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SOLUTION DESCRIBING THE NATURAL VIBRATIONS OF RECTANGULAR SHALLOW SHELLS WITH VARYING THICKNESS

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Natural vibrations of shallow cylindrical shells with rectangular plan and varying thickness are studied using a spline-approximation method developed previously. Computation is carried out for different types of boundary conditions. The effect of the curvature of the midsurface on the natural frequencies is examined. The natural frequencies of shells with constant and varying thickness are compared

Keywords: natural vibrations, shallow shell, spline-collocation

Introduction. Shallow shells of various shapes are widely used as structural members in modern engineering and building structures. The operating conditions for these structures impose certain requirements on their strength and reliability. In this connection, efficient numerical and experimental methods for the determination of the load-bearing capacity and, in particular, resonant frequencies of such structures take on special significance.

Of interest are the natural vibrations of rectangular (in plan) shallow shells with varying thickness and different boundary conditions. For shells of constant thickness with hinged edges, it is possible to find a closed-form solution [5, 6]. If the edges are clamped, then the variables in the original equations of motion cannot be separated and, therefore, numerical methods should be applied. There are just a few publications devoted to this class of problems [4, 11–13]. This is because their solution involves computational difficulties.

Spline functions have recently been used to study the mechanical behavior of plates and shells. Their main advantages are:

– stability against local perturbations; i.e., the local behavior of splines in the neighborhood of a point does not influence their overall behavior, in contrast to, for example, polynomial approximation;

– better convergence than that of polynomial approximation;

– simple and convenient computer implementation.

This paper proposes an efficient numerical technique for studying the natural frequencies and modes of shallow rectangular (in plan) shells of varying thickness. The technique is based on spline-approximation in one coordinate direction and solution of a boundary-value eigenvalue problem for systems of ordinary differential equations of high order with variable coefficients by the stable discrete-orthogonalization method in combination with step-by-step search. The shell material is generally anisotropic.

Noteworthy is the series of publications where the spline-approximation was used to analyze the stress–strain state of shells of different structure and the natural vibrations of plates [1, 7–10].

With such an approach, we can study the natural vibrations of a wide class of isotropic and anisotropic shallow shells with arbitrarily varying thickness and complex boundary conditions.

The objective of the present paper is to study the natural vibrations of elastic rectangular (in plan) shallow shells with varying thickness on the basis of spline-approximation.

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1. Original Relations. Consider a shallow and, generally, orthotropic shell with varying thickness $h(x, y)$ and rectangular plan (in the plane xOy of a Cartesian coordinate system). The geometry of the plan is approximately identified with the geometry of the mid-surface, i.e., $A \approx 1$, $B \approx 1$, and the principal curvatures satisfy the relation $k_1 \cdot k_2 \approx 0$.

According to the Donnell–Mushtari–Vlasov theory of shallow shells [3, 6], the natural transverse vibrations of shallow shells are described by the equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial S}{\partial y} = 0, \qquad \frac{\partial S}{\partial x} + \frac{\partial N_y}{\partial y} = 0,$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - k_1 N_x - k_2 N_y = \rho h \frac{\partial^2 w}{\partial t^2},$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} = Q_x, \qquad \frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x} = Q_y, \tag{1.1}$$

where x and y are the Cartesian coordinates of a point on the mid-surface ($0 \le x \le a$, $0 \le y \le b$), t is time, w is the deflection of the shell, and ρ is the density of the material.

The normal and shear forces N_x, N_y, and S and the bending and twisting moments M_x, M_y, and H satisfy the following relations:

$$N_x = C_{11}\left(\frac{\partial u}{\partial x} + k_1 w\right) + C_{12}\left(\frac{\partial v}{\partial y} + k_2 w\right), \qquad S = C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

$$N_y = C_{12}\left(\frac{\partial u}{\partial x} + k_1 w\right) + C_{22}\left(\frac{\partial v}{\partial y} + k_2 w\right), \qquad M_x = -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\right),$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial x \partial y}, \qquad M_y = -\left(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2}\right), \tag{1.2}$$

where $E_1, E_2, G_{12}, \nu_1, \nu_2$ are the elastic and shear moduli and Poisson's ratios.

The system of equations (1.1)–(1.2) yields three equivalent differential equations for the three displacements u, v, and w of the mid-surface:

$$C_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial C_{11}}{\partial x}\frac{\partial u}{\partial x} + C_{66}\frac{\partial^2 u}{\partial y^2} + \frac{\partial C_{66}}{\partial y}\frac{\partial u}{\partial y} + (C_{12} + C_{66})\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial C_{66}}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial C_{12}}{\partial x}\frac{\partial v}{\partial y}$$

$$+ (C_{11}k_1 + C_{12}k_2)\frac{\partial w}{\partial x} + \frac{\partial (C_{11}k_1 + C_{12}k_2)}{\partial x} w = 0,$$

$$C_{66}\frac{\partial^2 v}{\partial x^2} + \frac{\partial C_{66}}{\partial x}\frac{\partial v}{\partial x} + C_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial C_{22}}{\partial y}\frac{\partial v}{\partial y} + (C_{12} + C_{66})\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial C_{12}}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial C_{66}}{\partial x}\frac{\partial u}{\partial y}$$

$$+ (C_{12}k_1 + C_{22}k_2)\frac{\partial w}{\partial y} + \frac{\partial (C_{12}k_1 + C_{22}k_2)}{\partial y} w = 0,$$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + D_{22}\frac{\partial^4 w}{\partial y^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 2\frac{\partial D_{11}}{\partial x}\frac{\partial^3 w}{\partial x^3} + 2\frac{\partial D_{22}}{\partial y}\frac{\partial^3 w}{\partial y^3}$$

$$+2\frac{\partial}{\partial y}\left(D_{12} + 2D_{66}\right)\frac{\partial^3 w}{\partial x^2 \partial y} + 2\frac{\partial}{\partial x}\left(D_{12} + 2D_{66}\right)\frac{\partial^3 w}{\partial x \partial y^2} + \left(\frac{\partial^2 D_{11}}{\partial x^2} + \frac{\partial^2 D_{12}}{\partial y^2}\right)\frac{\partial^2 w}{\partial x^2}$$

$$+\left(\frac{\partial^2 D_{12}}{\partial x^2}+\frac{\partial^2 D_{22}}{\partial y^2}\right)\frac{\partial^2 w}{\partial y^2}+4\frac{\partial^2 D_{66}}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+(C_{11}k_1^2+2C_{12}k_1k_2+C_{22}k_2^2)w$$

$$+(C_{11}k_1+C_{12}k_2)\frac{\partial u}{\partial x}+(C_{12}k_1+C_{22}k_2)\frac{\partial v}{\partial y}-\rho h\frac{\partial^2 w}{\partial t^2}=0. \tag{1.3}$$

Boundary conditions for displacements are specified on the boundaries $x=0,a$ and $y=0,b$. If y = const, then:
(i) clamped boundary:

$$u=v=w=\frac{\partial w}{\partial y}=0 \quad \text{at} \quad y=0, \quad y=b, \tag{1.4}$$

(ii) hinged boundary:

$$u=\frac{\partial v}{\partial y}=w=\frac{\partial^2 w}{\partial y^2}=0 \quad \text{at} \quad y=0, \quad y=b, \tag{1.5}$$

(iii) one boundary hinged and the other clamped:

$$u=\frac{\partial v}{\partial y}=w=\frac{\partial^2 w}{\partial y^2}=0 \quad \text{at} \quad y=0,$$

$$u=v=w=\frac{\partial w}{\partial y}=0 \quad \text{at} \quad y=b. \tag{1.6}$$

Similar conditions can also be prescribed on the boundaries x = const (replacing y by x and v by u in Eqs. (1.4)–(1.6)).

2. Solution Technique. The solution of the system of equations (1.3) is sought in the form

$$u=\sum_{i=0}^{N}u_i(x)\varphi_i(y), \quad v=\sum_{i=0}^{N}v_i(x)\chi_i(y), \quad w=\sum_{i=0}^{N}w_i(x)\psi_i(y), \tag{2.1}$$

where $u_i(x)$, $v_i(x)$, and $w_i(x)$ $(i=0,\dots,N)$ are the unknown functions; $\varphi_i(y)$ and $\chi_i(y)$ are functions constructed using cubic B-splines ($N\ge 4$), and $\psi_i(y)$ are functions constructed using quintic B-splines ($N\ge 6$). The functions $\varphi_i(y)$, $\chi_i(y)$, and $\psi_i(y)$ are selected so as to satisfy the boundary conditions at y = const using linear combinations of cubic and quintic B-splines, respectively.

We write the system of equations (1.3) in the following form:

$$\frac{\partial^2 u}{\partial x^2}=a_1\frac{\partial u}{\partial x}+a_2\frac{\partial^2 u}{\partial y^2}+a_3\frac{\partial u}{\partial y}+a_4\frac{\partial^2 v}{\partial x\partial y}+a_5\frac{\partial v}{\partial x}+a_6\frac{\partial v}{\partial y}+a_7\frac{\partial w}{\partial x}+a_8 w,$$

$$\frac{\partial^2 v}{\partial x^2}=b_1\frac{\partial v}{\partial x}+b_2\frac{\partial^2 v}{\partial y^2}+b_3\frac{\partial v}{\partial y}+b_4\frac{\partial^2 u}{\partial x\partial y}+b_5\frac{\partial u}{\partial x}+b_6\frac{\partial u}{\partial y}+b_7\frac{\partial w}{\partial y}+b_8 w,$$

$$\frac{\partial^4 w}{\partial x^4}=c_1\frac{\partial^3 w}{\partial x^3}+c_2\frac{\partial^4 w}{\partial x^2\partial y^2}+c_3\frac{\partial^3 w}{\partial x^2\partial y}+c_4\frac{\partial^2 w}{\partial x^2}+c_5\frac{\partial^3 w}{\partial x\partial y^2}+c_6\frac{\partial^2 w}{\partial x\partial y}$$

$$+c_7\frac{\partial^4 w}{\partial y^4}+c_8\frac{\partial^3 w}{\partial y^3}+c_9\frac{\partial^2 w}{\partial y^2}+c_{10}w+c_{11}\frac{\partial u}{\partial x}+c_{12}\frac{\partial v}{\partial y}, \tag{2.2}$$

where $a_m=a_m(x,y)$, $b_m=b_m(x,y)$, $m=1,\dots,8$, $c_n=c_n(x,y)$, $n=1,\dots,9,11,12$, $c_{10}=c_{10}(x,y,\omega)$.

Substituting (2.1) into Eqs. (2.2), we require that they be satisfied at prescribed collocation points $\xi_k\in[0,b]$, $k=0,\dots,N$. If the mesh has an even number of nodes ($N=2n+1$, $n\ge 3$) and the collocation points are such that

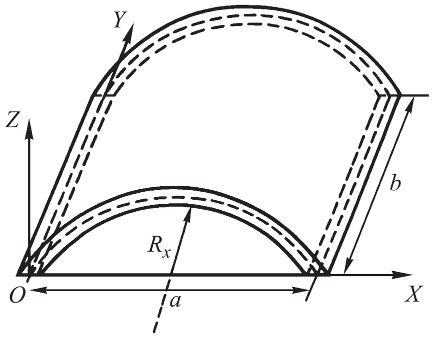


Fig. 1

$\xi_{2i} \in [y_{2i}, y_{2i+1}]$, $\xi_{2i+1} \in [y_{2i}, y_{2i+1}]$ $(i = 0,\dots,n)$, then there are two collocation points on the interval $[y_{2i}, y_{2i+1}]$ and no collocation points on the neighboring intervals $[y_{2i+1}, y_{2i+2}]$. On each of the intervals $[y_{2i}, y_{2i+1}]$, collocation points are selected as follows: $\xi_{2i} = y_{2i} + z_1 h$, $\xi_{2i+1} = y_{2i} + z_2 h$ $(i = 0,\dots,n)$, where z_1 and z_2 are the roots of a Legendre polynomial of the second order on the interval [0, 1], $z_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$ and $z_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$. This choice of collocation points is optimal and enhances significantly the accuracy of approximation. After all transformations, we obtain a system of $N+1$ linear differential equations for u_i, v_i, and w_i. With the notation

$$\Phi_l = [\varphi_i^{(l)}(\xi_k)], \quad \mathrm{X}_l = [\chi_i^{(l)}(\xi_k)], \quad \Psi_m = [\psi_i^{(m)}(\xi_k)],$$

$$i, k = 0,\dots,N, \quad l = 0,\dots,2, \quad m = 0,\dots,4,$$

$$\bar{u}^T = \{u_0,\dots,u_N\}, \quad \bar{v}^T = \{v_0,\dots,v_N\}, \quad \bar{w}^T = \{w_0,\dots,w_N\},$$

$$\bar{a}_r^T = \{a_r(x,\xi_0),\dots,a_r(x,\xi_N)\}, \quad \bar{b}_r^T = \{b_r(x,\xi_0),\dots,b_r(x,\xi_N)\}, \quad r = 1,\dots,8,$$

$$\bar{c}_p^T = \{c_p(x,\xi_0),\dots,c_p(x,\xi_N)\}, \quad p = 1,\dots,9,11,12, \quad \bar{c}_{10}^T = \{c_{10}(x,\xi_0,\omega),\dots,c_{10}(x,\xi_N,\omega)\}$$

and the notation $\bar{c} \cdot A$ of the matrix $[c_i a_{ij}]$, where $A = [a_{ij}]$ $(i, j = 0,\dots,N)$ is a matrix and $\bar{c} = \{c_0,\dots,c_N\}$ is a vector, the system of differential equations becomes

$$\bar{u}'' = \Phi_0^{-1}\Big\{\big(\bar{a}_1 \cdot \Phi_0\big)\bar{u}' + \big(\bar{a}_2 \cdot \Phi_2 + \bar{a}_3 \cdot \Phi_1\big)\bar{u} + \big(\bar{a}_4 \cdot \mathrm{X}_1 + \bar{a}_5 \cdot \mathrm{X}_0\big)\bar{v}'$$

$$+ \big(\bar{a}_6 \cdot \mathrm{X}_1\big)\bar{v} + \big(\bar{a}_7 \cdot \Psi_0\big)\bar{w}' + \big(\bar{a}_8 \cdot \Psi_0\big)\bar{w}\Big\},$$

$$\bar{v}'' = \mathrm{X}_0^{-1}\Big\{\big(\bar{b}_1 \cdot \mathrm{X}_0\big)\bar{v}' + \big(\bar{b}_2 \cdot \mathrm{X}_2 + \bar{b}_3 \cdot \mathrm{X}_1\big)\bar{v} + \big(\bar{b}_4 \cdot \Phi_1 + \bar{b}_5 \cdot \Phi_0\big)\bar{u}' + \big(\bar{b}_6 \cdot \Phi_1\big)\bar{u} + \big(\bar{b}_7 \cdot \Psi_1\big)\bar{w} + \big(\bar{b}_8 \cdot \Psi_0\big)\bar{w}\Big\},$$

$$\bar{w}^{\mathrm{IV}} = \Psi_0^{-1}\{(\bar{c}_1 \cdot \Psi_0)\bar{w}''' + (\bar{c}_2 \cdot \Psi_2 + \bar{c}_3 \cdot \Psi_1 + \bar{c}_4 \cdot \Psi_0)\bar{w}'' + (\bar{c}_5 \cdot \Psi_2 + \bar{c}_6 \cdot \Psi_1)\bar{w}'$$

$$+ (\bar{c}_7 \cdot \Psi_4 + \bar{c}_8 \cdot \Psi_3 + \bar{c}_9 \cdot \Psi_2 + \bar{c}_{10} \cdot \Psi_0)\bar{w} + \big(\bar{c}_{11} \cdot \Phi_0\big)\bar{u}' + \big(\bar{c}_{12} \cdot \mathrm{X}_1\big)\bar{v}\Big\}, \tag{2.3}$$

where $u_i^{(k)} = u^{(k)}(x,\eta_i)$, $v_i^{(k)} = v^{(k)}(x,\eta_i)$, $w_i^{(l)} = w^{(l)}(x,\eta_i)$, $k = 0,\dots,1$, $l = 0,\dots,3$, $i = 0,\dots,N$.

This system of ordinary differential equations can be written in a normal form:

$$\frac{d\bar{Y}}{dx} = A(x,\omega)\bar{Y} \qquad (0 \le x \le a), \tag{2.4}$$

where $Y^T = \{u_0,\dots,u_N, u'_0,\dots,u'_N, v_0,\dots,v_N, v'_0,\dots,v'_N, w_0,\dots,w_N, w'_0,\dots,w'_N, w''_0,\dots,w''_N, w'''_0,\dots,w'''_N\}$, $A(x,\omega)$ is a square matrix of dimension $8(N+1) \times 8(N+1)$.

TABLE 1

Boundary	Boundary conditions, BC			
	1	2	3	4
$x=0$	$u=v=w=\frac{\partial w}{\partial x}=0$	$u=v=w=\frac{\partial w}{\partial x}=0$	$u=v=w=\frac{\partial w}{\partial x}=0$	$u=v=w=\frac{\partial w}{\partial x}=0$
$x=a$	$u=v=w=\frac{\partial w}{\partial x}=0$	$u=v=w=\frac{\partial w}{\partial x}=0$	$u=v=w=\frac{\partial w}{\partial x}=0$	$\frac{\partial u}{\partial x}=v=w=\frac{\partial^2 w}{\partial x^2}=0$
$y=0$	$u=v=w=\frac{\partial w}{\partial y}=0$	$u=\frac{\partial v}{\partial y}=w=\frac{\partial^2 w}{\partial y^2}=0$	$u=\frac{\partial v}{\partial y}=w=\frac{\partial^2 w}{\partial y^2}=0$	$u=v=w=\frac{\partial w}{\partial y}=0$
$y=b$	$u=v=w=\frac{\partial w}{\partial y}=0$	$u=v=w=\frac{\partial w}{\partial y}=0$	$u=\frac{\partial v}{\partial y}=w=\frac{\partial^2 w}{\partial y^2}=0$	$u=\frac{\partial v}{\partial y}=w=\frac{\partial^2 w}{\partial y^2}=0$

The boundary conditions (1.4)–(1.6) for the system of equations (2.4) can be written in the form

$$B_1\bar{Y}(0)=\bar{0}, \qquad B_2\bar{Y}(a)=\bar{0}. \tag{2.5}$$

The eigenvalue problem for the system of ordinary differential equations (2.4) with the boundary conditions (2.5) has been solved by the discrete-orthogonalization method in combination with step-by-step search [1, 2].

3. Solution. The technique outlined above was used to analyze the spectrum of natural vibrations of shallow cylindrical shells square in plan (Fig. 1). The thickness of the shell varies as follows:

$$h=h_0\left[1+\alpha\left(6\zeta^2-6\zeta+1\right)\right], \tag{3.1}$$

where $0\le\zeta\le 1$, $|\alpha|<1$, $\zeta=x/a$; h_0 is the thickness of a shell of constant thickness and equivalent mass. For a shallow cylindrical shell, we have $k_1=1/R_x$ and $k_2=1/R_y=0$.

The following boundary conditions were used:

– the entire boundary is clamped (BC1);

– three sides are clamped and the fourth side is hinged (BC2);

– two opposite sides are clamped and the other sides are hinged (BC3);

– two adjacent sides are clamped and the other sides are hinged (BC4).

Expressions for these boundary conditions are presented in Table 1.

The results obtained by the spline-collocation method with different number of collocation points ($N=8$, $N=10$, $N=12$) practically coincide.

We will discuss the results obtained with $N=10$.

Tables 2, 3, 4, and 5 collect the dimensionless values of the first four resonant frequencies for shells of variable ($|\alpha|>0$) and constant ($\alpha=0$) thickness with the following radii of curvature of the mid-surface: r_x = 6.26000, 1.60250, 0.86125 ($r_x=R_x/a$ is the dimensionless radius of curvature).

Figures 2–4 show the natural frequencies of the shell as a function of the parameter α for BC1 (solid line), BC2 (dotted line), BC3 (dashed line), and BC4 (dash-and-dot line).

Figure 5 shows the first four natural modes of shells of varying thickness for α = 0.1, BC1, and different values of curvature.

To test the technique outlined above, we have calculated the natural frequencies of a shallow isotropic cylindrical shell with square plan and hinged edges (its thickness is constant, i.e., $\alpha=0$) using the following formula [6]:

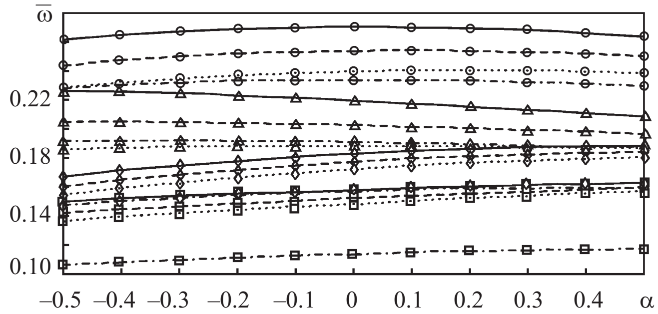


Fig. 2

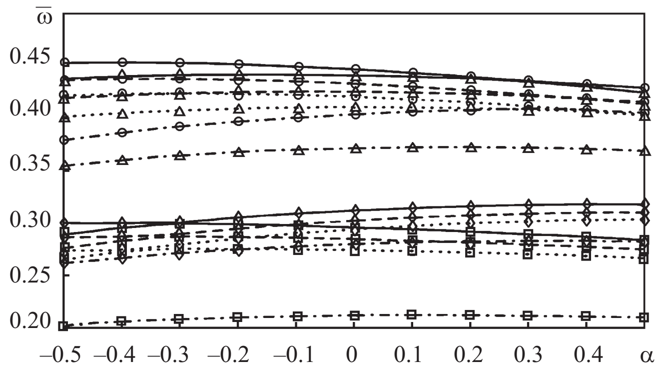


Fig. 3

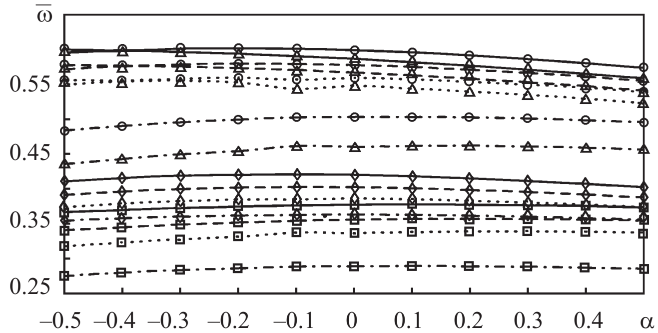
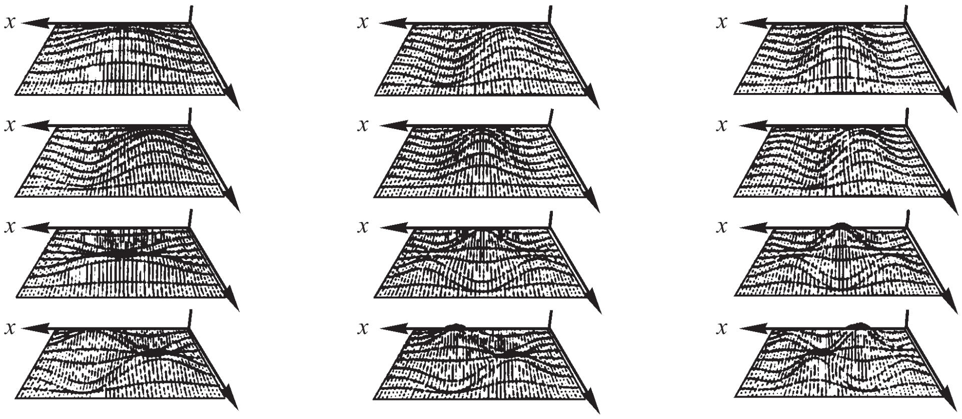


Fig. 4

$$\omega_{mn} = \sqrt{\frac{1}{\rho h}\left\{D_M\left(\lambda_m^2+\lambda_n^2\right)^2+\frac{Ehk_1^2\lambda_m^4}{\left(\lambda_m^2+\lambda_n^2\right)^2}\right\}}, \tag{3.2}$$

where $D_M = \dfrac{Eh^3}{12(1-\nu^2)}$ is the cylindrical stiffness of the shell; $\lambda_m = \dfrac{m\pi}{a}$, $\lambda_n = \dfrac{n\pi}{a}$; and m and n are the number of half-waves along the OX- and OY-axes, respectively.

Table 5 summarizes the dimensionless frequencies calculated using (3.2) (A) and using (2.3)–(2.5) with $N = 10$ (B). As is seen, the maximum difference between the analytic and calculated frequencies is less than 0.3%, which is indicative of good accuracy of the spline-collocation method.



$r_x = 6.26000$ $r_x = 1.60250$ $r_x = 0.86125$

Fig. 5

TABLE 2

BC	i	$\overline{\omega}_i = \omega_i \alpha \sqrt{\rho(1-\nu^2)/E}$										
		α										
		–0.5	–0.4	–0.3	–0.2	–0.1	0	0.1	0.2	0.3	0.4	0.5
1	1	0.1493	0.1509	0.1524	0.1537	0.1550	0.1562	0.1574	0.1585	0.1597	0.1608	0.1619
	2	0.1664	0.1702	0.1735	0.1764	0.1790	0.1811	0.1830	0.1846	0.1858	0.1867	0.1874
	3	0.2270	0.2258	0.2244	0.2226	0.2208	0.2188	0.2167	0.2146	0.2124	0.2102	0.2079
	4	0.2626	0.2654	0.2675	0.2689	0.2696	0.2699	0.2696	0.2688	0.2676	0.2660	0.2640
2	1	0.1411	0.1432	0.1452	0.1470	0.1487	0.1504	0.1520	0.1535	0.1550	0.1566	0.1580
	2	0.1590	0.1631	0.1668	0.1699	0.1727	0.1752	0.1773	0.1791	0.1806	0.1818	0.1827
	3	0.2039	0.2037	0.2032	0.2025	0.2017	0.2008	0.1998	0.1987	0.1977	0.1965	0.1954
	4	0.2440	0.2472	0.2496	0.2514	0.2525	0.2532	0.2534	0.2532	0.2526	0.2516	0.2501
3	1	0.1358	0.1383	0.1406	0.1427	0.1447	0.1466	0.1484	0.1503	0.1520	0.1537	0.1554
	2	0.1530	0.1574	0.1612	0.1646	0.1676	0.1704	0.1727	0.1747	0.1764	0.1778	0.1790
	3	0.1849	0.1855	0.1859	0.1861	0.1862	0.1862	0.1861	0.1859	0.1857	0.1855	0.1853
	4	0.2286	0.2320	0.2347	0.2367	0.2383	0.2393	0.2400	0.2402	0.2400	0.2395	0.2386
4	1	0.1058	0.1075	0.1088	0.1101	0.1112	0.1122	0.1132	0.1141	0.1149	0.1158	0.1166
	2	0.1464	0.1488	0.1508	0.1525	0.1539	0.1550	0.1560	0.1567	0.1572	0.1576	0.1579
	3	0.1901	0.1903	0.1903	0.1900	0.1896	0.1890	0.1885	0.1878	0.1872	0.1865	0.1858
	4	0.2284	0.2304	0.2318	0.2328	0.2333	0.2334	0.2333	0.2328	0.2320	0.2310	0.2296

TABLE 3

BC	i	$\overline{\omega}_i = \omega_i \alpha \sqrt{\rho(1-\nu^2)/E}$										
		α										
		–0.5	–0.4	–0.3	–0.2	–0.1	0	0.1	0.2	0.3	0.4	0.5
1	1	0.2875	0.2930	0.2962	0.2948	0.2932	0.2913	0.2893	0.2871	0.2847	0.2822	0.2794
	2	0.2980	0.2973	0.2977	0.3017	0.3051	0.3079	0.3102	0.3119	0.3131	0.3137	0.3138
	3	0.4388	0.4399	0.4405	0.4404	0.4398	0.4388	0.4372	0.4352	0.4323	0.4274	0.4220
	4	0.4533	0.4530	0.4518	0.4499	0.4474	0.4444	0.4408	0.4367	0.4327	0.4297	0.4263
2	1	0.2748	0.2809	0.2840	0.2830	0.2819	0.2805	0.2789	0.2772	0.2752	0.2731	0.2708
	2	0.2850	0.2847	0.2863	0.2909	0.2947	0.2980	0.3006	0.3027	0.3042	0.3052	0.3055
	3	0.4199	0.4218	0.4229	0.4235	0.4235	0.4231	0.4221	0.4207	0.4187	0.4156	0.4107
	4	0.4368	0.4372	0.4365	0.4352	0.4332	0.4306	0.4276	0.4240	0.4200	0.4163	0.4133
3	1	0.2640	0.2708	0.2725	0.2720	0.2713	0.2703	0.2692	0.2679	0.2664	0.2647	0.2627
	2	0.2727	0.2728	0.2766	0.2817	0.2860	0.2896	0.2926	0.2950	0.2968	0.2980	0.2986
	3	0.4023	0.4047	0.4064	0.4076	0.4082	0.4083	0.4079	0.4070	0.4055	0.4035	0.4004
	4	0.4223	0.4231	0.4230	0.4220	0.4204	0.4182	0.4156	0.4125	0.4089	0.4049	0.4010
4	1	0.2027	0.2050	0.2067	0.2079	0.2087	0.2090	0.2091	0.2089	0.2084	0.2077	0.2067
	2	0.2610	0.2653	0.2689	0.2718	0.2742	0.2760	0.2774	0.2782	0.2786	0.2785	0.2779
	3	0.3548	0.3598	0.3634	0.3661	0.3680	0.3692	0.3697	0.3697	0.3691	0.3680	0.3664
	4	0.3798	0.3862	0.3913	0.3955	0.3990	0.4017	0.4039	0.4054	0.4063	0.4055	0.4025

Conclusions. From Tables 2–4 it follows that the natural frequencies of the shell decrease more quickly with decrease in the stiffness of fixation than with variation in the parameter α.

From Figs. 2–4 it follows that the frequencies of shells of varying thickness can either increase or decrease with increase in α, depending on the curvature of the mid-surface and the stiffness of fixation. According to (3.1), the surfaces bounding the shell are symmetric about the mid-surface, which is likely the cause of such a behavior of the natural frequencies. The more the curvature of the bounding surfaces differs from the curvature of the mid-surface, the more different the frequencies of shells with varying and constant thickness. The rate of variation in the natural frequency with variation in α changes too: it is less for low frequencies.

From Fig. 5 it also follows that the natural modes of the shell are strongly dependent on the curvature of the mid-surface: when $r_x = 6.2$, the vibration modes of the shell are similar to those of a plate with the same plan (the first frequency corresponds to one half-wave in each coordinate direction, the number of half waves increasing with curvature). Moreover, when the curvature of the mid-surface is small, the lowest frequencies are almost linearly dependent on the parameter α (Fig. 2). As the curvature increases, the function $\overline{\omega} = f(\alpha)$ becomes nonlinear, and after a certain value of the curvature, the parameter α no longer has a significant effect on the lowest frequencies (Figs. 3 and 4). At highest frequencies, the function $\overline{\omega} = f(\alpha)$ is always nonlinear.

According to Fig. 5, the maximum amplitudes shift toward the center of the shell's plan because, according to (3.1), the thickness and hence the stiffness of the shell is minimum at the points ($a/2$; y).

TABLE 4

BC	i	$\overline{\omega}_i = \omega_i \alpha \sqrt{\rho(1-\nu^2)/E}$										
		α										
		–0.5	–0.4	–0.3	–0.2	–0.1	0	0.1	0.2	0.3	0.4	0.5
1	1	0.3650	0.3674	0.3693	0.3708	0.3719	0.3726	0.3727	0.3725	0.3718	0.3706	0.3690
	2	0.4094	0.4133	0.4156	0.4166	0.4164	0.4152	0.4131	0.4103	0.4069	0.4027	0.3981
	3	0.5966	0.5987	0.5957	0.5923	0.5884	0.5841	0.5793	0.5742	0.5685	0.5623	0.5556
	4	0.6013	0.5993	0.6004	0.6003	0.5990	0.5967	0.5934	0.5891	0.5839	0.5778	0.5709
2	1	0.3395	0.3428	0.3456	0.3479	0.3496	0.3509	0.3516	0.3519	0.3517	0.3510	0.3497
	2	0.3897	0.3940	0.3965	0.3979	0.3980	0.3973	0.3957	0.3934	0.3905	0.3870	0.3829
	3	0.5729	0.5761	0.5747	0.5718	0.5683	0.5644	0.5601	0.5554	0.5500	0.5443	0.5379
	4	0.5792	0.5772	0.5777	0.5780	0.5772	0.5753	0.5724	0.5686	0.5638	0.5583	0.5518
3	1	0.3174	0.3216	0.3252	0.3281	0.3336	0.3323	0.3336	0.3343	0.3345	0.3341	0.3332
	2	0.3718	0.3763	0.3792	0.3807	0.3798	0.3809	0.3798	0.3780	0.3757	0.3727	0.3692
	3	0.5502	0.5541	0.5536	0.5511	0.5408	0.5447	0.5408	0.5364	0.5315	0.5261	0.5200
	4	0.5572	0.5557	0.5564	0.5575	0.5539	0.5561	0.5539	0.5507	0.5466	0.5416	0.5357
4	1	0.2763	0.2797	0.2822	0.2840	0.2858	0.2856	0.2858	0.2855	0.2848	0.2838	0.2822
	2	0.3530	0.3553	0.3569	0.3579	0.3578	0.3583	0.3578	0.3568	0.3554	0.3536	0.3513
	3	0.4337	0.4408	0.4465	0.4508	0.4571	0.4560	0.4571	0.4572	0.4564	0.4547	0.4521
	4	0.4833	0.4893	0.4937	0.4966	0.4993	0.4993	0.4993	0.4986	0.4971	0.4949	0.4920

TABLE 5

i	$\overline{\omega}_i = \omega_i \alpha \sqrt{\rho(1-\nu^2)/E}$								
	$r_x = 6.26000$			$r_x = 1.60250$			$r_x = 0.86125$		
	A	B	Π, %	A	B	Π, %	A	B	Π, %
1	0.0888	0.0888	0.00	0.1648	0.1649	0.04	0.2491	0.2492	0.05
2	0.1180	0.1180	0.00	0.2355	0.2356	0.03	0.2534	0.2535	0.02
3	0.1669	0.1670	0.07	0.3011	0.3012	0.02	0.3928	0.3930	0.02
4	0.1976	0.1978	0.09	0.3483	0.3486	0.07	0.4516	0.4528	0.27

The technique presented here allows us to determine the natural frequencies and vibration modes over a wide range of mechanical and geometrical parameters of shallow shells, which is necessary to ensure the required load-bearing capacity of structural members.

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