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FORMATION OF THE PLASTIC ZONE IN AN ANISOTROPIC BODY WITH A CRACK

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The influence of the length of a mode I crack on the plastic zone in an anisotropic body under hard loading is studied. The case of a generalized plane stress state is examined. A boundary-value problem is solved numerically to study the behavior of the main plastic zone at the crack tip, the additional plastic zone on the lateral face of the body, and the merged plastic zone

Keywords: anisotropic body, mode I crack, plastic zone

Introduction. Various crack models are widely used in elastoplastic fracture mechanics [9–11, 13–16, 19]. To justify these models, it is necessary to know the size and shape of the plastic zone at a crack and to solve the corresponding boundary-value problems. The papers [4–6, etc.] solved (both analytically and numerically) many boundary-value problems for plane and antiplane strains and generalized plane stress state. However, they are all concerned with the plastic zone in an isotropic body. The plastic zone in an anisotropic body is still inadequately understood. There are just a few studies on the subject [17, 18], where several boundary-value problems have been solved (numerically) for the case of plane strain and the effect of anisotropy and loads along the crack on the size and shape of the plastic zone has been established.

This paper studies the plastic zone at a crack in an anisotropic body in the case of a generalized plane stress state. Strains are assumed small. The body is rectangular and thin and has a mode I crack at the center. The governing equations are written for the components of the displacement vector. By numerically solving the boundary-value problem, we can describe how the plastic zone forms and, in particular, can establish the effect of the crack length on the size and shape of the plastic zone.

1. Preliminaries. We assume that Poynting's effect is absent when a body is deformed. Hence, tensor-linear constitutive equations may be used to derive the governing equations.

1.1. Tensor-Linear Constitutive Equations. The following equations are derived in [2] to relate the components of the stress tensor $\boldsymbol{S}$ with the components of the strain tensor $\boldsymbol{D}$:

$$S^{\alpha\beta} = \frac{H}{Z} g^{\alpha\beta} + \sqrt{\frac{K - \dfrac{H^2}{Z}}{\Xi - \dfrac{E^2}{Z}}} \left(G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \frac{E}{Z} g^{\alpha\beta} \right), \tag{1.1}$$

where

$$E = g^{\alpha\beta} D_{\alpha\beta}, \quad Z = F_{\alpha\beta\gamma\delta} g^{\alpha\beta} g^{\gamma\delta}, \quad H = F_{\alpha\beta\gamma\delta} g^{\alpha\beta} S^{\gamma\delta},$$

$$K = F_{\alpha\beta\gamma\delta} S^{\alpha\beta} S^{\gamma\delta}, \quad \Xi = G^{\alpha\beta\gamma\delta} D_{\alpha\beta} D_{\gamma\delta}. \tag{1.2}$$

Note that the anisotropy tensors $\boldsymbol{F}$ and $\boldsymbol{G}$ are reciprocal, i.e.,

$$F_{\alpha\beta\gamma\delta} G^{\alpha\beta\varepsilon\zeta} = \delta_\gamma^\varepsilon \delta_\delta^\zeta (\varepsilon, \zeta), \tag{1.3}$$

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where

$$\delta_{\eta}^{\iota} = \begin{cases} 1, & \eta = \iota, \\ 0, & \eta \neq \iota. \end{cases} \quad (1.4)$$

The components of $\boldsymbol{F}$ are to be derived from experimental dependences of all the components of the strain tensor $\boldsymbol{D}$ on each component of the stress tensor $\boldsymbol{S}$. Using formulas (1.3) and the components of $\boldsymbol{F}$, we can calculate the components of $\boldsymbol{G}$.

The above-mentioned anisotropy tensors show high symmetry. In other words, the indices in any index pair and the index pairs themselves can be interchanged in these tensors.

We will show that Eqs. (1.1) go over into the Hencky–Nadai equations [12, 21] with certain restrictions imposed on the components of $\boldsymbol{F}$.

Let the components of $\boldsymbol{F}$ be expressed in terms of two constants (ρ and σ):

$$F_{\alpha\beta\gamma\delta} = \rho g_{\alpha\beta} g_{\gamma\delta} + \sigma g_{\alpha\gamma} g_{\beta\delta} \quad (\gamma, \delta). \quad (1.5)$$

With (1.5), formulas (1.3) become

$$\rho G^{\alpha\beta\varepsilon\zeta} g_{\alpha\beta} g_{\gamma\delta} + \sigma G^{\alpha\beta\varepsilon\zeta} g_{\alpha\gamma} g_{\beta\delta} = \delta_{\gamma}^{\varepsilon} \delta_{\delta}^{\zeta} \quad (\varepsilon, \zeta). \quad (1.6)$$

Contracting formulas (1.6) by $g^{\gamma\eta} g^{\delta\vartheta}$ and considering (1.4), we get

$$\rho G^{\alpha\beta\varepsilon\zeta} g_{\alpha\beta} g^{\eta\vartheta} + \sigma G^{\eta\vartheta\varepsilon\zeta} = g^{\varepsilon\eta} g^{\zeta\vartheta} \quad (\varepsilon, \zeta)$$

or

$$\rho G^{\alpha\beta\varepsilon\zeta} g_{\alpha\beta} g^{\eta\vartheta} + \sigma G^{\varepsilon\zeta\eta\vartheta} = g^{\varepsilon\eta} g^{\zeta\vartheta} \quad (\varepsilon, \zeta). \quad (1.7)$$

Contracting formulas (1.7) by $g_{\varepsilon\zeta}$, we arrive at

$$\rho G^{\alpha\beta\varepsilon\zeta} g_{\alpha\beta} g_{\varepsilon\zeta} g^{\eta\vartheta} + \sigma G^{\varepsilon\zeta\eta\vartheta} g_{\varepsilon\zeta} = g^{\eta\vartheta}. \quad (1.8)$$

Contracting formulas (1.8) by $g_{\eta\vartheta}$, we find

$$3\rho G^{\alpha\beta\varepsilon\zeta} g_{\alpha\beta} g_{\varepsilon\zeta} + \sigma G^{\varepsilon\zeta\eta\vartheta} g_{\varepsilon\zeta} g_{\eta\vartheta} = 3. \quad (1.9)$$

Formula (1.9) yields

$$G^{\alpha\beta\varepsilon\zeta} g_{\alpha\beta} g_{\varepsilon\zeta} = \frac{3}{3\rho + \sigma}. \quad (1.10)$$

Substituting formula (1.10) into (1.8), we derive

$$G^{\varepsilon\zeta\eta\vartheta} g_{\varepsilon\zeta} = \frac{1}{3\rho + \sigma} g^{\eta\vartheta}. \quad (1.11)$$

Formulas (1.11) yield

$$G^{\alpha\beta\varepsilon\zeta} g_{\alpha\beta} = \frac{1}{3\rho + \sigma} g^{\varepsilon\zeta}. \quad (1.12)$$

Based on (1.7) and (1.12), we obtain

$$G^{\varepsilon\zeta\eta\vartheta} = \frac{1}{\sigma} \left(g^{\varepsilon\eta} g^{\zeta\vartheta} - \frac{\rho}{3\rho + \sigma} g^{\varepsilon\zeta} g^{\eta\vartheta} \right) \quad (\varepsilon, \zeta). \quad (1.13)$$

Formulas (1.13) yield

$$G^{\alpha\beta\gamma\delta} = \frac{1}{\sigma}\left(g^{\alpha\gamma} g^{\beta\delta} - \frac{\rho}{3\rho+\sigma} g^{\alpha\beta} g^{\gamma\delta} \right) \quad (\gamma, \delta). \tag{1.14}$$

Using (1.14) and the first formula in (1.2), we find

$$G^{\alpha\beta\gamma\delta} D_{\gamma\delta} = \frac{1}{\sigma}\left(g^{\alpha\gamma} g^{\beta\delta} D_{\gamma\delta} - \frac{\rho}{3\rho+\sigma} E g^{\alpha\beta} \right). \tag{1.15}$$

We will transform the invariants Z, H, K, and Ξ as follows (see (1.2)).

In view of (1.5), the invariant Z becomes

$$Z = 3(3\rho + \sigma). \tag{1.16}$$

For the invariants H and K, formulas (1.5) yield

$$H = (3\rho+\sigma)\Phi, \quad K = \rho\Phi^2 + \sigma X \quad (\Phi = g_{\gamma\delta} S^{\gamma\delta}, \quad X = g_{\alpha\gamma} g_{\beta\delta} S^{\alpha\beta} S^{\gamma\delta}). \tag{1.17}$$

With (1.14) and the first formula in (1.2), the invariant Ξ becomes

$$\Xi = \frac{1}{\sigma}\left(Y - \frac{\rho}{3\rho+\sigma} E^2 \right), \tag{1.18}$$

where $Y = g^{\alpha\gamma} g^{\beta\delta} D_{\alpha\beta} D_{\gamma\delta}$.

Using formulas (1.15)–(1.18), we transform Eqs. (1.1) as follows:

$$S^{\alpha\beta} = \frac{\Phi}{3} g^{\alpha\beta} + \sqrt{\frac{X - \frac{\Phi^2}{3}}{Y - \frac{E^2}{3}}} \left(g^{\alpha\gamma} g^{\beta\delta} D_{\gamma\delta} - \frac{E}{3} g^{\alpha\beta} \right).$$

Hence, we arrive at the Hencky–Nadai equations [12, 21]. The invariants of the tensors ***S*** and ***D*** in Eqs. (1.1) should be related to each other.

According to [8], we may assume that

$$H = E. \tag{1.19}$$

A thermodynamic analysis of Eqs. (1.1) shows [3] that $\sqrt{K - H^2/Z}$ is a single-valued function of $\sqrt{\Xi - E^2/Z}$:

$$\sqrt{K - \frac{H^2}{Z}} = \varphi\left(\sqrt{\Xi - \frac{E^2}{Z}} \right).$$

Consider some positive constant υ. Assume that the equation

$$\sqrt{K - \frac{H^2}{Z}} = \sqrt{\Xi - \frac{E^2}{Z}} \tag{1.20}$$

holds if $\sqrt{\Xi - E^2/Z} < \upsilon$ and the equation

$$\sqrt{K - \frac{H^2}{Z}} = \sqrt{\Xi - \frac{E^2}{Z}} \left[1 - \tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}} \right) \right] \tag{1.21}$$

holds if $\sqrt{\Xi - E^2 / Z} \geq \upsilon$, where

$$\tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right) = \frac{\sqrt{\Xi - \frac{E^2}{Z}} - \upsilon + p - \sqrt[3]{\sqrt{\left(\frac{q}{3}\right)^3 + \left(\frac{r}{2}\right)^2} - \frac{r}{2}} + \sqrt[3]{\sqrt{\left(\frac{q}{3}\right)^3 + \left(\frac{r}{2}\right)^2} + \frac{r}{2}}}{\sqrt{\Xi - \frac{E^2}{Z}}}, \tag{1.22}$$

$$p = \frac{1}{3}\frac{b}{c}, \qquad q = \frac{1}{3}\frac{b^2}{c^2} + \frac{1}{c}, \qquad r = \frac{2}{27}\frac{b^3}{c^3} - \frac{1}{3}\frac{b}{c^2} - \frac{1}{c}\left(\sqrt{\Xi - \frac{\mathrm{E}^2}{\mathrm{Z}}} - \upsilon\right). \tag{1.23}$$

The constants b and c are to be derived from experimental dependence of $\sqrt{\Xi - E^2 / Z}$ on $\sqrt{K - H^2 / Z}$, which should be fitted by a polynomial:

$$\sqrt{\Xi - \frac{E^2}{Z}} = \sqrt{K - \frac{H^2}{Z}} + b\left(\sqrt{K - \frac{H^2}{Z}} - \upsilon\right)^2 + c\left(\sqrt{K - \frac{H^2}{Z}} - \upsilon\right)^3$$

when $\sqrt{K - H^2 / Z} \geq \upsilon$.

Considering (1.19) and (1.21), we write Eqs. (1.1) as follows:

$$S^{\alpha\beta} = G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\left(G^{\alpha\beta\gamma\delta} D_{\gamma\delta} - \frac{E}{Z} g^{\alpha\beta}\right). \tag{1.24}$$

1.2. Plasticity Criterion. If formulas (1.20) and (1.21) hold, we have the following plasticity criterion [17]:

$$\sqrt{\Xi - \frac{E^2}{Z}} = \upsilon. \tag{1.25}$$

Let us show that criterion (1.25) transforms into the von Mises criterion [20] with the same restrictions on the components of the tensor $\boldsymbol{F}$.

With (1.16) and (1.18), criterion (1.25) becomes

$$\sqrt{\frac{1}{\sigma}\left(Y - \frac{E^2}{3}\right)} = \upsilon. \tag{1.26}$$

Introducing a new constant υ', namely, $\upsilon' = \sigma\upsilon^2$, we represent criterion (1.26) as $Y - E^2 / 3 = \upsilon'$. Hence, we arrive at the von Mises criterion [20].

2. Generalities. Assume that the body is described in a Cartesian coordinate system x^1, x^2, x^3 such that

$$g^{\varepsilon\zeta} = \begin{cases} 1, & \varepsilon = \zeta, \\ 0, & \varepsilon \neq \zeta. \end{cases} \tag{2.1}$$

2.1. Constitutive Equations. Let us write the constitutive equations for the displacement vector $\boldsymbol{u}$. The components of the strain tensor $\boldsymbol{D}$ and the components of the displacement vector $\boldsymbol{u}$ are related as follows [7]:

$$D_{\varepsilon\zeta} = \frac{\partial u_\varepsilon}{\partial x^\zeta} \, (\varepsilon, \zeta). \tag{2.2}$$

In view of (2.2), Eqs. (1.24) become

$$S^{\alpha\beta} = G^{\alpha\beta\gamma\delta}\frac{\partial u_\gamma}{\partial x^\delta} - \tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\left(G^{\alpha\beta\gamma\delta}\frac{\partial u_\gamma}{\partial x^\delta} - \frac{E}{Z}g^{\alpha\beta}\right). \tag{2.3}$$

Let us dwell on the invariants Z, E, and Ξ (see (1.2)). With (2.1), the invariant Z becomes simpler:

$$Z = F_{1111} + F_{1122} + F_{1133} + F_{2211} + F_{2222} + F_{2233} + F_{3311} + F_{3322} + F_{3333}. \tag{2.4}$$

With (2.1) and (2.2), the invariant E becomes

$$E = \frac{\partial u_1}{\partial x^1} + \frac{\partial u_2}{\partial x^2} + \frac{\partial u_3}{\partial x^3}. \tag{2.5}$$

In view of (2.2), the invariant Ξ becomes

$$\Xi = G^{\alpha\beta\gamma\delta}\frac{\partial u_\alpha}{\partial x^\beta}\frac{\partial u_\gamma}{\partial x^\delta}. \tag{2.6}$$

Assume that the body is orthotropic and the principal axes of orthotropy are aligned with the x^1-, x^2-, and x^3-axes. Consider a generalized plane stress state:

$$S^{11} = S^{11}(x^1, x^2), \qquad S^{12} = S^{12}(x^1, x^2), \qquad S^{22} = S^{22}(x^1, x^2), \tag{2.7}$$

$$S^{13} = 0, \qquad S^{23} = 0, \qquad S^{33} = 0. \tag{2.8}$$

Considering the third equation in (2.8) and formula (2.1) and using Eqs. (2.3), we establish

$$0 = G^{3311}\frac{\partial u_1}{\partial x^1} + G^{3322}\frac{\partial u_2}{\partial x^2} + G^{3333}\frac{\partial u_3}{\partial x^3}$$

$$-\tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\left(G^{3311}\frac{\partial u_1}{\partial x^1} + G^{3322}\frac{\partial u_2}{\partial x^2} + G^{3333}\frac{\partial u_3}{\partial x^3} - \frac{E}{Z}\right). \tag{2.9}$$

Equation (2.9) yields

$$\frac{\partial u_3}{\partial x^3} = \frac{1}{G^{3333}}\left[\tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\left(G^{3311}\frac{\partial u_1}{\partial x^1} + G^{3322}\frac{\partial u_2}{\partial x^2} + G^{3333}\frac{\partial u_3}{\partial x^3} - \frac{E}{Z}\right) - G^{3311}\frac{\partial u_1}{\partial x^1} - G^{3322}\frac{\partial u_2}{\partial x^2}\right]. \tag{2.10}$$

Considering formulas (2.1) and Eq. (2.10) and using Eqs. (2.3), we find

$$S^{\alpha\beta} = \left(G^{\alpha\beta 11} - \frac{G^{\alpha\beta 33}}{G^{3333}}G^{3311}\right)\frac{\partial u_1}{\partial x^1} + \left(G^{\alpha\beta 22} - \frac{G^{\alpha\beta 33}}{G^{3333}}G^{3322}\right)\frac{\partial u_2}{\partial x^2}$$

$$-\tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\left[\left(G^{\alpha\beta 11} - \frac{G^{\alpha\beta 33}}{G^{3333}}G^{3311}\right)\frac{\partial u_1}{\partial x^1} + \left(G^{\alpha\beta 22} - \frac{G^{\alpha\beta 33}}{G^{3333}}G^{3322}\right)\frac{\partial u_2}{\partial x^2} - \left(1 - \frac{G^{\alpha\beta 33}}{G^{3333}}\right)\frac{E}{Z}\right]$$

$$(\alpha, \beta = 1, 2, \alpha = \beta) \tag{2.11}$$

Considering formulas (2.1) and using Eqs. (2.3), we obtain

$$S^{\alpha\beta} = G^{\alpha\beta 12}\frac{\partial u_1}{\partial x^2} + G^{\alpha\beta 21}\frac{\partial u_2}{\partial x^1} - \tilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\left(G^{\alpha\beta 12}\frac{\partial u_1}{\partial x^2} + G^{\alpha\beta 21}\frac{\partial u_2}{\partial x^1}\right)$$

$$(\alpha, \beta = 1, 2, \alpha \neq \beta). \tag{2.12}$$

Since $\widetilde{\varphi}(\sqrt{\Xi - E^2 / Z}) \neq 0$, considering formulas (2.1) and the first and second equations in (2.8) and using Eqs. (2.3), we get

$$\frac{\partial u_1}{\partial x^3} + \frac{\partial u_3}{\partial x^1} = 0, \quad \frac{\partial u_2}{\partial x^3} + \frac{\partial u_3}{\partial x^2} = 0. \tag{2.13}$$

Let

$$G^{1111} \equiv \mu_{AA}, \quad G^{1212} \equiv \mu_{BB}, \quad G^{1122} \equiv \mu_{AC}, \quad G^{2222} \equiv \mu_{CC},$$
$$G^{1133} \equiv \mu_{AF}, \quad G^{2233} \equiv \mu_{CF}, \quad G^{3333} \equiv \mu_{FF}. \tag{2.14}$$

With (2.13) and (2.14), formula (2.6) becomes

$$\Xi = \mu_{AA} \frac{\partial u_1}{\partial x^1}\frac{\partial u_1}{\partial x^1} + 2\mu_{AC} \frac{\partial u_1}{\partial x^1}\frac{\partial u_2}{\partial x^2} + \mu_{CC} \frac{\partial u_2}{\partial x^2}\frac{\partial u_2}{\partial x^2} + \mu_{BB}\left(\frac{\partial u_1}{\partial x^2}\frac{\partial u_1}{\partial x^2} + 2\frac{\partial u_1}{\partial x^2}\frac{\partial u_2}{\partial x^1} + \frac{\partial u_2}{\partial x^1}\frac{\partial u_2}{\partial x^1} \right)$$
$$+ 2\mu_{AF} \frac{\partial u_1}{\partial x^1}\frac{\partial u_3}{\partial x^3} + 2\mu_{CF} \frac{\partial u_2}{\partial x^2}\frac{\partial u_3}{\partial x^3} + \mu_{FF} \frac{\partial u_3}{\partial x^3}\frac{\partial u_3}{\partial x^3}. \tag{2.15}$$

The equilibrium equations can be written for the components of the stress tensor ***S*** as follows [7]:

$$\frac{\partial S^{\alpha\beta}}{\partial x^\beta} = 0. \tag{2.16}$$

With (2.7) and (2.8), Eqs. (2.16) yield

$$\frac{\partial S^{11}}{\partial x^1} + \frac{\partial S^{12}}{\partial x^2} = 0, \quad \frac{\partial S^{21}}{\partial x^1} + \frac{\partial S^{22}}{\partial x^2} = 0. \tag{2.17}$$

Then we suppose that

$$\frac{G^{1133}}{G^{3333}} \equiv \xi_{AF}, \quad \frac{G^{2233}}{G^{3333}} \equiv \xi_{CF}, \tag{2.18}$$

and

$$G^{1111} - \frac{G^{1133}}{G^{3333}} G^{3311} \equiv \widehat{\mu}_{AA}, \quad G^{1122} - \frac{G^{1133}}{G^{3333}} G^{3322} \equiv \widehat{\mu}_{AC},$$
$$G^{2211} - \frac{G^{2233}}{G^{3333}} G^{3311} \equiv \widehat{\mu}_{CA}, \quad G^{2222} - \frac{G^{2233}}{G^{3333}} G^{3322} \equiv \widehat{\mu}_{CC}. \tag{2.19}$$

It is obvious that $\widehat{\mu}_{AC} = \widehat{\mu}_{CA}$.

Let the anisotropy tensor ***G*** be independent of the coordinates x^1 and x^2. Then substituting Eqs. (2.11) and (2.12) into (2.17) and considering the second notation in (2.14) and notation (2.18) and (2.19), we find

$$\widehat{\mu}_{AA} \frac{\partial^2 u_1}{\partial x^1 \partial x^1} + \mu_{BB} \frac{\partial^2 u_1}{\partial x^2 \partial x^2} + (\widehat{\mu}_{AC} + \mu_{BB}) \frac{\partial^2 u_2}{\partial x^1 \partial x^2} = Q^1,$$
$$\mu_{BB} \frac{\partial^2 u_2}{\partial x^1 \partial x^1} + \widehat{\mu}_{CC} \frac{\partial^2 u_2}{\partial x^2 \partial x^2} + (\mu_{BB} + \widehat{\mu}_{CA}) \frac{\partial^2 u_1}{\partial x^1 \partial x^2} = Q^2, \tag{2.20}$$

where

$$Q^1 = \frac{\partial \widehat{T}_A}{\partial x^1} + \frac{\partial T_B}{\partial x^2}, \quad Q^2 = \frac{\partial T_B}{\partial x^1} + \frac{\partial \widehat{T}_C}{\partial x^2}. \tag{2.21}$$

Formulas (2.21) contain

$$\widehat{T}_A = \widetilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\left[\widehat{\mu}_{AA}\frac{\partial u_1}{\partial x^1} + \widehat{\mu}_{AC}\frac{\partial u_2}{\partial x^2} - (1-\xi_{AF})\frac{E}{Z}\right],$$

$$T_B = \widetilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\mu_{BB}\left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1}\right),$$

$$\widehat{T}_C = \widetilde{\varphi}\left(\sqrt{\Xi - \frac{E^2}{Z}}\right)\left[\widehat{\mu}_{CA}\frac{\partial u_1}{\partial x^1} + \widehat{\mu}_{CC}\frac{\partial u_2}{\partial x^2} - (1-\xi_{CF})\frac{E}{Z}\right]. \tag{2.22}$$

Thus, we have derived differential equations with the second-order partial derivatives of the components u_1 and u_2 with respect to the coordinates x^1 and x^2.

Either the stress vector $\boldsymbol{P}$ or the displacement vector u^* can be specified on the surface of the body.

The boundary conditions can be written for the components of the stress vector $\boldsymbol{P}$ as follows [7]:

$$S^{\alpha\beta} n_\beta = P^\alpha, \tag{2.23}$$

where n_β are the components of the unit outward normal vector to the body surface.

Using (2.23) and (2.8), we get

$$S^{11} n_1 + S^{12} n_2 = P^1, \quad S^{21} n_1 + S^{22} n_2 = P^2. \tag{2.24}$$

Substituting Eqs. (2.11) and (2.12) into (2.24) and considering the second notation in (2.14), notation (2.18) and (2.19), and formulas (2.22), we obtain

$$\left(\widehat{\mu}_{AA}\frac{\partial u_1}{\partial x^1} + \widehat{\mu}_{AC}\frac{\partial u_2}{\partial x^2}\right)n_1 + \mu_{BB}\left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1}\right)n_2 = P^1 + R^1,$$

$$\mu_{BB}\left(\frac{\partial u_1}{\partial x^2} + \frac{\partial u_2}{\partial x^1}\right)n_1 + \left(\widehat{\mu}_{CA}\frac{\partial u_1}{\partial x^1} + \widehat{\mu}_{CC}\frac{\partial u_2}{\partial x^2}\right)n_2 = P^2 + R^2, \tag{2.25}$$

where

$$R^1 = \widehat{T}_A n_1 + T_B n_2, \quad R^2 = T_B n_1 + \widehat{T}_C n_2. \tag{2.26}$$

Thus, we have derived differential equations with first-order partial derivatives of the components u_1 and u_2 with respect to the coordinates x^1 and x^2.

Equations (2.20) and (2.25) can be integrated using Il'yushin's method of successive approximations [1]. To this end, Q^1, Q^2 and R^1, R^2 should be set at zero as a first approximation and then calculated in all the subsequent approximations from the components u_1 and u_2 determined in the previous approximation.

2.2. Statement of the Boundary-Value Problem. Consider a thin rectangular body with a central crack. The symmetry axes of the body are aligned with the x^1- and x^2-axes.

The components P^1 and P^2 are prescribed on the lower and upper crack surfaces and on the lateral surfaces of the body. The components u_1^* and u_2^* are prescribed on the lower and upper surfaces of the body symmetrically about the x^1- and x^2-axes. In this connection, it suffices to consider a quarter the body (Fig. 1).

On the upper crack surface, we have $-n_1 = 1$ and $n_2 = 0$. Equations (2.25) become

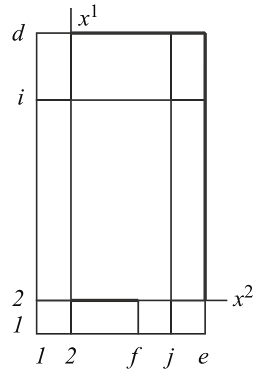

Fig. 1

$$-\left(\hat{\mu}_{AA}\frac{\partial u_1}{\partial x^1}+\hat{\mu}_{AC}\frac{\partial u_2}{\partial x^2}\right)=P^1+R^1,\qquad -\mu_{BB}\left(\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}\right)=P^2+R^2. \tag{2.27}$$

Formulas (2.26) yield

$$-R^1=\hat{T}_A,\qquad -R^2=T_B. \tag{2.28}$$

On the lateral surface of the body, $n_1=0$ and $n_2=1$. Equations (2.25) become

$$\mu_{BB}\left(\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}\right)=P^1+R^1,\qquad \hat{\mu}_{CA}\frac{\partial u_1}{\partial x^1}+\hat{\mu}_{CC}\frac{\partial u_2}{\partial x^2}=P^2+R^2. \tag{2.29}$$

Formulas (2.26) yield

$$R^1=T_B,\qquad R^2=\hat{T}_C. \tag{2.30}$$

On the upper surface of the body

$$u_1=u_1^*,\qquad u_2=u_2^*. \tag{2.31}$$

Due to the symmetry about the x^1-axis, we have

$$u_1(x^1,-x^2)-u_1(x^1,+x^2)=0,\qquad u_2(x^1,-x^2)+u_2(x^1,+x^2)=0. \tag{2.32}$$

Due to the symmetry about the x^2-axis, we have

$$u_1(-x^1,x^2)+u_1(+x^1,x^2)=0,\qquad u_2(-x^1,x^2)-u_2(+x^1,x^2)=0, \tag{2.33}$$

and

$$u_1=0,\qquad \frac{\partial u_2}{\partial x^1}=0 \tag{2.34}$$

at the crack tip.

Note that Eqs. (2.20), (2.27), (2.29), (2.31), and (2.32)–(2.34) are the governing equations for the components u_1 and u_2.

2.3. Discretization of Variables to Transform the Governing Equations. Let us form a mesh of coordinates (with a spacing h):

$$\underset{i}{x}^1=(i-2)h\quad (i=1,\ldots,d),\qquad \underset{j}{x}^2=(j-2)h\quad (j=1,\ldots,e).$$

Denote

$$u_1(\underset{i}{x^1},\underset{j}{x^2}) \equiv y_s, \quad u_2(\underset{i}{x^1},\underset{j}{x^2}) \equiv y_t, \tag{2.35}$$

where

$$s = 2\big[(i-1)e + j - f\big] + 1, \quad t = 2\big[(i-1)e + j - f\big] + 2. \tag{2.36}$$

Considering notation (2.35), expressing the partial derivatives of u_1 and u_2 with respect to x^1 and x^2 (at the point $(\underset{i}{x^1},\underset{j}{x^2})$) in terms of finite differences, and using Eqs. (2.20), (2.27), (2.29), and (2.31)–(2.34), we obtain n linear algebraic equations ($n = 2(\mathrm{de} - f + 1)$) with variables $y_1,\dots,y_n$:

$$A_{ss}y_s + A_{ss+2e}y_{s+2e} + A_{ss-2e}y_{s-2e} + A_{ss+2}y_{s+2} + A_{ss-2}y_{s-2}$$

$$+A_{st+2(e+1)}y_{t+2(e+1)} + A_{st+2(e-1)}y_{t+2(e-1)} + A_{st-2(e-1)}y_{t-2(e-1)} + A_{st-2(e+1)}y_{t-2(e+1)} \approx B_s,$$

$$A_{tt}y_t + A_{tt+2e}y_{t+2e} + A_{tt-2e}y_{t-2e} + A_{tt+2}y_{t+2} + A_{tt-2}y_{t-2}$$

$$+A_{ts+2(e+1)}y_{s+2(e+1)} + A_{ts+2(e-1)}y_{s+2(e-1)} + A_{ts-2(e-1)}y_{s-2(e-1)} + A_{ts-2(e+1)}y_{s-2(e+1)} \approx B_t$$

$$(i=2, \quad j=f+1,\dots,e-1, \quad i=3,\dots,d-1, \quad j=2,\dots,e-1),$$

$$A_{ss}y_s + A_{ss+2e}y_{s+2e} + A_{ss+4e}y_{s+4e} + A_{st+2}y_{t+2} + A_{st-2}y_{t-2} \approx B_s,$$

$$A_{ts+2}y_{s+2} + A_{ts-2}y_{s-2} + A_{tt}y_t + A_{tt+2e}y_{t+2e} + A_{tt+4e}y_{t+4e} \approx B_t$$

$$(i=2, \quad j=2,\dots,f-1),$$

$$A_{ss}y_s + A_{ss-2}y_{s-2} + A_{ss-4}y_{s-4} + A_{st+2e}y_{t+2e} + A_{st-2e}y_{t-2e} \approx B_s,$$

$$A_{ts+2e}y_{s+2e} + A_{ts-2e}y_{s-2e} + A_{tt}y_t + A_{tt-2}y_{t-2} + A_{tt-4}y_{t-4} \approx B_t$$

$$(i=2,\dots,d-1, \quad j=e),$$

$$A_{ss}y_s = B_s, \quad A_{tt}y_t = B_t \quad (i=d, \quad j=2,\dots,e),$$

$$A_{s-2s-2}y_{s-2} + A_{s-2s+2}y_{s+2} = B_{s-2}, \quad A_{t-2t-2}y_{t-2} + A_{t-2t+2}y_{t+2} = B_{t-2} \quad (i=2,\dots,d, \quad j=2),$$

$$A_{s-2es-2e}y_{s-2e} + A_{s-2es+2e}y_{s+2e} = B_{s-2e}, \quad A_{t-2et-2e}y_{t-2e} + A_{t-2et+2e}y_{t+2e} = B_{t-2e} \quad (i=2, \quad j=f,\dots,e),$$

$$A_{ss}y_s = B_s, \quad A_{tt}y_t + A_{tt+2e}y_{t+2e} + A_{tt+4e}y_{t+4e} \approx B_t \quad (i=2, \quad j=f), \tag{2.37}$$

where

$$-A_{ss} = 8(\hat{\mu}_{AA} + \mu_{BB}), \quad A_{ss+2e} = 4\hat{\mu}_{AA}, \quad A_{ss-2e} = 4\hat{\mu}_{AA}, \quad A_{ss+2} = 4\mu_{BB}, \quad A_{ss-2} = 4\mu_{BB},$$

$$A_{st+2(e+1)} = \hat{\mu}_{AC} + \mu_{BB}, \quad -A_{st+2(e-1)} = \hat{\mu}_{AC} + \mu_{BB},$$

$$-A_{st-2(e-1)} = \hat{\mu}_{AC} + \mu_{BB}, \quad A_{st-2(e+1)} = \hat{\mu}_{AC} + \mu_{BB}, \quad B_s = 4h^2 Q^1(\underset{i}{x^1},\underset{j}{x^2}),$$

$$-A_{tt} = 8(\mu_{BB} + \hat{\mu}_{CC}), \quad A_{tt+2e} = 4\mu_{BB}, \quad A_{tt-2e} = 4\mu_{BB}, \quad A_{tt+2} = 4\hat{\mu}_{CC}, \quad A_{tt-2} = 4\hat{\mu}_{CC},$$

$$A_{ts+2(e+1)} = \mu_{BB} + \hat{\mu}_{CA}, \quad -A_{ts+2(e-1)} = \mu_{BB} + \hat{\mu}_{CA},$$

$$-A_{ts-2(e-1)} = \mu_{BB} + \hat{\mu}_{CA}, \quad A_{ts-2(e+1)} = \mu_{BB} + \hat{\mu}_{CA}, \quad B_t = 4h^2 Q^2(\underset{i}{x^1},\underset{j}{x^2})$$

$$(i=2,\quad j=f+1,\ldots,e-1,\quad i=3,\ldots,d-1,\quad j=2,\ldots,e-1),$$

$$A_{ss}=3\widehat{\mu}_{AA},\quad -A_{ss+2e}=4\widehat{\mu}_{AA},\quad A_{ss+4e}=\widehat{\mu}_{AA},\quad -A_{st+2}=\widehat{\mu}_{AC},\quad A_{st-2}=\widehat{\mu}_{AC},$$

$$B_s=2h[P^1(x_i^1,x_j^2)+R^1(x_{i^1},x_{j^2})],$$

$$-A_{ts+2}=\mu_{BB},\quad A_{ts-2}=\mu_{BB},\quad A_{tt}=3\mu_{BB},\quad -A_{tt+2e}=4\mu_{BB},\quad A_{tt+4e}=\mu_{BB},$$

$$B_t=2h[P^2(x_i^1,x_j^2)+R^2(x_i^1,x_j^2)]\quad (i=2,\quad j=2,\ldots,f-1),$$

$$A_{ss}=3\mu_{BB},\quad -A_{ss-2}=4\mu_{BB},\quad A_{ss-4}=\mu_{BB},\quad A_{st+2e}=\mu_{BB},\quad -A_{st-2e}=\mu_{BB},$$

$$B_s=2h[P^1(x_i^1,x_j^2)+R^1(x_i^1,x_j^2)],$$

$$A_{ts+2e}=\widehat{\mu}_{CA},\quad -A_{ts-2e}=\widehat{\mu}_{CA},\quad A_{tt}=3\widehat{\mu}_{CC},\quad -A_{tt-2}=4\widehat{\mu}_{CC},\quad A_{tt-4}=\widehat{\mu}_{CC},$$

$$B_t=2h[P^2(x_i^1,x_j^2)+R^2(x_i^1,x_j^2)]\quad (i=2,\ldots,d-1,\quad j=e),$$

$$A_{ss}=1,\quad B_s=u_1^*(x_i^1,x_j^2),\qquad A_{tt}=1,\quad B_t=u_2^*(x_i^1,x_j^2)\quad (i=d,\quad j=2,\ldots,e),$$

$$A_{s-2s-2}=1,\quad -A_{s-2s+2}=1,\quad B_{s-2}=0,\quad A_{t-2t-2}=1,\quad A_{t-2t+2}=1,\quad B_{t-2}=0$$

$$(i=2,\ldots,d,\quad j=2),$$

$$A_{s-2es-2e}=1,\quad A_{s-2es+2e}=1,\quad B_{s-2e}=0,\quad A_{t-2et-2e}=1,\quad -A_{t-2et+2e}=1,\quad B_{t-2e}=0$$

$$(i=2,\quad j=f,\ldots,e),$$

$$A_{ss}=1,\quad B_s=0,\quad -A_{tt}=3,\quad A_{tt+2e}=4,\quad -A_{tt+4e}=1,\quad B_t=0\quad (i=2,\quad j=f). \tag{2.38}$$

It is also needed to express $Q^1(x_i^1,x_j^2), Q^2(x_i^1,x_j^2)$ and $R^1(x_i^1,x_j^2), R^2(x_i^1,x_j^2)$ in terms of $y_1,\ldots,y_n$.

Expressing the partial derivatives of $\widehat{T}_A, T_B$, and $\widehat{T}_C$ with respect to x^1 and x^2 in terms of finite differences and using formulas (2.21), we establish

$$Q^1(x_i^1,x_j^2)\approx\frac{1}{2h}\left(\widehat{T}_A\big|_{(x_{i+1}^1,x_j^2)}-\widehat{T}_A\big|_{(x_{i-1}^1,x_j^2)}+T_B\big|_{(x_{i+1}^1,x_j^2)}-T_B\big|_{(x_{i-1}^1,x_j^2)}\right),$$

$$Q^2(x_i^1,x_j^2)\approx\frac{1}{2h}\left(T_B\big|_{(x_{i+1}^1,x_j^2)}-T_B\big|_{(x_{i-1}^1,x_j^2)}+\widehat{T}_C\big|_{(x_i^1,x_{j+1}^2)}-\widehat{T}_C\big|_{(x_i^1,x_{j-1}^2)}\right)$$

$$(i=2,\quad j=f+1,\ldots,e-1,\quad i=3,\ldots,d-1,\quad j=2,\ldots,e-1). \tag{2.39}$$

Formulas (2.28) yield

$$-R^1(x_i^1,x_j^2)=\widehat{T}_A\big|_{(x_i^1,x_j^2)},\qquad -R^2(x_i^1,x_j^2)=T_B\big|_{(x_i^1,x_j^2)}$$

$$(i=2,\quad j=2,\ldots,f-1), \tag{2.40}$$

and formulas (2.30) yield

$$R^1(x_i^1, x_j^2) = T_B\big|_{(x_i^1, x_j^2)}, \qquad R^2(x_i^1, x_j^2) = \hat{T}_C\big|_{(x_i^1, x_j^2)}$$

$$(i = 2, \ldots, d-1, \quad j = e). \tag{2.41}$$

Due to the symmetry about the x^1-axis, we have

$$-T_B\big|_{(x_i^1, x_1^2)} = T_B\big|_{(x_i^1, x_3^2)}, \qquad -\hat{T}_C\big|_{(x_i^1, x_1^2)} = \hat{T}_C\big|_{(x_i^1, x_3^2)}$$

$$(i = 3, \ldots, d-1). \tag{2.42}$$

and due to the symmetry about the x^2-axis, we have

$$-\hat{T}_A\big|_{(x_1^1, x_j^2)} = \hat{T}_A\big|_{(x_3^1, x_j^2)}, \qquad -T_B\big|_{(x_1^1, x_j^2)} = T_B\big|_{(x_3^1, x_j^2)}$$

$$(j = f+1, \ldots, e-1). \tag{2.43}$$

The first-order partial derivatives of u_1 and u_2 with respect to x^1 and x^2 appearing in the formulas for $\hat{T}_A$, T_B, and $\hat{T}_C$ should be expressed in terms of finite differences considering (2.35).

Equations (2.37) are solved using the modified Gaussian elimination proposed in [17].

3. Studying the Formation of the Plastic Zone. Let us examine the effect of the length of a mode I crack on the plastic zone in a body under hard loading in the case of a generalized plane stress state.

3.1. Solution of the Boundary-Value Problem. To solve the boundary-value problem, we used data for D16 alloy reported in the paper [2], which has determined the components of $\boldsymbol{F}$ of which important ones are

$$F_{1111} = 0.193 \cdot 10^{-10}\ \text{Pa}^{-1}, \quad -F_{1122} = 0.045 \cdot 10^{-10}\ \text{Pa}^{-1}, \quad -F_{1133} = 0.049 \cdot 10^{-10}\ \text{Pa}^{-1},$$

$$F_{1212} = 0.107 \cdot 10^{-10}\ \text{Pa}^{-1}, \quad F_{1313} = 0.121 \cdot 10^{-10}\ \text{Pa}^{-1}, \quad F_{2222} = 0.142 \cdot 10^{-10}\ \text{Pa}^{-1},$$

$$-F_{2233} = 0.045 \cdot 10^{-10}\ \text{Pa}^{-1}, \quad F_{2323} = 0.107 \cdot 10^{-10}\ \text{Pa}^{-1}, \quad F_{3333} = 0.193 \cdot 10^{-10}\ \text{Pa}^{-1},$$

and the components of $\boldsymbol{G}$ of which important are

$$G^{1111} = 6.395 \cdot 10^{10}\ \text{Pa}, \quad G^{1122} = 2.744 \cdot 10^{10}\ \text{Pa}, \quad G^{1133} = 2.263 \cdot 10^{10}\ \text{Pa},$$

$$G^{1212} = 2.336 \cdot 10^{10}\ \text{Pa}, \quad G^{1313} = 2.066 \cdot 10^{10}\ \text{Pa}, \quad G^{2222} = 8.781 \cdot 10^{10}\ \text{Pa},$$

$$G^{2233} = 2.744 \cdot 10^{10}\ \text{Pa}, \quad G^{2323} = 2.336 \cdot 10^{10}\ \text{Pa}, \quad G^{3333} = 6.395 \cdot 10^{10}\ \text{Pa}.$$

The paper [2] has also established the dependence of $\sqrt{\Xi - E^2 / Z}$ on $\sqrt{K - H^2 / Z}$ such that $\upsilon = 3.25 \cdot 10^2\ \text{Pa}^{1/2}$ and $b = 0.1964347 \cdot 10^{-2}\ \text{Pa}^{-1/2}$, $c = 0.5632820 \cdot 10^{-4}\ \text{Pa}^{-1}$ for $h = 0.2$ mm, $d = 302, e = 152, f = 62, 42, 22$.

Note that with such values of h and f, the crack length L is equal to 24, 16, 8 mm.

The boundary-value problem has been solved for

$$P^1(x_i^1, x_j^2) = 0, \quad P^2(x_i^1, x_j^2) = 0 \quad (i = 2, j = 2, \ldots, f-1),$$

$$P^1(x_i^1, x_j^2) = 0, \quad P^2(x_i^1, x_j^2) = 0 \quad (i = 2, \ldots, d-1, j = e),$$

$$u_1^*(x_i^1, x_j^2) > 0, \quad u_2^*(x_i^1, x_j^2) = 0 \quad (i = d, j = 2, \ldots, e).$$

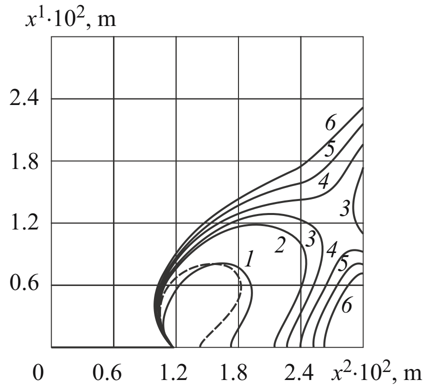

Fig. 2

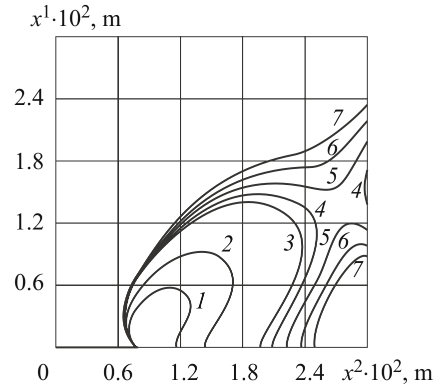

Fig. 3

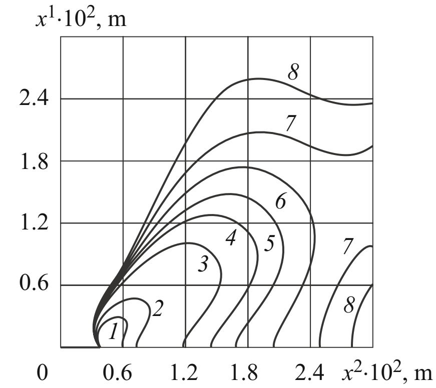

Fig. 4

The values of $y_1, \ldots, y_n$ have been found in nine approximations. In doing so, $Q^1(x^1_i, x^2_j)$, $Q^2(x^1_i, x^2_j)$ and $R^1(x^1_i, x^2_j)$, $R^2(x^1_i, x^2_j)$ were set at zero in the first approximation and calculated in subsequent approximations (by formulas (2.39)–(2.41) and (2.42), (2.43), (2.22), (1.22), (1.23), (2.4), (2.5), (2.15), and (2.10)) from the values of $y_1, \ldots, y_n$ found in the previous approximation.

Note that the ninth approximation has revealed values of the indices *i* and *j* at which the radical $\sqrt{\Xi - E^2 / Z}$ on the left-hand side of (1.25) becomes larger and smaller than the constant υ. This made it possible to calculate the coordinates of points on the boundary of the plastic zone.

3.2. Analysis of the Results. The plastic zone is illustrated in Figs. 2 ($L = 2.4 \cdot 10^{-2}$ m), 3 ($L = 1.6 \cdot 10^{-2}$ m), and 4 ($L = 0.8 \cdot 10^{-2}$ m) for the following values of $u_1^*(x^1_i, x^2_j) \cdot 10^6$, m: 70 (curve *1*), 80 (curve *2*), 82 (curve *3*), 84 (curve *4*), 86 (curve *5*), 88 (curve *6*).

The dashed line represents the boundary of the plastic zone in the case of plane strain ($u_1^*(x^1_i, x^2_j) = 70 \cdot 10^{-6}$ m) [17].

Note that the main plastic zone occurring at the crack tip when $u_1^*(x^1_i, x^2_j) = 70 \cdot 10^{-6}$ m in the case of a generalized plane stress state is different from that observed in the case of plane strain. Indeed, it has shifted far along the x^2-axis and become larger (more than twice as long along the x^2-axis as before). The subsequent behavior of this plastic zone remained unchanged. It expanded under tension and bent toward the lateral surface of the body.

An additional plastic zone formed near the point ($x^1_{70} = 13.6$ mm, $x^2_{152} = 30$ mm) on the lateral surface at some $u_1^*(x^1_i, x^2_j)$ larger than $80 \cdot 10^{-6}$ m but smaller than $82 \cdot 10^{-6}$ m. Under further tension, both plastic zones expanded and merged, forming a single plastic zone. The merged plastic zone expanded further. It is noteworthy that its shape near the lateral surface of the body substantially changed.

The curves in Fig. 3 ($L = 16$ mm) correspond to the following values of $u_1^*(x^1_i, x^2_j) \cdot 10^6$, m: 70 (curve *1*), 80 (curve *2*), 88 (curve *3*), 89 (curve *4*), 90 (curve *5*), 91 (curve *6*), 92 (curve *7*).

Note that the main plastic zone formed at the crack tip at $u_1^*(x^1_i, x^2_j) = 70 \cdot 10^{-6}$ m became much smaller. As before, the main plastic zone expanded and bent toward the lateral surface of the body under tension. An additional plastic zone formed near the point ($x^1_{78} = 15.2$ mm, $x^2_{152} = 30$ mm) on the lateral surface at some $u_1^*(x^1_i, x^2_j)$ larger than $88 \cdot 10^{-6}$ m but smaller than $89 \cdot 10^{-6}$ m. Under further tension, the main and additional plastic zones expanded and merged, forming a single plastic zone, which further expanded. Its shape near the lateral surface also significantly changed.

TABLE 1

$x^1_i \cdot 10^2$, m	$x^2_j \cdot 10^2$, m				
	2.28	2.46	2.64	2.82	3.00
1.08	1.5840 –0.0048 –0.4408	1.5597 0.0281 –0.4082	1.5393 0.0394 –0.3811	1.5330 0.0303 –0.3639	1.5526 0.0000 –0.3620
1.26	1.5408 –0.0464 –0.4230	1.5360 –0.0103 –0.3994	1.5335 0.0085 –0.3786	1.5438 0.0114 –0.3661	1.5791 0.0000 –0.3682
1.44	1.4973 –0.0781 –0.4000	1.5074 –0.0423 –0.3852	1.5191 –0.0183 –0.3716	1.5415 –0.0053 –0.3645	1.5853 0.0000 –0.3696
1.62	1.4567 –0.1007 –0.3768	1.4771 –0.0665 –0.3694	1.4986 –0.0395 –0.3624	1.5285 –0.0185 –0.3601	1.5748 0.0000 –0.3671
1.80	1.4201 –0.1156 –0.3558	1.4468 –0.0833 –0.3542	1.4741 –0.0546 –0.3525	1.5074 –0.0279 –0.3540	1.5520 0.0000 –0.3618

TABLE 2

$x^1_i \cdot 10^2$, m	$x^2_j \cdot 10^2$, m				
	2.28	2.46	2.64	2.82	3.00
1.08	1.5972 0.0201 –0.4045	1.5772 0.0286 –0.3869	1.5622 0.0284 –0.3735	1.5544 0.0194 –0.3650	1.5578 0.0000 –0.3632
1.26	1.5838 –0.0034 –0.4029	1.5720 0.0094 –0.3872	1.5641 0.0141 –0.3749	1.5640 0.0110 –0.3675	1.5762 0.0000 –0.3675
1.44	1.5662 –0.0244 –0.3958	1.5621 –0.0090 –0.3835	1.5610 –0.0002 –0.3736	1.5671 0.0024 –0.3681	1.5850 0.0000 –0.3695
1.62	1.5471 –0.0415 –0.3860	1.5492 –0.0248 –0.3774	1.5536 –0.0130 –0.3704	1.5641 –0.0052 –0.3670	1.5848 0.0000 –0.3695
1.80	1.5279 –0.0544 –0.3755	1.5345 –0.0374 –0.3703	1.5428 –0.0234 –0.3661	1.5559 –0.0115 –0.3646	1.5771 0.0000 –0.3677

TABLE 3

$x^1_i \cdot 10^2$, m	$x^2_j \cdot 10^2$, m				
	2.28	2.46	2.64	2.82	3.00
1.08	1.6044 0.0048 –0.3828	1.5966 0.0064 –0.3774	1.5905 0.0064 –0.3733	1.5866 0.0044 –0.3706	1.5859 0.0000 –0.3697
1.26	1.6017 –0.0021 –0.3833	1.5954 0.0009 –0.3779	1.5908 0.0024 –0.3738	1.5886 0.0021 –0.3712	1.5901 0.0000 –0.3707
1.44	1.5975 –0.0090 –0.3823	1.5928 –0.0047 –0.3774	1.5896 –0.0018 –0.3736	1.5888 –0.0003 –0.3712	1.5917 0.0000 –0.3711
1.62	1.5923 –0.0152 –0.3801	1.5890 –0.0101 –0.3760	1.5868 –0.0060 –0.3727	1.5870 –0.0027 –0.3708	1.5906 0.0000 –0.3708
1.80	1.5866 –0.0204 –0.3774	1.5842 –0.0148 –0.3741	1.5829 –0.0097 –0.3714	1.5834 –0.0049 –0.3698	1.5870 0.0000 –0.3700

The curves in Fig. 4 ($L = 8$ mm) correspond to the following values of $u_1^*(x^1_i, x^2_j) \cdot 10^6$, m: 70 (curve *1*), 80 (curve *2*), 90 (curve *3*), 92 (curve *4*), 93 (curve *5*), 94 (curve *6*), 95 (curve *7*), 96 (curve *8*). Note that the main plastic zone occurred at the crack tip at $u_1^*(x^1_i, x^2_j) = 70 \cdot 10^{-6}$ m became even smaller. With increasing $u_1^*(x^1_i, x^2_j)$, the main plastic zone expanded and bent toward the lateral surface, but to a smaller extent. An additional plastic zone formed near the point ($x^1_{75} = 14.6$ mm, $x^2_{152} = 30$ mm) on the lateral surface at some $u_1^*(x^1_i, x^2_j)$ larger than $94 \cdot 10^{-6}$ m but smaller than $95 \cdot 10^{-6}$ m. Under further tension, the main and additional plastic zones expanded and merged, forming a single plastic zone, which further expanded. Its shape near the lateral surface did not change so significantly.

Noteworthy is that the position of the point near which the additional plastic zone formed on the lateral face is weakly dependent on the crack length L. For example, while the coordinate x^1_i of this point did increase from 1.36 mm to 1.52 mm as the crack length L decreased from 24 mm to 16 mm, it even decreased from 15.2 mm to 14.6 mm as the crack length L decreased from 16 mm to 8 mm. The region ($i = 56, ..., 92, j = 116, ..., 152$) around the point of occurrence of the additional plastic zone has been examined. In particular, the components $D_{11}(x^1_i, x^2_j)$, $D_{12}(x^1_i, x^2_j)$, and $D_{22}(x^1_i, x^2_j)$ have been calculated for $u_1^*(x^1_i, x^2_j) = 80 \cdot 10^{-6}$ m ($L = 24$ mm), $u_1^*(x^1_i, x^2_j) = 88 \cdot 10^{-6}$ m ($L = 16$ mm), and $u_1^*(x^1_i, x^2_j) = 94 \cdot 10^{-6}$ m ($L = 8$ mm), i.e., for the displacements that precede the occurrence of the additional plastic zone. To this end, we used relations (2.2) according to which

$$D_{11}(\underset{i}{x^1},\underset{j}{x^2}) = \left.\frac{\partial u_1}{\partial x^1}\right|_{(\underset{i}{x^1},\underset{j}{x^2})}, \qquad D_{12}(\underset{i}{x^1},\underset{j}{x^2}) = \frac{1}{2}\left(\left.\frac{\partial u_1}{\partial x^2}\right|_{(\underset{i}{x^1},\underset{j}{x^2})} + \left.\frac{\partial u_2}{\partial x^1}\right|_{(\underset{i}{x^1},\underset{j}{x^2})}\right),$$

$$D_{22}(\underset{i}{x^1},\underset{j}{x^2}) = \left.\frac{\partial u_2}{\partial x^2}\right|_{(\underset{i}{x^1},\underset{j}{x^2})}. \tag{3.1}$$

The partial derivatives in (3.1) are calculated from the values of $y_1, \ldots, y_n$. Tables 1 ($L = 24$ mm), 2 ($L = 16$ mm), and 3 ($L = 8$ mm) summarize the calculated values of $D_{11}(\underset{i}{x^1},\underset{j}{x^2})\cdot 10^3$, $D_{12}(\underset{i}{x^1},\underset{j}{x^2})\cdot 10^3$, and $D_{22}(\underset{i}{x^1},\underset{j}{x^2})\cdot 10^3$.

These tables show that the component $D_{11}(\underset{i}{x^1},\underset{j}{x^2})$, which is predominant, is greater at points on the lateral surface than at neighboring points inside the body. This confirms that an additional plastic zone naturally forms on the lateral surface.

The tables also show that as the crack length L decreases, the values of $D_{11}(\underset{i}{x^1},\underset{j}{x^2})$ at different points of the region in question become close to each other.

Therefore, the decrease of the crack length causes more rapid expansion of the main and additional plastic zones and formation of the merged plastic zone.

Conclusions. A boundary-value problem has been numerically solved to study the effect of the length of a mode I crack on the plastic zone in a body under hard loading in the case of a generalized plane stress state. It has been established that as the crack length decreases, the main plastic zone at the crack tip substantially decreases, an additional plastic zone occurs on the lateral surface later, and the main and additional plastic zones expand more rapidly to form the merged plastic zone.

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