$L_{2j}(\overline{U}^n_{l,m}) = \frac{T^n_{22j,m+1/2} - T^n_{22j,m-1/2}}{\Delta s_2} + k_{2j,m}\overline{T}^n_{23j,m} + [S]^n_j,$$

$$L_{3j}(\overline{U}^n_{l,m}) = \frac{\overline{T}^n_{23j,m+1/2} - \overline{T}^n_{23j,m-1/2}}{\Delta s_2} - k_{2j,m}T^n_{22j,m} + [\overline{T}_{13}]^n_j,$$

$$L_{4j}(\overline{U}^n_{l,m}) = \frac{M^n_{21j,m+1/2} - M^n_{21j,m-1/2}}{\Delta s_2} \pm h_{cj}\frac{\overline{T}^n_{21j,m+1/2} - \overline{T}^n_{21j,m-1/2}}{\Delta s_2} + [M_{11}]^n_j,$$

$$L_{5j}(\overline{U}^n_{l,m}) = \frac{M^n_{22j,m+1/2} - M^n_{22j,m-1/2}}{\Delta s_2} - T^n_{23j,m} \pm h_{cj}\left(\frac{T^n_{22j,m+1/2} - T^n_{22j,m-1/2}}{\Delta s_2} + k_{2j,m}\overline{T}^n_{23j,m}\right) + [H]^n_j, \tag{4.4}$$

where $L_k(\overline{U}^n_{l,m})$ and $L_{kj}(\overline{U}^n_{l,m})$, $k = \overline{1,5}$, are the difference operators of the left-hand sides of Eqs. (3.7)–(3.9) with respect to the discrete space coordinates s_{1l} and s_{1l} in the $(n–m)$th time slice. The difference derivatives are denoted as in [14].

The second stage is finite-difference approximation of forces, moments, and strains such that the finite-difference energy equation is valid [12].

In analyzing the linearized difference equations for stability, use is made of the necessary stability condition

$$\Delta t \le 2/\omega,$$

where $\omega = \max(\omega_0, \omega_j)$, $j = \overline{1,J}$, are the maximum natural frequencies of the discrete-difference system of the casing and the jth rib.

5. Numerical Example. Let us analyze the forced vibrations of a transversely reinforced ellipsoidal shell with clamped edges in the region $D = \{\alpha_{10} \le \alpha_1 \le \alpha_{1N},\ \alpha_{20} \le \alpha_2 \le \alpha_{2N}\}$ under a distributed normal load $P_3(\alpha_1, \alpha_2, t)$. The boundary conditions are $\overline{U}(\alpha_{10}, \alpha_2) = \overline{U}(\alpha_{1N}, \alpha_2) = 0$ and $\overline{U}(\alpha_1, \alpha_{20}) = \overline{U}(\alpha_1, \alpha_{2N}) = 0$. The initial conditions are zero at $t = 0$ for all the components of the generalized displacement vector:

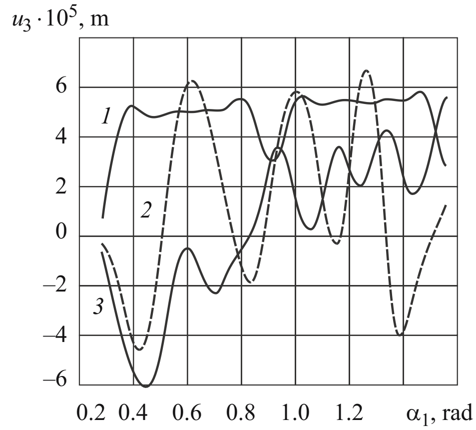

Fig. 1

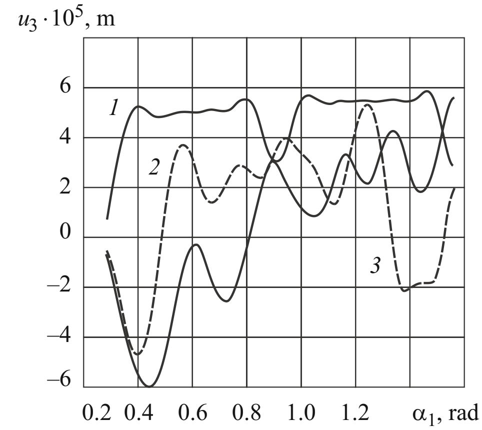

Fig. 2

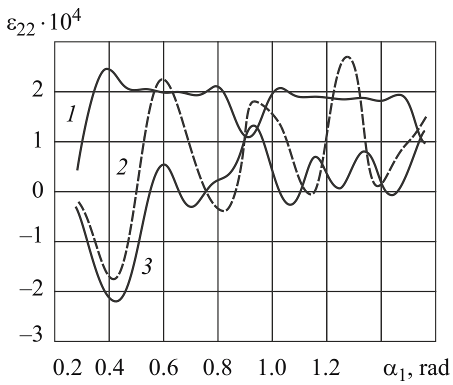

Fig. 3

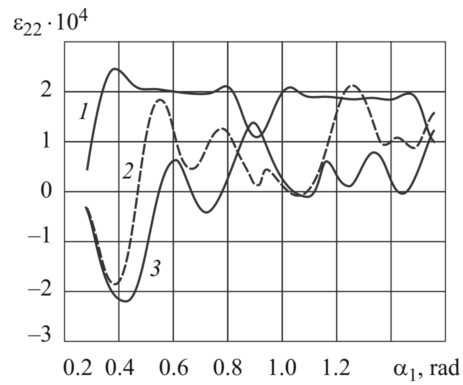

Fig. 4

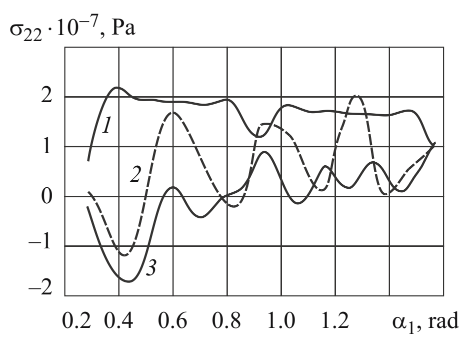

Fig. 5

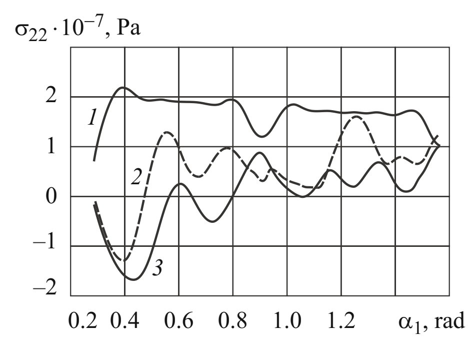

Fig. 6

$$u_1(\alpha_1, \alpha_2) = u_2(\alpha_1, \alpha_2) = u_3(\alpha_1, \alpha_2) = \varphi_1(\alpha_1, \alpha_2) = \varphi_2(\alpha_1, \alpha_2) = 0,$$

$$\frac{\partial u_1(\alpha_1, \alpha_2)}{\partial t} = \frac{\partial u_2(\alpha_1, \alpha_2)}{\partial t} = \frac{\partial u_3(\alpha_1, \alpha_2)}{\partial t} = \frac{\partial \varphi_1(\alpha_1, \alpha_2)}{\partial t} = \frac{\partial \varphi_2(\alpha_1, \alpha_2)}{\partial t} = 0.$$

The distributed normal load $P_3(\alpha_1, \alpha_2, t)$ is given by

$$P_3(\alpha_1, \alpha_2, t) = A \cdot \sin \frac{\pi t}{T} [\eta(t) - \eta(t - T)],$$

where A is the amplitude of the load; T is the duration of the load. It is assumed that $A = 10^6$ Pa and $T = 50$ μsec.

We will use a geometrically linear problem formulation and the following geometrical, physical, and mechanical parameters of the original structure (isotropic case): $\alpha_{10} = \pi/12$, $\alpha_{1N} = \pi - \pi/12$, $\alpha_{20} = -\pi/2$, $\alpha_{2N} = \pi/2$, $a/h = 30$, $E = 7\cdot 10^{10}$ Pa, $\nu_{12} = \nu_{21} = 0.33$, $\rho = 2.7\cdot 10^3$ kg/m^3.

The transverse ribs are arranged in the cross-sections $\alpha_{1j} = \frac{7}{24}\pi + \frac{5}{24}\pi j$, $j = 0, 1, 2$, along the α_2-axis.

Figures 1–6 show the most typical curves for u_3, ε_{22}, and σ_{22} on the time interval $t = 10T$. They can be used to analyze the stress–strain state of the structure. Figures 1 (for external ribs) and 2 (for internal ribs) show u_3 as a function of the spatial coordinate α_1 in the section $\alpha_2 = 0$ in the range $\alpha_{10} \le \alpha_1 \le \pi/2$ (due to symmetry) at $t_1 = 1T$, $t_2 = 3T$, and $t_3 = 8T$. Figures 3 and 4 present similar curves for ε_{22}, and Figs. 5 and 6 for σ_{22}. It can be visually seen where the ribs are located in the sections α_{1j} ($j = 0, 1, 2$). The maximum amplitudes of u_3 and strains ε_{22} obtained with internal or external ribs differ by 30% in some cases (curve *2* in Figs. 1–4).

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