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DAMPING THE VIBRATIONS OF A RECTANGULAR PLATE WITH PIEZOELECTRIC ACTUATORS

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The paper addresses the active damping of nonstationary vibrations of a hinged rectangular plate with distributed piezoelectric actuators. The problem is solved by two methods: (i) the classical method of balancing the fundamental vibration modes by applying the appropriate potential difference to the actuator and (ii) the dynamic-programming method that reduces the problem to an algebraic Riccati equation. The results produced by both approaches are presented and compared

Keywords: piezoelectric actuator, damping, hinged plate, dynamic-programming method, algebraic Riccati equation

Introduction. Thin plates are widely used in many fields of modern engineering including space technology, aircraft engineering, automotive industry, shipbuilding, mechanical engineering, radio electronics, etc. Such plates are often subjected to nonstationary and harmonic mechanical loads. Especially dangerous are resonant vibrations occurring when the frequency of the harmonic force becomes equal to the natural frequency of the plate. Intensive mechanical nonstationary vibrations are no less dangerous. This brings about the task of damping stationary and nonstationary vibrations of thin plates. For this purpose, passive damping (elements with high hysteresis losses embedded into the plate) is widely used. Numerous Ukrainian and foreign publications on passive damping of vibrations of thin-walled elements are reviewed in [7–9, 15].

Recently, active damping with distributed piezoelectric inclusions (so-called sensors and actuators) has been used. The essence of active-damping methods is in the following. The use of eigenfunction expansion reduces many vibration problems for thin-walled elements to a system of ordinary differential equations and sometimes to even one ordinary differential equation (this is so, for example, for rectangular plates and cylindrical panels with hinged ends), i.e., to a one-degree-of-freedom system. The solution describing the forced vibrations of a one-degree-of-freedom system with viscous friction under a harmonic force consists of two terms.

The first term describes accompanying natural vibrations that exponentially decay and depend on the initial conditions. The damping rate depends on the damping factor: the greater this coefficient, the quicker the vibrations decay.

The second term describes purely forced vibrations with the frequency of the exciting force. The natural vibrations eventually decay and only the purely forced vibrations remain. The intensity of the latter depends on the amplitude and frequency of the exciting force and the damping factor. The higher the amplitude of the force, the closer its frequency to the natural frequency, the less the damping factor, the higher the amplitude of forced vibrations. There are two damping methods. One employs only actuators to which a potential difference is applied to balance the mechanical load. This decreases the amplitude of the exciting force and, hence, the amplitude of the forced vibrations. The primary task here is to calculate the necessary potential difference to be applied to the actuator(s). Thus, the former method substantially reduces the intensity of forced vibrations but cannot change the damping factor and, hence, the intensity of natural vibrations. This method is efficient when there is detailed information on the external load and especially effective in damping resonant vibrations with a few or even one actuator. The other method employs both actuators and sensors. The actuator is subjected to a potential difference proportional to the first time derivative of the potential difference or current of the sensor. The sensor indications are proportional to the deflection of the

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thin-walled element; therefore, the method changes the dissipative characteristics of this element, making it possible to considerably decrease the amplitude of vibrations and increase the decay rate of the natural vibrations. This method does not need detailed information on the external load, but cannon change the magnitude of the load.

The efficiency of both methods depends on many factors: the geometrical and electromechanical characteristics of the sensors and actuators, the geometrical and mechanical characteristics of the passive plate, the mechanical and electric boundary conditions, and temperature [3–6]. The achievements on active damping with piezoelectric inclusions are reviewed in [12, 13, 19, 20–24]. The vibrations of piezoelectric and viscoelastic plates are studied in [11, 15–18].

For effective damping of the nonstationary vibrations of plates by the first method, i.e., by using only piezoelectric actuators, it is possible to proceed classically, i.e., to apply a potential difference to the actuators to balance the most intensive modes of the mechanical load. As shown in [3], balancing of even one (fundamental) mode leads to an abrupt decrease in the amplitude of vibrations.

To better damp nonstationary vibrations, it is possible to use optimum-control methods [2].

Here we address the issue of damping the nonstationary vibrations of a hinged rectangular plate under a normal surface load with distributed piezoelectric actuators. The mechanical load varies stepwise with time. To damp flexural vibrations, piezoelectric layers with opposite polarization are pairwise applied on the surfaces $z=\pm h/2$ of the plate. Two damping methods can be used: (i) the potential difference applied to the actuator balances only the most intensive modes and (ii) the potential difference applied to the actuator varies in an optimal-control manner. The main objective of the present paper is to compare these two methods.

1. Problem Formulation and Solution Method. Consider a hinged rectangular elastic plate undergoing nonstationary vibrations under external normal pressure $p(x, y, t)$. There are piezoelectric actuators, laid as described above, on the surfaces $z=\pm h/2$ of the plate. A potential difference causing flexural vibrations is applied to the actuators. It is necessary to find the potential difference that would balance the mechanical load and, thus, reduce the amplitude of vibrations. Using the basic equations of electromechanics [1, 10], the Kirchhoff–Love hypotheses, and similar hypotheses for the electric field [1], we obtain the following equation describing the transverse vibrations of the plate [1, 3–5]:

$$\rho h \frac{\partial^2 W}{\partial t^2} + D\left[\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4}\right] = p(x, y, t) + \frac{\partial^2 M_0}{\partial x^2} + \frac{\partial^2 M_0}{\partial y^2}, \tag{1}$$

where W is the deflection; ρ is the density of the plate material; h is the thickness of the plate; $D = Eh^3/[12(1-\nu^2)]$ is the flexural stiffness; ν is Poisson's ratio; and M_0 is the bending moment generated by the energized actuator.

According to [3–5], the moment M_0 is defined as

$$M_0 = \sum_{i=1}^{N} M_0^i = \sum_{i=1}^{N} \gamma_{31}^i (h + h_i) V_a^i, \tag{2}$$

where N is the number of actuators; γ_{31}^i is the piezoelectric constant of the ith actuator; h_i is the thickness of the ith actuator; and V_a^i is the potential difference applied to the ith actuator;

The solution of Eq. (1) is sought in the form of series of natural modes $F_{mn}(x, y)$:

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(t) F_{mn}(x, y). \tag{3}$$

The moment M_0 is expanded into a series in the same manner:

$$M_0 = \sum_{i=1}^{N} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}^i(t) F_{mn}(x, y), \tag{4}$$

where

$$M_{mn}^{i}=\frac{M_0^i\iint\limits_{S_i}F_{mn}dxdy}{K_{mn}},\quad K_{mn}=\iint\limits_{S}F_{mn}^2dxdy, \tag{5}$$

S_i is the area of the ith actuator; S is the area of the whole plate.

For a hinged rectangular plate with the side lengths a and b, we have

$$F_{mn}(x,y)=\sin\frac{\pi mx}{a}\sin\frac{\pi ny}{b},\quad K_{mn}=\frac{1}{4}ab. \tag{6}$$

Expanding the surface load $p(x,y,t)$ into a series of functions (6), we get

$$p(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}p_{mn}\sin\frac{\pi mx}{a}\sin\frac{\pi ny}{b}. \tag{7}$$

Next, expanding W and M_0^i in (1) into series of functions (6) and considering (7), we obtain a system of ordinary differential equations:

$$\ddot{W}_{mn}+\omega_{mn}^2W_{mn}=\frac{1}{\rho h}p_{mn}-\sqrt{\frac{1}{D\rho h}}\omega_{mn}\sum_{i=1}^{N}M_{mn}^{i},\quad m,n=\overline{1,\infty}, \tag{8}$$

where

$$\omega_{mn}=\sqrt{\frac{D}{\rho h}}\left(\frac{\pi^2m^2}{a^2}+\frac{\pi^2n^2}{b^2}\right). \tag{9}$$

Considering (2), (5), and (9) and introducing the notation

$$L_{mn}^{i}=\sqrt{\frac{1}{D\rho h}}\,\omega_{mn}\frac{\gamma_{31}^{i}(h+h_i)\iint\limits_{S_i}F_{mn}dxdy}{K_{mn}}, \tag{10}$$

we rearrange relations (8) as

$$\ddot{W}_{mn}+\omega_{mn}^2W_{mn}=\frac{1}{\rho hK_{mn}}p_{mn}-\sum_{i=1}^{N}L_{mn}^{i}V_a^i,\quad m,n=\overline{1,\infty}. \tag{11}$$

We retain Q harmonics in $m=m_1,m_2,\ldots\ldots m_Q$ and S harmonics in $n=n_1,n_2,\ldots,n_S$ and introduce the notation

$$x_{2k-1}=W_{m_qn_s},\quad x_{2k}=\dot{W}_{m_qn_s},\quad k=\overline{1,QS},\quad q=\overline{1,Q},\quad s=\overline{1,S}. \tag{12}$$

Then (11) yields a system of linear differential equations of order $2QS$:

$$\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{BV}_a+\mathbf{F}, \tag{13}$$

where **A** and **B** are $2QS\times 2QS$ and $2QS\times N$ matrices, respectively; **F** is a $2QS$-dimensional column vector,

$$\mathbf{A}=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -\omega^2_{m_1 n_1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -\omega^2_{m_2 n_1} & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & -\omega^2_{m_q n_s} & 0 & \cdots & 0 \\ \cdots & & & & & \\ \cdots & & & & & \\ 0 & \cdots & & & 0 & 1 \\ 0 & \cdots & & & -\omega^2_{m_Q n_S} & 0 \end{pmatrix}, \quad (14)$$

$$\mathbf{B}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ L^1_{m_1 n_1} & L^2_{m_1 n_1} & \cdots & L^N_{m_1 n_1} \\ 0 & 0 & \cdots & 0 \\ L^1_{m_2 n_1} & L^2_{m_2 n_1} & \cdots & L^N_{m_2 n_1} \\ \cdots & & & \\ 0 & 0 & \cdots & 0 \\ L^1_{m_q n_s} & L^2_{m_q n_s} & \cdots & L^N_{m_q n_s} \\ \cdots & & & \\ 0 & 0 & \cdots & 0 \\ L^1_{m_Q n_S} & L^2_{m_Q n_S} & \cdots & L^N_{m_Q n_S} \end{pmatrix} \quad \mathbf{F}=\begin{pmatrix} 0 \\ \frac{1}{\rho h} p_{m_1 n_1} \\ 0 \\ \frac{1}{\rho h} p_{m_2 n_1} \\ 0 \\ \cdots \\ 0 \\ \frac{1}{\rho h} p_{m_q n_s} \\ \cdots \\ 0 \\ \frac{1}{\rho h} p_{m_Q n_S} \end{pmatrix}. \quad (15)$$

Let us formulate a problem of optimizing the potential differences V_a^i applied to the actuators to damp the vibrations of the plate caused by the external load and the initial conditions. As follows from (3), the behavior of the deflection with time at any point of the plate is determined by the behavior of W_{mn}. Then, minimum-energy damping can be provided by minimizing the quadratic functional

$$J=\int_0^\infty \left(\sum_{i=1}^{2QS} q_i x_i^2 + \sum_{i=1}^{N} r_i V_a^{i2} \right) dt,$$

where q_i and r_i are weight coefficients that set the priority for the minimization of some phase coordinate or control.

The functional J can be expressed in matrix form:

$$J=\int_0^\infty \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{V}_a^T \mathbf{R} \mathbf{V}_a \right) dt, \quad (16)$$

where $\mathbf{Q}$ and $\mathbf{R}$ are square diagonal matrices with q_i and r_i on the leading diagonals, respectively.

Thus, expressions (13)–(16) allow synthesizing an optimal controller for a linear stationary system of differential equations for the quadratic performance criterion

$$J=\int_0^\infty \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{V}_a^T \mathbf{R} \mathbf{V}_a \right) dt \to \min, \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{V}_a + \mathbf{F}, \quad \mathbf{x}(0)=\mathbf{x}_0. \quad (17)$$

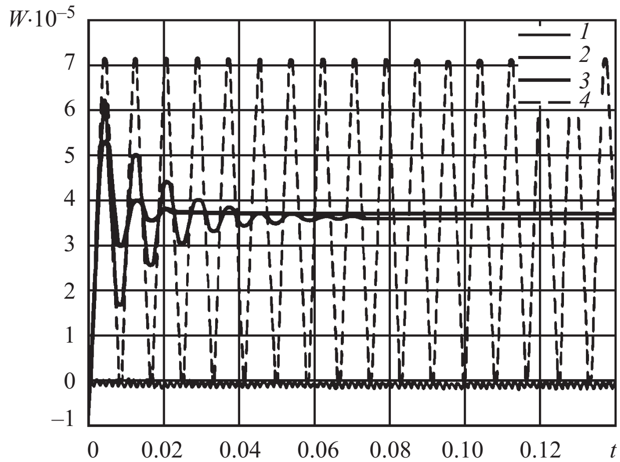

Fig. 1

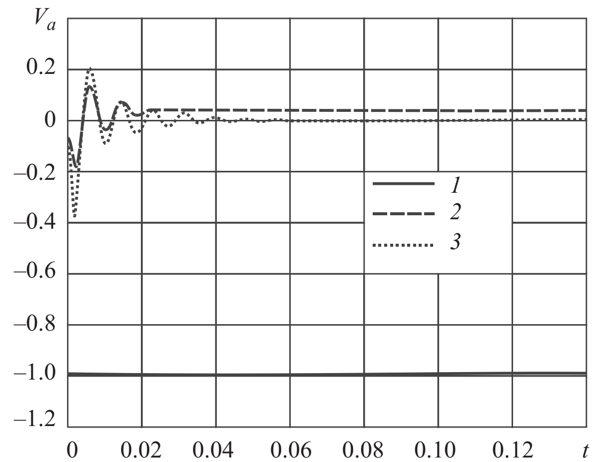

Fig. 2

According to [2], the dynamic-programming method gives the following formula for the optimal control in problem (17):

$$\mathbf{V}_a = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}\mathbf{x}, \tag{18}$$

where the matrix $\mathbf{K}$ follows from the algebraic Riccati equation

$$\mathbf{K}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{K}^T - \mathbf{K}\mathbf{A} - \mathbf{A}^T\mathbf{K}^T - \mathbf{Q} = 0. \tag{19}$$

2. Calculated Results. Comparison of Two Methods. Consider, as an example, a hinged square metal plate with side length $a = b = 20\,\text{cm}$; thickness h = 1 mm; density $\rho = 7850\,\text{kg/m}^3$; flexural stiffness D = 18.315. A square piezoelectric actuator is laid on the plate. Its side length is equal to that of the plate; thickness h = 10 mm; coefficient $\gamma_{31} = -18.2857$ (TsTS$_\text{T}$BS-2 material). The centers of the actuator and the plate coincide, and their sides are parallel. The plate is subject to constant load p = 100 N/m^2 distributed over the plate surface. The plate is initially undeformed. Problem (17) has been solved for the harmonics m, n = 1, 3, 5. The Riccati equation has been solved numerically, using a standard MATLAB routine. Figure 1 shows the deflection at the center of the plate (curve *4* corresponds to the case without control; curve *2* to $q_i = 50, i = \overline{1,18}, r = 1$; and curve *3* to $q_i = 350, i = \overline{1,18}, r = 1$). Figure 2 shows the behavior of the optimized control V_a (curve *2* corresponds to $q_i = 50$, $i = \overline{1,18}$, $r = 1$; curve *3* to $q_i = 350, i = \overline{1,18}, r = 1$). As is seen, just one actuator substantially reduces the amplitudes of vibrations in a rather short time.

Curves *1* in Figs. 1 and 2 show W and the required potential difference at zero initial conditions that has been determined not by minimizing a functional, but by equating the right-hand side of Eq. (11) to zero for the lowest harmonic $m = n = 12$ when only one actuator is used. It can be seen that the classical method considerably decreases the deflection, but increases by an order of magnitude the required potential difference. The chief shortcoming of this method is that it cannot be applied when the initial conditions are nonzero because the external force is balanced without due regard to the natural vibrations of the plate.

REFERENCES

1. V. T. Grinchenko, A. F. Ulitko, N. A. Shul'ga, *Electroelasticity*, Vol. 5 of the six-volume series *Mechanics of Coupled Fields in Structural Members* [in Russian], Naukova Dumka, Kyiv (1989).
2. V. A. Ivanov and N. V. Faldin, *Theory of Optimal Automatic Control Systems* [in Russian], Nauka, Moscow (1981).
3. V. G. Karnaukhov, A. V. Kozlov, and E. V. Pyatetskii, "Damping the vibrations of viscoelastic plates with distributed piezoelectric inclusions," *Akust. Vestn*., **5**, No. 4, 15–32 (2002).

4. V. G. Karnaukhov, V. I. Kozlov, and O. V. P'yatets'ka, "Contribution of boundary conditions to the active damping of the vibrations of a viscoelastic rectangular plate with distributed piezoelectric inclusions," *Visn. Kyiv. Univ.*, *Fiz.-Mat. Nauky*, No. 1, 75–85 (2003).
5. V. G. Karnaukhov, V. I. Kozlov, and E. V. Pyatetskaya, "Active damping of the vibrations of a rectangular plate with distributed sensors and actuators," *Teor. Prikl. Mekh.*, **37**, 136–140 (2003).
6. V. G. Karnaukhov and V. V. Mikhailenko, *Nonlinear Thermomechanics of Piezoelectric Inelastic Bodies under Monoharmonic Loading* [in Russian], ZhGTU, Zhitomir (2005).
7. B. G. Korenev and L. M. Reznikov, *Dynamic Dampers: Theory and Applications* [in Russian], Nauka, Moscow (1988).
8. V. V. Matveev, *Damping Vibrations of Deformable Bodies* [in Russian], Naukova Dumka, Kyiv (1985).
9. A. D. Nashif, D. J. Johnes, and J. P. Henderson, *Vibration Damping*, John Wiley & Sons, New York (1985).
10. N. A. Shul'ga and A. M. Bolkisev, *Vibrations of Piezoelectric Bodies* [in Russian], Naukova Dumka, Kyiv (1990).
11. B. Kh. Éshmatov, "Nonlinear vibration analysis of viscoelastic plates based on a refined Timoshenko theory," *Int. Appl. Mech.*, **42**, No. 5, 596–605 (2006).
12. U. Gabbert and H. S. Tzou, *Smart Structures and Structronic Systems*, Kluwer Acad. Pub., London (2001).
13. U. Gabbert, H. Koppe, and F. Laugwitz, "Numerical and experimental investigations of adaptive plate and shell structures," in: J. Holnicki-Szulk and J. Rodellar (eds.), *Smart Structures*, Kluwer Academic Pub., London (1999), pp. 7–13.
14. B. Lazan, *Damping of Materials and Members in Structural Mechanics*, Pergamon Press, Oxford (1968).
15. V. L. Karlash, "Resonant electromechanical vibrations of piezoelectric plates," *Int. Appl. Mech.*, **41**, No. 7, 709–747 (2005).
16. V. L. Karlash, "Planar electroelastic vibrations of piezoceramic rectangular plate and half-disk," *Int. Appl. Mech.*, **43**, No. 5, 547–553 (2007).
17. V. L. Karlash, "Evolution of the planar vibrations of a rectangular piezoceramic plate as its aspect ratio is changed," *Int. Appl. Mech.*, **43**, No. 7, 786–793 (2007).
18. I. F. Kirichok, "Resonant flexural vibrations and dissipative heating of a piezoceramic ring plate with nonuniformly electroded surfaces," *Int. Appl. Mech.*, **42**, No. 3, 336–341 (2006).
19. C. K. Lee and F. C. Moon, "Laminated piezopolymer plates for torsion and bending sensors and actuators," *J. Acoust. Soc. Amer.*, **85**, No. 6, 2432–2439 (1989).
20. C. K. Lee, "Theory of laminated piezoelectric plates for the design of distributed sensor/actuators. Part I: Governing equations and reciprocal relations," *J. Acoust. Soc. Amer.*, **87**, No. 3, 1144–1158 (1990).
21. S. S. Rao and M. Sunar, "Piezoelectricity and its use in disturbance sensing and control of flexible structures: A survey," *Appl. Mech. Rev.*, **47**, No. 4, 113–123 (1994).
22. J. Tani, T. Takagi, and J. Qiu, "Intelligent material systems: Applications of functional materials," *Appl. Mech. Rev.*, **51**, No. 8, 505–521 (1998).
23. H. S. Tzou and L. A. Bergman, *Dynamics and Control of Distributed Systems*, Cambridge Univ. Press, Cambridge (1998).
24. H. S. Tzou, *Piezoelectric Shells* (*Distributed Sensing and Control of Continua*),: Kluwer Acad. Pub., London (1993).