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CONTACT INTERACTION OF AN ELASTIC PUNCH AND AN ELASTIC HALF-SPACE WITH INITIAL (RESIDUAL) STRESSES

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The contact interaction of an elastic punch of arbitrary cross-section and an elastic semi-space with initial (residual) stresses is studied. A general method to solve the problem is proposed. It allows solving contact problems for bodies with initial (residual) stresses when the solution of the corresponding elastic problem is known

Keywords: elastic punch, elastic half-space, initial stress, residual stress

1. Introduction. The formulation of contact problems for elastic bodies with initial (residual) stresses under the action of an elastic cylindrical punch is addressed in [1, 3–6, 9]. Extensive reviews of studies on the contact interaction between elastic bodies with initial (residual) stresses and elastic punches can be found in [7, 8, 13]. Studies on linear contact problems for bodies without initial (residual) stresses treated under the classical theory of elasticity are reviewed in the monograph [6]. Some contact problems for elastic bodies (shells, plates) are addressed in [12, 15–17]. Based on the results from [1, 2–6], we will formulate and outline a method to solve stress–strain problems for compressible and incompressible elastic bodies with initial (residual) stresses contacting with elastic punches that have initial (residual) stresses too.

The present paper attacks an axisymmetric spatial problem for a half-space with initial (residual) stresses acted upon by a punch. It is assumed that the punch is elastic, has initial (residual) stresses, and is bounded by a cylindrical surface with arbitrary cross-section and generatrices perpendicular to the boundary of the half-space (Fig. 1).

We will outline a general method to solve the problem. The method is based on the linearized theory of elasticity [2, 3–6] and reduces the general mixed problem for an elastic half-space with initial (residual) stresses and an elastic punch with initial (residual) stresses to a classical problem for an elastic punch and a half-space in the harmonic-potential theory [10]. We will address a specific problem for an elastic cylindrical punch with initial (residual) stresses forced into a half-space also with initial (residual) stresses in the case of arbitrary elastic potentials of general form for the theory of large (finite) initial strains and various versions of the theory of small initial strains.

The basic relations of the problem are written using the theory of finite strains. To pass to theories of small initial strains, the simplifications adopted in [2] should be introduced.

Following the studies [3, 10], we will use the coordinates $\{y_i\}$ of the initial strain state [2], which are related to the Lagrange coordinates of the natural state by

$$y_i = \lambda_i x_i, \quad \lambda_i = \text{const} \quad (i = 1, 2, 3), \tag{1.1}$$

where λ_i are the elongation coefficients defining the displacements of the initial state.

We will write all the relations of the linearized theory of elasticity for prestressed bodies in the coordinates $\{y_i\}$ and will divide all quantities by elementary areas in the initial strain state, mainly following the results from [2].

According to [2], the method of description used corresponds to a method where the initial stress–strain state of the body is chosen to be the reference configuration, expressions (1.1) being mapping onto the reference configuration. For all

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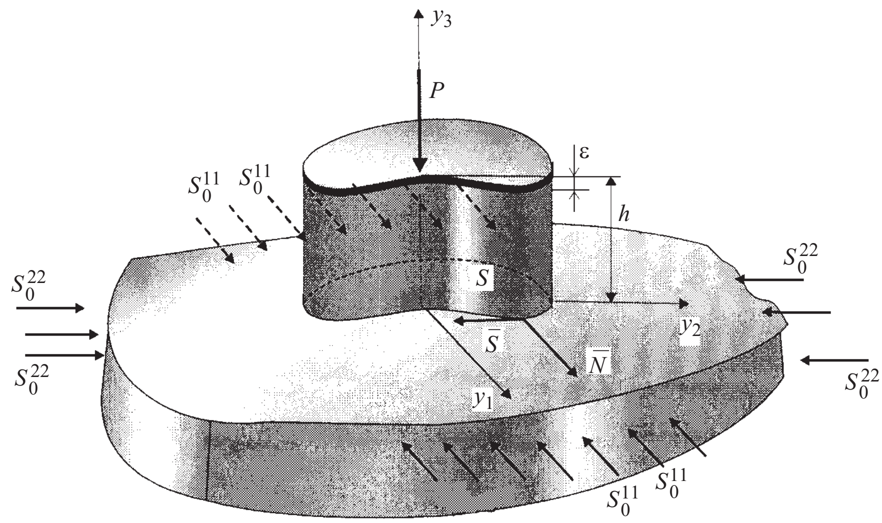

Fig. 1

quantities in the reference-configuration method, we introduce, as in [2], the following notation: $\tilde{Q}_{ij}$ are the components of the stress vector along the Oy_i-axis on the area element y_i =const, which are measured per unit area in the initial strain state.

Assume that the initial states of the half-space and punch are equal along the Oy_1- and Oy_2-axes. Then, as in [2], the initial (residual) stress–strain state of an isotropic body is

$$\lambda_1 = \lambda_2 \neq \lambda_3, \quad S_0^{11} = S_0^{22} \neq S_0^{33}. \tag{1.2}$$

2. Problem Formulation and Basic Equations. Consider an elastic punch of height h and arbitrary cross-section indented by a force P applied to its upper end into an elastic half-space (Fig. 1). The external load is applied so that the free end of the punch deforms along the punch axis by the same amount $-\varepsilon$, while its lateral surface beyond the contact region is free from external forces.

Assume that the punch comes into contact with the half-space after the initial (residual) stress–strain state (1.1) [1, 2, 4–6] occurs.

It is convenient to apply, along with the Cartesian coordinates $\{y_i\}$, i = 1, 2, 3, an arbitrary cylindrical coordinate system aligned with the Oy_3-axis. The following notation is used here: $\bar{N}$ is a normal unit vector to the boundary of the contact region S^*; S is the tangent to this boundary; N_1 and N_2 are the components of the unit vector $\bar{N}$ along the y_1- and y_2-axes; u_N and u_S are the normal and tangential components of the displacement vector; and $\partial/\partial N$ and $\partial/\partial S$ are derivatives along the normal and tangent.

We will use the superscripts "(1)" and "(2)" to refer to the cylinder and the half-space, respectively. Also, we will no longer specially mention that the half-space and punch have initial (residual) stresses.

Let the cylinder and half-space be made of different compressible or incompressible materials. In the case of arbitrary elastic potentials in general form for the theory of large (finite) initial strains, we have the following boundary conditions to determine the components of the displacement vector and stress tensor in the cylinder and half-space:

at the end $y_3 = H$ of the punch:

$$u_z^{(1)} = -\varepsilon, \quad \tilde{Q}_{3N}^{(1)} = 0, \quad \tilde{Q}_{3S}^{(1)} = 0, \quad \forall\, (y_1, y_2) \in S^*; \tag{2.1}$$

on the boundary $y_3 = 0$ of the elastic half-space in the contact region:

$$u_3^{(1)} = u_3^{(2)}, \quad \tilde{Q}_{33}^{(1)} = \tilde{Q}_{33}^{(2)}, \quad \tilde{Q}_{3N}^{(2)} = 0, \quad \tilde{Q}_{3S}^{(2)} = 0,$$

$$\tilde{Q}_{3N}^{(1)} = 0, \quad \tilde{Q}_{3S}^{(1)} = 0, \quad \forall\, (y_1, y_2) \in S^*; \tag{2.2}$$

on the boundary $y_3 = 0$ of the half-space beyond the contact region:

$$\widetilde{Q}_{3S}^{(2)} = 0, \quad \widetilde{Q}_{3N}^{(2)} = 0, \quad \widetilde{Q}_{33} = 0, \quad \forall\,(y_1, y_2) \in S^*; \tag{2.3}$$

on the lateral surface of the punch:

$$\widetilde{Q}_{N3}^{(1)} = 0, \quad \widetilde{Q}_{NN}^{(1)} = 0, \quad \widetilde{Q}_{NS}^{(1)} = 0, \quad \forall \in [0, H]. \tag{2.4}$$

Following [2], we will examine two general representations of the solution for compressible and incompressible bodies.

Unequal Roots. For compressible and incompressible bodies, the displacements and the components of the stress vector at $y_3 =$const are expressed as follows [2, formulas (4.21)–(4.22)]:

$$u_N^{(i)} = \frac{\partial}{\partial N}\left(\varphi_1^{(i)} + \varphi_2^{(i)}\right) - \frac{\partial}{\partial S}\varphi_3^{(i)}, \tag{2.5}$$

$$u_S^{(i)} = \frac{\partial}{\partial S}\left(\varphi_1^{(i)} + \varphi_2^{(i)}\right) - \frac{\partial}{\partial N}\varphi_3^{(i)}, \tag{2.6}$$

$$u_3^{(i)} = \frac{m_1^{(i)}}{\sqrt{n_1^{(i)}}}\frac{\partial^2}{\partial z_1^2} + \frac{m_2^{(i)}}{\sqrt{n_2^{(i)}}}\frac{\partial^2}{\partial z_2^2}\varphi_2^i, \tag{2.7}$$

$$Q_{33}^{(i)} = C_{44}^{(i)}\left[\left(1+m_1^{(i)}\right) l_1^{(i)}\frac{\partial^2}{\partial z_1^2}\varphi_1^{(i)} + \left(1+m_2^{(i)}\right) l_2^{(i)}\frac{\partial^2}{\partial z_2^2}\varphi_2^{(i)}\right], \tag{2.8}$$

$$\widetilde{Q}_{3N}^{(i)} = C_{44}\left[\frac{1+m_1^{(i)}}{\sqrt{n_1^{(i)}}}\frac{\partial_2}{\partial N \partial z_1}\varphi_1^{(i)} + \frac{1+m_2^{(i)}}{\sqrt{n_2^{(i)}}}\frac{\partial^2}{\partial N \partial z_2}\varphi_2^{(i)} - \frac{1}{\sqrt{n_3^{(i)}}}\frac{\partial_2}{\partial S \partial z_3}\varphi_3^{(i)}\right], \tag{2.9}$$

$$\widetilde{Q}_{3S}^{(i)} = C_{44}^{(i)}\left[\frac{1+m_1^{(i)}}{\sqrt{n_1^{(i)}}}\frac{\partial_2}{\partial S \partial z_1}\varphi_1^{(i)} + \frac{1+m_2^{(i)}}{\sqrt{n_2^{(i)}}}\frac{\partial^2}{\partial S \partial z_2}\varphi_2^{(i)} + \frac{1}{\sqrt{n_3^{(i)}}}\frac{\partial_2}{\partial N \partial z_3}\varphi_3^{(i)}\right], \tag{2.10}$$

where the functions φ_i, in view of

$$z_i = \frac{1}{\sqrt{n_j^{(i)}}}y_3, \quad i = 1, 2, \quad j = 1, 2, 3, \tag{2.11}$$

are solutions of the differential equation

$$\left(\Delta_1 + \frac{\partial^2}{\partial z_j^2}\right)\varphi_j^{(i)}(y_1, y_2, z_i) = 0, \quad \Delta_1 = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}. \tag{2.12}$$

For the punch, they are

$$\varphi_1^{(1)}(\gamma_i z_j) = A_0 z_1 + B_0 - \frac{1}{\sqrt{n_1}}\sum_{k=1}^{\infty}\beta_k I_0(\beta_k r)\left[C_k \cos(\gamma_k z_1) - D_k \sin(\gamma_k z_1)\right]$$

$$-\frac{1}{\sqrt{n_1}}\sum_{k=1}^{\infty}\alpha_k J_0(\alpha_k r)\left[E_k \cosh(\alpha_k z_1) + F_k \sinh(\alpha_k z_1)\right], \tag{2.13}$$

$$\varphi_2^{(1)} = -D_0\left(r^2 - 2z_2^2\right) - \frac{1}{\sqrt{n_2}} \sum_{k=1}^{\infty} \beta_k I_0(\beta_k r)\left[C_k \sin(\beta_k z_2) + B_k \cos(\beta_k z_2)\right]$$

$$-\frac{1}{\sqrt{n_2}} \sum_{k=1}^{\infty} \alpha_k J_0(\alpha_k r)\left[N_k \cosh(\alpha_k z_2) + \mu_k \sinh(\alpha_k z_2)\right]. \tag{2.14}$$

Moreover, for incompressible bodies we obtain an expression for the scalar p [2]:

$$p = -\lambda_1 q_1^{-1}\left\{\left[\tilde{\kappa}_{1111} - \frac{\lambda_1 q_1}{\lambda_3 q_3}(\tilde{\kappa}_{1133} + \kappa_{1313})\,\Delta_1\left(\varphi_1^{(i)} + \varphi_2^{(i)}\right)\right] - \tilde{\kappa}_{3113}\Delta_1\left(n_1^{-1}\varphi_1 + n_2^{-1}\varphi_2\right)\right\}. \tag{2.15}$$

Equal Roots. As in the case of unequal roots, all derivations will be carried out in general form for compressible and incompressible bodies. The components of the displacements vector and the stress tensor are expressed as follows [8, formulas (4.40)–(4.41)]:

$$u_N = \frac{\partial}{\partial N}\left(\varphi_1^{(i)} + \varphi_2^{(i)}\right) + z_1 \frac{\partial^2}{\partial N \partial z_1}\varphi_2^{(i)} - \frac{\partial}{\partial S}\varphi_3^{(i)}, \tag{2.16}$$

$$u_S = \frac{\partial}{\partial S}\left(\varphi_1^{(i)} + \varphi_2^{(i)}\right) + z_1 \frac{\partial^2}{\partial S \partial z_1}\varphi_2^{(i)} - \frac{\partial}{\partial N}\varphi_3^{(i)}, \tag{2.17}$$

$$u_3 = \frac{m^{(i)}}{\sqrt{n_1^{(i)}}}\left(\frac{\partial}{\partial z_1}\varphi_1^{(i)} + z_1 \frac{\partial^2}{\partial z_1^2}\varphi_2^{(i)}\right) + \frac{m_1 - 1}{\sqrt{n_1^{(i)}}}\frac{\partial}{\partial z_1}\varphi_2^{(i)}, \tag{2.18}$$

$$\tilde{Q}_{33} = C_{44}^{(i)}\left\{\frac{\partial}{\partial z_1^2}\left[\left(1+m_1^{(i)}\right) l_1^{(i)} \varphi_1^{(i)} + \left(1+m_2^{(i)}\right) l_2^{(i)} \varphi_2^{(i)}\right] + \left(1+m_1^{(i)}\right) l_1^{(i)} z_1 \frac{\partial_3}{\partial z_1^3}\varphi_2^{(i)}\right\}, \tag{2.19}$$

$$\tilde{Q}_{3N} = C_{44}^{(i)}\left\{\frac{1}{\sqrt{n_1^{(i)}}}\frac{\partial^2}{\partial S \partial z_1}\left[\left(1+m_1^{(i)}\right)\varphi_1^{(i)} + \left(1+m_2^{(i)}\right)\varphi_2^{(i)}\right]\right.$$

$$\left. + \sqrt{n_1^{(i)}}\left(1+m_1^{(i)}\right) z_1 \frac{\partial}{\partial N \partial z_1^2}\varphi_2^{(i)} - \frac{1}{\sqrt{n_3^{(i)}}}\frac{\partial^2}{\partial S \partial z_3}\right\}, \tag{2.20}$$

$$\tilde{Q}_{3S}^{(i)} = C_{44}^{(i)}\left\{\frac{1}{\sqrt{n_1^{(i)}}}\frac{\partial_2}{\partial S \partial z_1}\left[\left(1+m_1^{(i)}\right)\varphi_1^{(i)} + \left(1+m_2^{(i)}\right)\varphi_2^{(i)}\right]\right.$$

$$\left. + \frac{m^{(i)}+1}{\sqrt{n_1^{(i)}}} z_1 \frac{\partial^3}{\partial S \partial z_1^2}\varphi_2^{(i)} + \frac{1}{\sqrt{n_3^{(i)}}}\frac{\partial^2}{\partial N \partial y_3}\varphi_3^{(i)}\right\}. \tag{2.21}$$

The functions $\varphi_1^{(1)}$ and $\varphi_2^{(1)}$ describing the stress–strain state of the punch satisfy the differential equations

$$\left(\Delta_1 + \frac{\partial^2}{\partial z_1^2}\right)\varphi_1^{(1)} = 0, \quad \left(\Delta_1 + \frac{\partial^2}{\partial z_1^2}\right)\varphi_2^{(i)} = 0 \tag{2.22}$$

and have the form

$$\varphi_1^{(1)}=A_0z_1+B_0-\frac{1}{\sqrt{n_1}}\sum_{k=1}^{\infty}\beta_kI_0(\beta_kr)\left[C_k\cos(\gamma_kz_1)-D_k\sin(\gamma_kz_1)\right]$$

$$-\frac{1}{\sqrt{n_1}}\sum_{k=1}^{\infty}\alpha_kJ_0(\alpha_kr)\left[E_k\cosh(\alpha_kz_1)+F_k\sinh(\alpha_kz_1)\right], \tag{2.23}$$

$$\varphi_2^{(2)}=-D_0\left(r-2z_1^2\right)-\sum_{k=1}^{\infty}I_0(\beta_kr)\left[C_k\sin(\gamma_kz_1)+D_k\cos(\gamma_kz_1)\right]$$

$$-\sum_{k=1}^{\infty}J_0(\alpha_kr)\left[N_k\cosh(\alpha_kz_2)+M_k\sinh(\alpha_kz_2)\right], \tag{2.24}$$

where $\gamma_k=n\pi/h$ are the eigenvalues for $\sin(\gamma_kh)=0$; $\alpha_k=\mu_k/p$ are the eigenvalues for $J_0(\mu_k)=0$; $\beta_k=\sqrt{n_1^{(1)}}\gamma_k$.

For incompressible bodies, as well as for unequal roots, we have an expression for the scalar p:

$$p=-\lambda_1q_1^{-1}\left\{\left[\tilde{\kappa}_{1111}-\lambda_1q_1^{-1}q_3^{-1}(\tilde{\kappa}_{1133}+\tilde{\kappa}_{1313})\right]\Delta_1+n_1\tilde{\kappa}_{3113}\frac{\partial^2}{\partial z_1}\right\}\left(\varphi_1+z_1\frac{\partial}{\partial z_1}\varphi_2+\varphi_2\right). \tag{2.25}$$

Expressions (2.5)–(2.24) have been derived in general form for compressible and incompressible bodies. These expressions include the parameters $n_j^{(i)}, m_j^{(i)}, C_{44}^{(i)}, l_j^{(i)}$ and the tensors $\tilde{\omega}_{ij\alpha\beta}$, which describe the initial (residual) stress–strain states of both half-space and punch. The parameters $n_j^{(i)}, m_j^{(i)}, C_{44}^{(i)}, l_j^{(i)}$ and the tensors $\tilde{\kappa}_{ij\alpha\beta}$ and $\tilde{\omega}_{ij\alpha\beta}$ are functions of the elongation along the coordinate axes $\{y_1, y_2, y_3\}$ in the homogeneous initial (residual) state defined by the displacements

$$u_m=(\lambda_m-1)x_m,\quad \lambda_m=\text{const},\quad m=1,2,3, \tag{2.26}$$

and corresponding to elongation (contraction), compression ($\lambda_1<1$), and tension ($\lambda_1>1$) along the coordinate axes.

3. Method of Solution. Using the technique proposed in [3, 10], we introduce new potentials for the half-space, separately for the cases of equal and unequal roots of the constitutive equation:

$$\varphi_1^{(1)}(y_1,y_2,z_j)=\left(1+m_1^{(2)}\right)^{-1}f(y_1,y_2,z_1),$$

$$\varphi_2^{(1)}(y_1,y_2,z_j)=-\left(1+m_2^{(2)}\right)^{-1}f(y_1,y_2,z_1) \tag{3.1}$$

for equal roots and

$$\varphi_1^{(1)}(y_1,y_2,z_j)=n^{1/2}\left(1+m_1^{(1)}\right)^{-1}f(y_1,y_2,z_1),$$

$$\varphi_2^{(i)}(y_1,y_2,z_j)=-n_2^{1/2}\left(1+m_2^{(2)}\right)f(y_1,y_2,z_2) \tag{3.2}$$

for unequal roots.

To determine the stress–strain state of the half-space, it is necessary to determine the elastic potential $f(y_1, y_2, y_3)$ by substituting (3.1) and (3.2) into (2.12) and (2.22), respectively. Then we arrive at the solution of the differential equations

$$\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}+\frac{\partial^2}{\partial y_j^2}\right)f(y_1,y_2,z_j)=0,\quad j=1,2, \tag{3.3}$$

with the following boundary conditions for potentials (3.3) and (2.5)–(2.10), (2.16)–(2.21):

at the end $y_3 = h$ of the punch:

$$u_3^{(2)} = -\varepsilon, \quad \tilde{Q}_{3S}^{(1)} = 0, \quad \tilde{Q}_{3N}^{(1)} = 0, \quad \forall (y_1, y_2) \in S^*; \tag{3.4}$$

on the boundary $y_3 = 0$ of the half-space in the contact region S^*:

$$\frac{\partial f}{\partial y_3} f(y_1, y_2, y_3) = A u_3^{(2)}, \quad \frac{\partial^2 f}{\partial y_3^2} = B \tilde{Q}_{33}^{(1)}, \quad \forall (y_1, y_2) \in S^*;$$

$$\tilde{Q}_{3N}^{(1)} = 0, \quad \tilde{Q}_{3S}^{(1)} = 0, \quad \forall (y_1, y_2) \in S, \tag{3.5}$$

on the boundary $y_3 = 0$ of the half-space beyond the contact region:

$$\frac{\partial^2 f}{\partial y_3^2} = 0, \quad \tilde{Q}_{3N}^{(2)} = 0, \quad \tilde{Q}_{3S}^{(2)} = 0, \quad \forall (y_1, y_2) \overline{\in} S, \tag{3.6}$$

on the lateral surface of the punch:

$$\tilde{Q}_{N3}^{(1)} = 0, \quad \tilde{Q}_{NN}^{(1)} = 0, \quad \tilde{Q}_{NS}^{(1)} = 0, \quad \forall (y_3) \in [0, H]. \tag{3.7}$$

It should be noted that on the boundary $y_3 = 0$ of the half-space, we can use the following relations in view of (2.11) for equal and unequal roots:

$$f(y_1, y_2, y_3) \equiv f(y_1, y_2, y_3), \quad \frac{\partial}{\partial z_j} f(y_1, y_2, z_j) \equiv \frac{\partial}{\partial y_3} f(y_1, y_2, y_3). \tag{3.8}$$

Considering (2.11) and (3.8), we derive expressions for the components of the displacement vector and stress tensor from (2.16)–(2.24) on the boundary $y_3 = 0$ of the half-space $y_3 \le 0$ for equal roots $n_1 = n_2$:

$$u_3^{(2)} = \frac{1 + 2m_1^{(2)} - m_2^{(2)}}{\sqrt{n_1}\left(1 + m_1^{(2)}\right)\left(1 + m_2^{(1)}\right)} \frac{\partial}{\partial y_3} f(y_1, y_2, y_3), \quad \tilde{Q}_{31}^{(2)} = 0,$$

$$Q_{33}^{(2)} = C_{44}^{(2)}\left(l_1^{(2)} - l_2^{(2)}\right) \frac{\partial^2}{\partial y_3^{(2)}} f(y_1, y_2, y_3), \quad \tilde{Q}_{32}^{(2)} = 0. \tag{3.9}$$

Considering (2.12) and (3.8) and replacing z_1 by y_3, according to [2, formula (4.52)], we arrive at the following differential equation for the potential $f(y_1, y_2, y_3)$:

$$\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3}\right) f(y_1, y_2, y_3) = 0; \tag{3.10}$$

in the case of unequal roots, we have

$$u_3^{(2)} = \frac{m_1^{(2)} - m_2^{(2)}}{\left(1 + m_1^{(2)}\right)\left(1 + m_2^{(2)}\right)} \frac{\partial}{\partial y_3} f(y_1, y_2, y_3), \quad \tilde{Q}_{31} = 0,$$

$$\tilde{Q}_{33}^{(2)} = C_{44}^{(2)}\left(\sqrt{n_1}\, l_1^{(2)} - \sqrt{n_2}\, l_2^{(2)}\right) \frac{\partial^2}{\partial y_3} f(y_1, y_2, y_3), \quad \tilde{Q}_{32} = 0. \tag{3.11}$$

As in the case of equal roots, we have Eq. (3.10) for the function $f(y_1, y_2, y_3)$.

Comparing (3.9)–(3.11) and (3.5), we find the values of the coefficients A and B:

$$A=\begin{cases}\dfrac{(1+m_1^{(2)})(1+m_2^{(2)})\sqrt{n_2}}{1-2m_1^{(2)}-m_2^{(2)}}, & n_1=n_2,\\[2ex] \dfrac{(1+m_1^{(2)})(1+m_2^{(2)})}{m_1^{(2)}-m_2^{(2)}}, & n_1\neq n_2,\end{cases} \tag{3.12}$$

$$B=\begin{cases}\dfrac{1}{C_{44}^{(2)}(l_2^{(2)}-l_1^{(2)})}, & n_1=n_2,\\[2ex] \dfrac{1}{C_{44}^{(2)}\left(l_1^{(2)}\sqrt{n_1}-l_2\sqrt{n_2}\right)}, & n_1\neq n_2.\end{cases} \tag{3.13}$$

Let us introduce a new potential:

$$A^{-1}\frac{\partial f(y_1,y_2,y_3)}{\partial y_3}=V. \tag{3.14}$$

From formulas (3.4)–(3.7) with (2.11) and (3.8), it follows that for $y_3=0$ the boundary conditions for the potential V at the end of the lateral surface of the punch are identical to expressions (3.4) and (3.7). On the boundary of the half-space:

in the contact region of the punch and the half-space $y_3=0$:

$$V=u_z^{(1)},\quad \frac{\partial V}{\partial y_3}=\frac{B}{A}\widetilde{Q}_{33}^{(1)},\quad \widetilde{Q}_{3N}=0,\quad \widetilde{Q}_{3S}=0,\quad \forall(y_1,y_2)\in S^*; \tag{3.15}$$

beyond the contact region of the punch and the half-space:

$$\frac{\partial V}{\partial y_3}=0,\quad Q_{3N}^{(1)}=0,\quad V(y_1,y_2)\in S^*,\quad \forall(y_1,y_2)\in S^*. \tag{3.16}$$

Thus, we have arrived at the classical mixed problem (3.10), (3.15) in the harmonic-potential theory for the half-space $y_3\le 0$, which completely coincides with a similar problem for harmonic elastic potential without prestresses [14, p. 79]). This mixed problem (3.10), (3.15) for the elastic half-space $y_3\le 0$ will go over into the problem [13] if

$$\frac{B}{A}=\frac{\lambda_2-2\mu_2}{2\mu_2(\lambda_2+\mu_2)}, \tag{3.17}$$

where λ_2 and μ_2 are the Lamé constants of the elastic half-space without prestresses.

Thus, in the case of half-space without prestresses (classical contact problem), the problem reduces to the determination of the harmonic potential (3.1), (3.2) for the half-space and the potentials $\varphi_1^{(1)}$ and $\varphi_2^{(1)}$ for the elastic punch, which, in view of (2.12), (2.22), and (3.17), depend on

$$\frac{B}{A}=\frac{\lambda_2+2\mu_2}{2\mu_2(\lambda_2+\mu_2)}, \tag{3.18}$$

where λ_1 and μ_1 are the Lamé constants of the elastic punch without initial (residual) stresses.

The results for an elastic punch of arbitrary cross-section forced into an elastic half-space make it possible to formulate a general method to solve such problems in linearized elasticity as the following statement.

Statement. Let the classical mixed problem of the harmonic-potential theory for a half-space corresponding to the contact problem for an elastic punch of arbitrary cross-section forced into an elastic half-space have been solved. The solution is a potential $f(y_1,y_2,y_3,\mu_2)$, where μ_2 is the shear modulus of a linear elastic body without initial (residual) stresses. Then, to determine the displacements and stresses in a prestressed elastic body, it is necessary to:

(i) use (3.15) with notation (3.12)–(3.13) and expressions (2.16)–(2.20) with (3.1) in the case of equal roots;

(ii) use (3.15) with notation (3.12)–(3.13) and expressions (2.5)–(2.10) with (3.2) in the case of unequal roots.

The method formulated above as a statement can be used to obtain an exact solution to a wide class of three-dimensional contact problems for prestressed elastic bodies for which there are solutions of similar contact problems of linear elasticity for isotropic bodies.

4. Elastic Contact of a Circular Punch and a Half-Space. We will use well-known solutions in the classical theory of elasticity, setting the elongation coefficient $\lambda_1 = 1$ for the elastic punch (as in the classical theory of elasticity) [11] to illustrate the above-stated general method intended to solve the three-dimensional contact problem for an elastic punch with arbitrary cross-section, height *h*, and initial (residual) stresses indented into a half-space with initial (residual) stresses. Let us consider the axisymmetric case. It is convenient to express the potentials $\varphi_j^{(i)}(y_1, y_2, y_3)$ and $f(y_1, y_2, y_3)$ in circular cylindrical coordinates (γ, θ, y_3) and (r, θ, z_j). In view of (2.12) and (2.22), the boundary conditions (2.1)–(2.4) become:

at the end $y_3 = h$ of the punch:

$$u_z^{(1)} = -\varepsilon, \quad \widetilde{Q}_{r3}^{(1)} = 0, \quad r \in [0, R]; \tag{4.1}$$

on the boundary $y_3 = 0$ of the half-space in the contact region:

$$\frac{\partial f}{\partial y_3} = A u_z^{(1)}, \quad \frac{\partial^2 f}{\partial y_3^2} = B \widetilde{Q}_{33}^{(1)}, \quad \widetilde{Q}_{r3}^{(1)} = 0, \quad r \in [0, R]; \tag{4.2}$$

on the boundary $y_3 = 0$ of the half-space beyond the contact region:

$$\frac{\partial^2 f}{\partial y_3^2} = 0, \quad \widetilde{Q}_{3r}^{(1)} = 0, \quad r \in [0, \infty]; \tag{4.3}$$

on the lateral surface of the cylinder:

$$\widetilde{Q}_{3r}^{(1)} = 0, \quad \widetilde{Q}_{rr}^{(1)} = 0, \quad \forall y_3 \in [0, h]. \tag{4.4}$$

According to (3.1), the components of the displacement and stress vectors on the boundary $y_3 = 0$ of the half-space can be expressed as

$$u_3 = A^{-1} \frac{\partial f}{\partial y_3}, \quad \widetilde{Q}_{33}^{(2)} = B^{-1} \frac{\partial^2 f}{\partial y_3^2}, \quad Q_{3r}^{(2)} = 0, \tag{4.5}$$

where the elastic potential $f(r, y_3)$ in circular cylindrical coordinates is the solution of the differential equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial y_3^2} \right) f(r, y_3) = 0. \tag{4.6}$$

Applying the Hankel transform [3, 4, 6], we get

$$f(r, y_3) = \int_0^\infty \eta F(\eta)\, e^{\eta y_3} J_0(\eta r)\, d\eta, \tag{4.7}$$

where the function $F(\eta)$ is defined as follows [3, 4, 6]:

$$F(\eta) = -\frac{\varepsilon}{\pi} \Omega \left[(B_0 - 1) \frac{\sin \eta R}{\eta^4} + \frac{R\left(\mu_1^{(1)} + 2\mu_1^{(1)}\right)}{2\mu_1^{(1)}}\, \Omega \cdot \sum_{k=1}^{\infty} B_k \int_0^1 \cos(\eta R y) \cos(\mu_k y)\, dy \right], \tag{4.8}$$

where μ_k are the roots of the equation $J_1(\mu_k)=0$; B_0 and B_k are determined from an infinite system of algebraic equations [5, formula (3.79)] with coefficients strongly dependent on the parameters of the initial (residual) stress state; $\Omega=\dfrac{\sqrt{2A}-2\sqrt{2A-BE_2}}{\sqrt{2A}-\sqrt{2A-BE_2}}$.

After finding the coefficients B_k from (3.11) for compressible and incompressible bodies in the cases of equal and unequal roots, we can determine the components of the displacement and stress vectors in the half-space [5].

5. Numerical Example. According to (2.5)–(2.25), we determine the displacements and stresses on the boundary $y_3=0$ of the half-space from

$$u_3=-\frac{\lambda_2+2\mu_2}{\lambda_2+\mu_2}\frac{\partial f}{\partial y_3},\quad \tilde{Q}_{33}^{(1)}=-2\mu_1\frac{\partial^2 f}{\partial y_3^2}. \tag{5.1}$$

Let us consider an elastic punch with $\lambda_1=1$ (as in the classical theory of elasticity) forced into an elastic half-space. We assume that the external load applied to the upper end of the punch (resultant main vector) is the same as in the problem of linearized elasticity. Then, according to the statement in Sec. 3, we have

$$u_3^{(2)}(y_1,y_2)=ku_3^{(1)}(y_1,y_2). \tag{5.2}$$

Comparing (5.1) and (3.11), we find the values of k in the cases of equal and unequal roots:

$$k=2\mu_2\frac{\lambda_2+\mu_2}{\lambda_2+\mu_2}\cdot\frac{B}{A}, \tag{5.3}$$

where A and B are defined by (3.8) and (3.9) in the cases of equal and unequal roots of the constitutive equation for compressible bodies [2, formula (2.12)] and incompressible bodies [2, formula (2.85)]. The results (5.2) and (5.3) have been obtained for an elastic punch in the form of a cylinder of arbitrary cross-section. Let us consider the following potentials.

Harmonic Potential. In this case, the parameters $m_j^{(2)}, l_j^{(2)}, n_j^{(2)}$, and $s_j^{(2)}$ are defined as in [5, formulas (3.108)]. According to [2, formulas (2.53)–(2.55)], we obtain the case of equal roots:

$$n_1=n_2=\left(\frac{2\mu_2+3\lambda_2}{2\mu_2+\lambda_2}\frac{1}{\lambda_1^{(2)}}-\frac{2\lambda_2}{2\mu_2+\lambda_2}\right)^2,$$

$$m_1^{(2)}=\frac{2\mu_2+3\lambda_2}{2\mu_2+\lambda_2}-\frac{2\lambda_2}{2\mu_2+\lambda_2},\qquad m_2=\frac{(\lambda_2-2\mu_2)(2\mu_2+3\lambda_2)+(6\mu_2-\lambda_2)\lambda_2\lambda_1}{(\lambda_2+2\mu_2)(2\mu_2+3\lambda_2)-\lambda_2\lambda_1},$$

$$l_1^{(2)}=\left(\frac{2\mu_2+3\lambda_2}{2\mu_2+\lambda_2}\frac{1}{\lambda_1}-\frac{2\lambda_2}{2\mu_2+\lambda_2}\right)^{-1}, \tag{5.4}$$

$$c_{44}^{(2)}=\frac{2\mu_2}{\lambda_1}\frac{2\mu_2+3\lambda_2-2\lambda_1\lambda_1}{2\mu_2+3\lambda_2(2\mu_2-\lambda_2)\lambda_1},\qquad s_0^{(2)}=-\frac{4\mu_2\lambda_1}{2\mu_2+3\lambda_2-\lambda_2\lambda_1},$$

$$s^{(2)}=\frac{2\mu_2+3\lambda_2-(3\lambda_2+4\mu_2)\lambda_1}{2\mu_2+3\lambda_2-\lambda_2\lambda_1},\qquad s_1^{(2)}=-\frac{4\mu_2\lambda_1}{2\mu_2+3\lambda_2-\lambda_2\lambda_1}. \tag{5.5}$$

Substituting (5.5) into (5.3) and omitting transformations, we write

$$k=\frac{2(\lambda_2+\mu_2)}{4\mu_2+5\lambda_2}\frac{\lambda_1^2}{\lambda_1-\dfrac{2\mu_2+3\lambda_2}{4\mu_2+5\lambda_2}}=\frac{1}{\nu_2+2}\frac{6\lambda_1^2}{\lambda_1-\dfrac{1+\nu_2}{2+\nu_2}}. \tag{5.6}$$

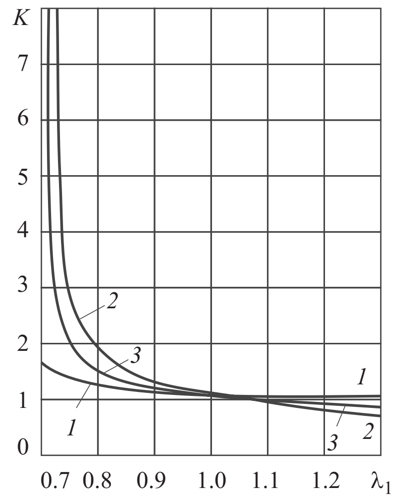

Fig. 2

Note that k tends to infinity if

$$\lambda_1 = \lambda_{cr} = \frac{2\mu_2 + 3\lambda_2}{4\mu_2 + 5\lambda} = \frac{1+\nu_2}{2+\nu_2}. \tag{5.7}$$

The value $\lambda_1 = \lambda_{cr}$ (5.7) represents the surface instability of the half-space under uniform biaxial compression [9] and $\lambda_{cr} = 0.5652$ when Poisson's ratio $\nu_2 = 0.3$. As $\lambda_1 \to \lambda_{cr}$, the displacements under the punch increase to infinity.

Bartenev–Khazanovich Potential. Expressions for this potential are given in [5, formulas (3.114)–(3.115)]. From [5, formulas (2.112)–(2.113)] it follows that we have the case of equal roots. After some transformations, we find

$$k = \frac{\lambda_2 + \mu_2}{\lambda_2 + 2\mu_2} 2\lambda_2^{7/2} \left(3\lambda_1^3 - 1\right)^{-1}. \tag{5.8}$$

Since a body described by the Bartenev–Khazanovich potential is incompressible, it is necessary that $\lambda_2 \to \infty$ in (5.8). Passing to the limit in (5.8) as $\lambda_2 \to \infty$, we obtain

$$k = 2\lambda_1^{7/2} (3\lambda_1^3 - 1)^{-1}, \tag{5.9}$$

whence it follows that expression (5.9) tends to infinity if

$$3\lambda_1^3 - 1 = 0 \quad (\lambda_1 = \lambda_{cr} = 0.69336). \tag{5.10}$$

Equation (5.10) describes surface instability [9]. Thus, as the prestress tends to the level corresponding to the surface instability of the half-space, the displacements under the punch tend to infinity, according to (5.2) and (5.9). Hence, when $\lambda_1 = \lambda_{cr}$, the Bartenev–Khazanovich potential exhibits resonant behavior.

Treloar Potential (Neo-Hookean Body). Expressions for the coefficients $n_j^{(2)}, m_j^{(2)}, l_j^{(2)}$, and $s_j^{(2)}$ are given in [5, formulas 2.114–2.116], i.e., we have the case of unequal roots $n_1 \neq n_2$. After some transformations, we obtain

$$k = \frac{\mu_2}{2c_{44}} \frac{\lambda_2 + \mu_2}{\lambda_2 + 2\mu} 2\lambda_1^4 \left(\lambda_1^3 + 1\right) \left(\lambda_1^9 + \lambda_1^6 + 3\lambda_1^3 - 1\right)^{-1}. \tag{5.11}$$

Since a neo-Hookean body is incompressible, it is necessary that $\lambda \to \infty$ in (5.11). After transformations for the Treloar potential, we get

$$k = 2\lambda_1^4 (\lambda_1^3 + 1)(\lambda_1^9 + \lambda_1^6 + 3\lambda_1^3 - 1)^{-1}, \tag{5.12}$$

where k tends to infinity if

$$\lambda_1^9 + \lambda_1^6 + 3\lambda_1^3 - 1 = 0. \tag{5.13}$$

Equation (5.13) describes the surface instability of the half-space under uniform biaxial compression [9]. The value $\lambda_1 = \lambda_{cr} = 0.667$ corresponds to the initial elongation at which a resonance occurs.

Thus, as the prestress tends to the level corresponding to the surface instability of the half-space, a resonance occurs, which is a mechanical effect similar to that observed in the mechanics of brittle fracture of prestressed materials [2] and in contact problems for rigid punches [6].

Figure 2 shows the dependence of k on λ_1 obtained using expressions (5.6), (5.9), and (5.12) in the cases of compression ($\lambda_1 < 1$) and tension ($\lambda_1 > 1$). Curves *1*, *2*, and *3* represent the harmonic (ν_2 = 0.3), Bartenev–Khazanovich, and Treloar potentials.

It can be seen that the prestresses affect the displacements under the punch as follows: the displacements increase under compression for both compressible and incompressible bodies; this increase is insignificant under tension.

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