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STUDYING THE HARMONIC VIBRATIONS OF A CYLINDRICAL SHELL MADE OF A NONLINEAR ELASTIC PIEZOELECTRIC MATERIAL

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The paper examines the harmonic vibrations of an infinitely long thin cylindrical shell made of a nonlinear elastic piezoceramic material and subjected to periodic electric loading. Amplitude–frequency characteristics are plotted for different amplitudes of the load. Points of these characteristics are analyzed for stability. The transients occurring while harmonic vibrations attain the steady state are studied

Keywords: piezoceramics, cylindrical shell, harmonic vibrations, amplitude–frequency characteristics, nonlinear elastic piezoceramic material

Introduction. Effects typical of nonlinear systems, such as the amplitude dependence of resonant frequencies, the hysteretic frequency dependence of the amplitude, etc., are observed when piezoelectric bodies are excited at resonant frequencies even in weak electric fields. Nonlinearities of two types are observed in the vibrations of thin-walled elements.

One type is due to the heating generated during the dissipation of electromechanical energy. This nonlinearity results from the temperature dependence of the material properties and the nonlinear dependence of the dissipation function on strains and temperature. It is adequately detailed in [6–10]. The vibrations of piezoelectric and viscoelastic plates are studied in [3–5].

The other type is associated with the dependence of the material characteristics on the amplitudes of independent field quantities. This nonlinearity has been inadequately studied. The papers [11–13] deal with some simple problems for piezoceramic bodies made of a material with second-type nonlinearity.

Since thin piezoelectric cylindrical shells are widely used in various engineering applications, this paper employs the harmonic balance method to examine harmonic vibrations in such shells made of materials with second-type nonlinearity. The amplitude–frequency characteristics of the shell will be plotted for different levels of electric loading. It will be shown that these characteristics may differ both quantitatively and qualitatively for different amplitudes of the load. Points of the amplitude–frequency characteristics will be analyzed for stability, depending on load amplitudes.

1. Problem Statement. Consider a radially polarized, infinitely long, piezoceramic, cylindrical shell with thickness h and mid-radius R. The shell is coated with infinitely thin electrodes to which an electric potential $V_0(t)$ is applied.

The momentless equation of motion of the shell is as follows:

$$\frac{\sigma_\varphi}{R} + \rho \ddot{w} = 0. \tag{1.1}$$

Its strain is determined as

$$\varepsilon_\varphi = \frac{w}{R}. \tag{1.2}$$

The equation below follows from (1.1) and (1.2):

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$$\rho R^2 \ddot{\varepsilon}_{\varphi} + \sigma_{\varphi} = 0, \tag{1.3}$$

where ρ is the density of the shell material. In the sequel, we omit φ. The constitutive equations of nonlinear electroelasticity become [8]

$$\sigma = E_c^{(0)} \varepsilon + \beta E_c^{(0)} \dot{\varepsilon} + E_c^{(1)} \varepsilon^2 + E_c^{(2)} \varepsilon^3 - \gamma_0 E_r - \gamma_1 \varepsilon E_r - \gamma_2 \varepsilon^2 E_r,$$

$$D_r = \gamma_0 \varepsilon + \frac{1}{2} \gamma_1 \varepsilon^2 + \frac{1}{3} \gamma_2 \varepsilon^3 + \nu_0 E_r. \tag{1.4}$$

If the material behaves equally under both compression and tension, Eqs. (1.4) become simpler:

$$\sigma = E_c^{(0)} \varepsilon + \beta E_c^{(0)} \dot{\varepsilon} + E_c^{(2)} \varepsilon^3 - \gamma_0 E_r - \gamma_2 \varepsilon^2 E_r,$$

$$D_r = \gamma_0 \varepsilon + \frac{1}{3} \gamma_2 \varepsilon^3 + \nu_0 E_r. \tag{1.5}$$

Since $\frac{\partial D_r}{\partial r} = 0$, we have $D_r = C(t)$, where C is a function of time independent of the radial coordinate.

Integrating the second equation in (1.5) over the shell thickness, we obtain

$$Ch = \gamma_0 \varepsilon h + \frac{1}{3} \gamma_2 \varepsilon^3 h - \nu_0 V_0(t)$$

or

$$D_r = C = \gamma_0 \varepsilon + \frac{1}{3} \gamma_2 \varepsilon^3 - \frac{\nu_0 V_0(t)}{h}. \tag{1.6}$$

Substituting the expression for D_r (1.6) into the second equation in (1.5), we find

$$E_r = -\frac{V_0(t)}{h}.$$

The first equation in (1.5) yields

$$\sigma = E_c^{(0)} \varepsilon + \beta E_c^{(0)} \dot{\varepsilon} + E_c^{(2)} \varepsilon^3 + (\gamma_0 + \gamma_2 \varepsilon^2) \frac{V_0(t)}{h}. \tag{1.7}$$

Substituting (1.7) into Eq. (1.3), we obtain

$$\rho R^2 \ddot{\varepsilon} + \beta E_c^{(0)} \dot{\varepsilon} + E_c^{(0)} \varepsilon + E_c^{(2)} \varepsilon^3 + (\gamma_2 \varepsilon^2 + \gamma_0) \frac{V_0(t)}{h} = 0. \tag{1.8}$$

The numerical values and expressions for calculating the material constants from (1.8) are given in [8]:

$$\gamma_0 = E_c^{(0)} d_{31}^{(0)}, \quad \gamma_2 = E_c^{(0)} d_{31}^{(2)} + E_c^{(2)} d_{31}^{(0)},$$

$$\rho = 7790 \text{ kg/m}^3, \quad E_c^{(0)} = 0.667 \cdot 10^{11} \text{ N/m}^2, \quad E_c^{(2)} = -1.4 \cdot 10^{18} \text{ N/m}^2,$$

$$d_{31}^{(0)} = -2.1 \cdot 10^{-10} \text{ m/V}, \quad d_{31}^{(2)} = -0.03596 \text{ m/V}, \quad \beta = 3.165 \cdot 10^{-6}. \tag{1.9}$$

2. Problem Solution and Analysis. Let us rearrange Eq. (1.8) as follows:

$$\rho R^2 \ddot{\varepsilon} + \beta E_c^{(0)} \dot{\varepsilon} + E_c^{(0)} \varepsilon + \gamma_2 \frac{V_0(t)}{h} \varepsilon^2 + E_c^{(2)} \varepsilon^3 + \gamma_0 \frac{V_0(t)}{h} = 0. \tag{2.1}$$

We introduce the following notation:

$$\omega_0^2 = \frac{E_c^{(0)}}{\rho R^2}, \qquad \tau = \omega_0 t \qquad \left(\frac{d}{dt} = \omega_0 \frac{d}{d\tau}, \quad \frac{d^2}{dt^2} = \omega_0^2 \frac{d^2}{d\tau^2} \right), \tag{2.2}$$

where τ is dimensionless time. The derivative with respect to τ is further primed.

We divide the strain ε into a small characteristic quantity μ,

$$\bar{\varepsilon} = \varepsilon / \mu. \tag{2.3}$$

Substituting (2.2)–(2.3) into (2.1), we obtain

$$\bar{\varepsilon}'' + \beta\omega_0 \bar{\varepsilon}' + \bar{\varepsilon} + \bar{\gamma}_2 V_0\left(\frac{\tau}{\omega_0}\right)\bar{\varepsilon}^2 + E_2 \bar{\varepsilon}^3 = G_0 V_0\left(\frac{\tau}{\omega_0}\right), \tag{2.4}$$

where

$$E_2 = \frac{\mu^2 E_c^{(2)}}{\rho R^2 \omega_0^2}, \qquad G_0 = -\frac{\gamma_0}{\mu h \rho R^2 \omega_0^2}, \qquad \bar{\gamma}_2 = \frac{\mu \gamma_2}{h \rho R^2 \omega_0^2}. \tag{2.5}$$

Let us analyze the harmonic vibrations and their amplitude–frequency characteristics. We use the harmonic balance method [2]. Suppose

$$V_0\left(\frac{\tau}{\omega_0}\right) = B \sin \bar{\omega}\tau, \qquad \bar{\omega} = \frac{\omega}{\omega_0}, \tag{2.6}$$

where ω is the frequency of electric excitation.

Neglecting subharmonic and ultraharmonic vibrations, we will analyze harmonic vibrations. The periodic solution to Eq. (2.4) is approximated as follows:

$$\bar{\varepsilon} = A_1 \cos \bar{\omega}\tau + A_2 \sin \bar{\omega}\tau. \tag{2.7}$$

Substituting (2.7) into (2.4) and equating the coefficients of $\cos \bar{\omega}\tau$ and $\sin \bar{\omega}\tau$, we obtain a nonlinear system of algebraic equations:

$$A_1 + \frac{3}{4} A_1 E_2 (A_1^2 + A_2^2) + \frac{1}{2} A_1 A_2 B \bar{\gamma}_2 + A_2 \omega_0 \beta \bar{\omega} - A_1 \bar{\omega}^2 = 0,$$

$$A_2 + \frac{3}{4} A_2 E_2 (A_1^2 + A_2^2) + \frac{1}{4} B \bar{\gamma}_2 (A_1^2 + 3A_2^2) - A_1 \omega_0 \beta \bar{\omega} - A_2 \bar{\omega}^2 = BG_0.$$

Hence, we find

$$A_1 = \frac{4\omega_0 \beta \bar{\omega} x}{-4BG_0 + B\bar{\gamma}_2 x}, \qquad A_2 = \frac{4(1-\bar{\omega}^2)x + 3E_2 x^2}{4BG_0 - 3B\bar{\gamma}_2 x}, \tag{2.8}$$

where $x = A^2 = A_1^2 + A_2^2$ is the squared amplitude.

We take the squares of Eq. (2.8) and add them to obtain the following equation for the squared amplitude:

$$B^2 x = \left(\frac{4\omega_0 \beta \bar{\omega} x}{-4G_0 + \bar{\gamma}_2 x} \right)^2 + \left(\frac{4(1-\bar{\omega}^2)x + 3E_2 x^2}{4G_0 - 3\bar{\gamma}_2 x} \right)^2. \tag{2.9}$$

We discard the trivial root of this equation and, after transformation, obtain the following algebraic equation of the fifth order in x:

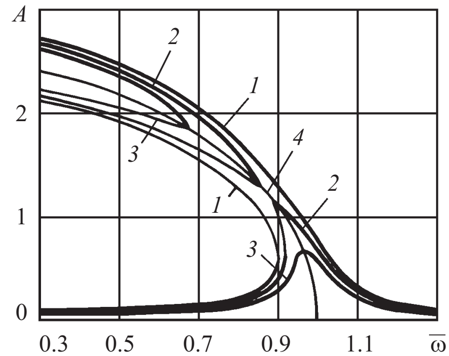

Fig. 1

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 = 0, \tag{2.10}$$

where

$$a_0 = -16 B^2 G_0^4, \quad a_1 = 16 G_0^2 \left(1 + 2B^2 G_0 \overline{\gamma}_2 + (\omega_0^2 \beta^2 - 2)\overline{\omega}^2 + \overline{\omega}^4\right),$$

$$a_2 = -2G_0 \left(12 E_2 G_0 (\overline{\omega}^2 - 1) + \overline{\gamma}_2 \left(11 B^2 G_0 \overline{\gamma}_2 + 4\left(1 + (3\omega_0^2 \beta^2 - 2)\overline{\omega}^2 + \overline{\omega}^4\right)\right)\right),$$

$$a_3 = 9 E_2^2 G_0^2 + 12 E_2 G_0 \overline{\gamma}_2 (\overline{\omega}^2 - 1) + \overline{\gamma}_2^2 \left(1 + 6 B^2 G_0 \overline{\gamma}_2 + (9\omega_0^2 \beta^2 - 2)\overline{\omega}^2 + \overline{\omega}^4\right),$$

$$a_4 = \frac{3}{16} \overline{\gamma}_2 \left(24 E_2^2 G_0 + 3 B^2 \overline{\gamma}_2^3 + 8 E_2 \overline{\gamma}_2 (\overline{\omega}^2 - 1)\right), \quad a_5 = \frac{9}{16} E_2^2 \overline{\gamma}_2^2 .$$

Solving Eq. (2.10) with the external frequency $\overline{\omega}$ kept constant and discarding the negative and complex roots, we obtain the squares of possible amplitudes. Hence, we can plot amplitude–frequency characteristics. Figure 1 shows the amplitude–frequency characteristics of the harmonic vibrations of a shell with radius $R = 0.1$ m and thickness $h = 0.005$ m for $\mu = 10^{-4}$ and different amplitudes of the exciting electric potential: 175 V (curve *1*), 150 V (curve *2*), and 120 V (curve *3*). Curve *4* is a skeletal curve [1] obtained from Eq. (2.9) when $B = \beta = 0$ and $x = \dfrac{4(1-\overline{\omega}^2)}{3E_2}$.

The heavy lines correspond to the stable points of the amplitude–frequency characteristics and the thin ones to the unstable points.

The points of the amplitude–frequency characteristic were analyzed for stability as follows. We add a small deviation ε_1 to the steady-state motion (2.7) $\overline{\varepsilon}_0 = A_1 \cos \overline{\omega}\tau + A_2 \sin \overline{\omega}\tau$. Then we substitute $\overline{\varepsilon}_0 + \varepsilon_1$ into Eq. (2.4). Taking into account Eq. (2.6), considering that $\overline{\varepsilon}_0$ obeys Eq. (2.4), and discarding the terms of the second and higher orders, we obtain the following equation for variations of ε_1:

$$\varepsilon_1'' + \beta\omega_0 \varepsilon_1' + \varepsilon_1 + 2\overline{\gamma}_2 B \sin \overline{\omega}\tau \overline{\varepsilon}_0 \varepsilon_1 + 3E_2 \overline{\varepsilon}_0^2 \varepsilon_1 = 0. \tag{2.11}$$

Substituting $\overline{\varepsilon}_0$ into this equation, we find

$$\varepsilon_1'' + \beta\omega_0 \varepsilon_1' + (\widetilde{\theta}_0 + \theta_c \cos 2\overline{\omega}\tau + \theta_s \sin 2\overline{\omega}\tau)\varepsilon_1 = 0, \tag{2.12}$$

where $\widetilde{\theta}_0 = 1 + \overline{\gamma}_2 B A_2 + \frac{3}{2} E_2 (A_1^2 + A_2^2)$, $\theta_c = -\overline{\gamma}_2 B A_2 + \frac{3}{2} E_2 (A_1^2 - A_2^2)$, $\theta_s = \overline{\gamma}_2 B A_1 + 3E_2 A_1 A_2$.

Changing variables

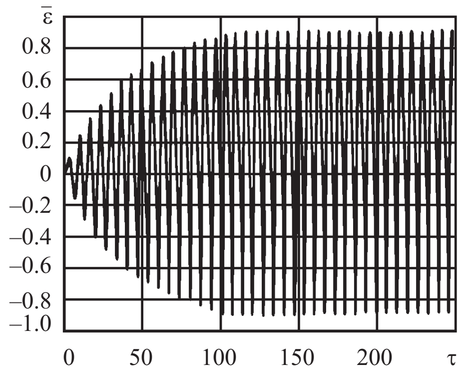

Fig. 2

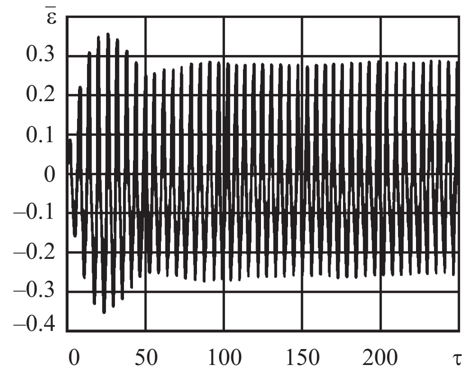

Fig. 3

$$\varepsilon_1 = \exp\left(-\frac{1}{2}\beta\omega_0\tau\right)\eta, \tag{2.13}$$

we reduce Eq. (2.14) to the form

$$\eta'' + (\theta_0 + \theta_c \cos 2\overline{\omega}\tau + \theta_s \sin 2\overline{\omega}\tau)\eta = 0, \qquad \theta_0 = \widetilde{\theta}_0 - \frac{1}{4}\beta^2\omega_0^2. \tag{2.14}$$

Equation (2.14) is a particular case of the generalized Hill equation. According to [2], Eq. (2.14) has a stable solution if

$$(\theta_0 - \overline{\omega}^2)^2 + \frac{1}{2}(\theta_0 - \overline{\omega}^2)\omega_0^2\beta^2 + \frac{1}{16}\omega_0^4\beta^4 - (\theta_c^2 + \theta_s^2) > 0. \tag{2.15}$$

Taking Eq. (2.13) into account, we can state that the steady-state motion (2.7), in which the constants A_1 and A_2 are determined from Eq. (2.8) and the squared amplitude is obtained from (2.10), will be stable if condition (2.15) is satisfied.

Figure 2 shows the numerical solution of Eq. (2.4) for $V_0\left(\frac{\tau}{\omega_0}\right) = B\sin\overline{\omega}\tau$, amplitude B = 150 V, and frequency $\overline{\omega} = 0.95$. Figure 3 shows a similar curve for $\overline{\omega} = 1.1$. These curves illustrate how nonstationary vibrations attain the steady state, which agrees well with the relevant (curves *2* in Fig. 1) amplitude–frequency characteristic.

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