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PHYSICALLY AND GEOMETRICALLY NONLINEAR DEFORMATION OF THIN-WALLED CONICAL SHELLS WITH A CURVILINEAR HOLE

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The elastoplastic state of thin conical shells with a curvilinear (circular) hole is analyzed assuming finite deflections. The distribution of stresses, strains, and displacements along the hole boundary and in the zone of their concentration are studied. The stress–strain state around a circular hole in shells subject to internal pressure of prescribed intensity is analyzed taking into account two nonlinear factors

Keywords: nonlinear theory of shells, stress concentration, conical shell, curvilinear (circular) hole, internal pressure, stress, strain, displacement

Introduction. The stress–strain state of simply connected conical shells made of metals or composites and having curvilinear and rectangular holes was analyzed only for the elastic stage of their deformation [1, 2, 6–9, etc.].

Some experimental data and a review on dynamic problems for thin-walled shells and plates with holes can be found in [4]. Experimental data for multiply connected conical shells are given in [6, 16] with reference to static problems of elasticity. Theoretical results are also reported in the paper [3], which studies the frequencies and circular modes of conical shells with a circular hole that is not loaded and not reinforced.

In [5], a perforated conical shell under axial compression was analyzed for stability. Axisymmetric buckling was studied by partitioning the shell into longitudinal strips regarded as compressed rods on an elastic foundation. The critical loads for a shell with (n) square holes in its middle part were determined. Numerical results were presented for a shell with four holes.

Note that the elastoplastic state of conical shells with curvilinear holes was analyzed in just a few studies [6, 7]. Therefore, it is of importance to solve static and dynamic problems for conical shells with both physical and geometrical nonlinearities. Note also that a generalized formulation of physically and geometrically nonlinear problems for arbitrary thin isotropic shells with one or several curvilinear holes was given in the paper [10], which also presents governing equations and a numerical method for solving static boundary-value problems for thin shells with allowance for several nonlinear factors (elastoplastic strains and finite deflections). Numerical results for spherical and cylindrical shells with a curvilinear hole under uniform pressure of prescribed intensity are used in [11–13] to analyze the distribution of stresses (strains, displacements) in shells and the effect of several nonlinearities on their stress–strain state.

Expanding upon [10–15], we will discuss specific results on the elastoplastic stress–strain state of thin-walled flexible conical shells with a circular hole under distributed surface and edge loads.

1. Let us analyze the elastoplastic state of a flexible conical shell with a curvilinear (circular, elliptic) hole in the side wall [6, 10]. The shell is thin-walled, deep, made of an isotropic homogeneous material with known mechanical characteristics, and subjected to surface and edge static loads.

We assume that large loads cause large deflections in the shell (along the normal to its surface) and elastoplastic deformation of its material [6].

To derive the governing equations, we will use, in the general case, the geometrical relations of the second-order theory of thin shells [6, 10] and the physical relations of the theory of flow with isotropic hardening (combined loading). In the case of simple loading, nonlinear physical relations are usually taken from the theory of small elastoplastic strains.

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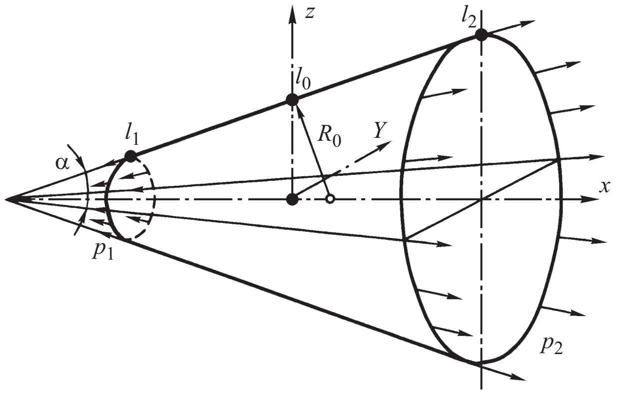

Fig. 1

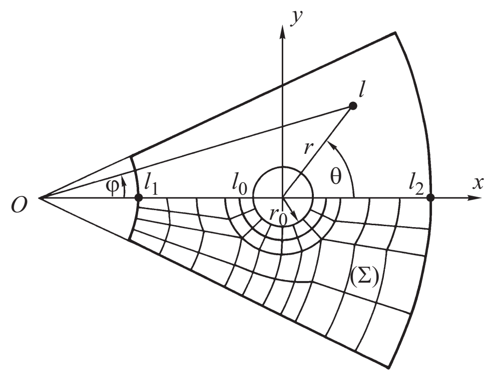

Fig. 2

Consider a nonlinear elastic thin-walled conical shell with a curvilinear hole of radius r_0 (Fig. 1). Introduce three coordinate systems on its mid-surface: polar (r, θ) and Cartesian (x, y) frames with the origins at the center of the hole and a polar frame (l, φ) with the origin at the vertex of the cone and axes coinciding with the lines of principal curvatures of the shell. These coordinate systems are related by

$$x = r\cos\theta = l\cos\varphi - l_0,$$

$$y = r\sin\theta = l\sin\varphi, \tag{1}$$

where l_0 is the distance from the vertex of the cone to the center of the hole.

The geometry of the mid-surface of the shell is described in a global Cartesian coordinate system (X, Y, Z) using the following equalities (Fig. 2):

$$X = (l - l_0)\cos\alpha, \quad Y = l\sin\alpha\sin\beta, \quad Z = l\sin\alpha\cos\beta, \tag{2}$$

where 2α is the cone angle and $\beta = \varphi / \sin\alpha$.

The curvatures and torsion of the shell are defined by

$$k_l = 0, \quad k_\varphi = l_0 / R_0 l, \quad k_{l\varphi} = 0, \quad k_x = k_\varphi \sin^2\varphi, \quad k_y = k_\varphi \cos^2\varphi, \quad k_{xy} = -k_\varphi \sin\varphi\cos\varphi,$$

$$k_r = \frac{\sin^2\theta}{R_0}\frac{l_0^3}{l^3}, \quad k_\theta = \frac{l_0}{lR_0} - \frac{l_0^3}{l^3}\frac{\sin^2\theta}{R_0}, \quad k_{r\theta} = -\frac{\sin 2\beta}{R}, \quad \sin\beta = \frac{l_0\sin\theta}{l}, \quad R = \frac{R_0 l}{l_0}, \tag{3}$$

where R_0 is the radius of curvature of the normal section coming through the center of the hole. Using Eqs. (1)–(3), we can write nonlinear geometrical and physical relations in the coordinate system (r, θ, z):

the geometrical equations

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} + k_r w + \frac{1}{2}\Theta_r^2, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v}{\partial\theta} + u\right) + k_\theta w + \frac{1}{2}\Theta_\theta^2,$$

$$\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) - 2k_{r\theta} w + \Theta_r\Theta_\theta, \quad \kappa_{rr} = \frac{\partial\Theta_r}{\partial r},$$

$$\kappa_{\theta\theta} = \frac{1}{r}\left(\Theta_r + \frac{\partial}{\partial\theta}\Theta_\theta\right), \quad 2\kappa_{r\theta} = \frac{1}{r}\left(\frac{\partial}{\partial\theta}\Theta_r - \Theta_\theta\right) + \frac{\partial}{\partial r}\Theta_\theta, \tag{4}$$

where $\Theta_r = -\frac{\partial w}{\partial r} + k_r u - k_{r\theta} v$ and $\Theta_\theta = -\frac{1}{r}\frac{\partial w}{\partial\theta} - k_{r\theta} u + k_\theta v$, the physical equations

$$\sigma_{rr} = \frac{2G}{1-\nu}(e_{rr} + \nu e_{\theta\theta}) + \sigma^*_{rr} \ (r \leftrightarrow \theta), \qquad \sigma_{r\theta} = G e_{r\theta} + \sigma^*_{r\theta}, \tag{5}$$

where σ^*_{rr}, $\sigma^*_{\theta\theta}$, and $\sigma^*_{r\theta}$ are nonlinear terms, which are defined as follows in the case of small elastoplastic strains [8]:

$$\sigma^*_{rr} = -\frac{2}{3}\frac{G}{1-\nu}\omega_i[(2-\nu)e_{rr} - (1-2\nu)e_{\theta\theta} - (1+\nu)e_{zz}] \quad (r \leftrightarrow \theta),$$

$$\sigma^*_{r\theta} = -G\omega_i e_{r\theta}, \tag{6}$$

where

$$e_{rr} = \varepsilon_{rr} + z\kappa_{rr} \quad (r \leftrightarrow \theta), \qquad e_{r\theta} = \varepsilon_{r\theta} + 2z\kappa_{r\theta}, \qquad e_{zz} = -\mu_i(e_{rr} + e_{\theta\theta}),$$

$$\omega_i = 1 - \frac{\sigma_i}{3Ge_i}, \qquad \mu_i = -\frac{3\nu + (1-2\nu)\omega_i}{3(1-\nu) - 2(1-2\nu)\omega_i},$$

where ω_i is the yield function; σ_i and e_i are the stress and strain intensities [6, 10]; G and ν are the shear modulus and Poisson's ratio; and ($\leftrightarrow$) indicates that the missing formulas can be obtained by cyclic permutation of the indices.

The expressions for internal forces and moments are derived from Eqs. (6) separating linear and nonlinear parts [6, 10].

2. The nonlinear governing equations describing the elastoplastic state of conical shells with a curvilinear hole and finite deflections are obtained on the basis of the virtual-displacement principle. An approximate method for solving doubly nonlinear problems is based on the procedure of step-by-step loading and iterative methods (Newton–Kantorovich, additional stresses) and a numerical method (finite-element) [10] applied in succession.

Using relations (3)–(6) to derive the variational equation and linearizing and discretizing it, we arrive (applying the stationarity conditions for the linearized total energy of the shell) at a system of algebraic equations, which has the following matrix form at the nth step of loading:

$$[S]\{q\} = \{\Delta V\}, \tag{7}$$

where $[S] = [S_0] + [S_\alpha] + [S_\sigma]$, $[S_0]$ is the stiffness matrix for linear elastic shells, $[S_\alpha]$ and $[S_\sigma]$ are the influence matrices of the initial angles of rotation Θ_r and Θ_0 and stresses; $\{q\}$ is the column vector of nodal displacements; $\{\Delta V\} = \{\Delta R\} - \{\Delta N\} + \{\Delta\Phi\}$, $\{\Delta R\}$, $\{\Delta N\}$, and $\{\Delta\Phi\}$ are the vectors of loads, nonlinearities, and residuals of the equilibrium equations at the end of the (n – 1)th step of loading.

Note that for conical shells with a circular hole under a surface load (internal pressure of intensity $p = p_0 \cdot 10^5$ Pa), we assume that only shearing forces $Q_r = pr_0 / 2$ act on the boundary of the hole and that the stress state is momentless at a sufficiently large distance from the hole. In this case, we have the following conditions:

at the edges $l = l_1$ ($l = l_2$):

$$T_k = \frac{l_i R_0}{2l_0} p, \qquad S_k = 0, \qquad Q_k = 0, \qquad M_k = 0, \tag{8}$$

at the edge $\varphi = \varphi_k$:

$$T_k = \frac{lR_0}{l_0} p, \qquad S_k = 0, \qquad M_k = 0, \qquad w = \frac{R_0^2}{2Eh} p \left(1.5\frac{l^2}{l_0^2} + 0.5 - \nu\right). \tag{9}$$

To solve specific problems for the shell under consideration, Eqs. (7) should be supplemented with the appropriate boundary conditions [6], which can be written for displacements, forces, or in a mixed form.

TABLE 1

θ, deg	w^*			
	LP	PNP	GNP	PGNP
0	0.661	1.692	0.834	1.256
45	1.785	5.167	1.141	1.600
81	2.939	7.180	1.580	2.011
90	3.014	7.211	1.630	2.047
99	2.964	7.047	1.626	2.031
135	1.691	4.564	1.137	1.511
180	0.227	0.335	0.640	0.892

TABLE 2

θ, deg	ξ	σ_θ^*			
		LP	PNP	GNP	PGNP
0	0.5 –0.5	3162 5591	1847 2203	3576 3611	1861 1836
45	0.5 –0.5	3973 2834	2032 1330	3058 2492	1752 1667
90	0.5 –0.5	2936 –2915	1831 –1866	2140 –728	1544 –760
135	0.5 –0.5	3396 1495	1833 –334	2596 1716	1685 1400
180	0.5 –0.5	2376 6186	1443 2271	3330 3970	1825 1884

3. We have analyzed the elastoplastic state of a flexible conical shell with a circular hole using our method, associated algorithm, and its implementing software. The following geometrical parameters have been used: $\kappa = r_0 / \sqrt{Rh} = 1.5$, $l_0 = 6r_0$, $\alpha = 66°$.

The shell is made of an AMg-6 material with $\sigma_n = 140$ MPa, $\varepsilon_n = 0.002$, $E = 70$ GPa, $\nu = 0.3$–0.5 [6]. Its stress–strain curve has a smooth transition to the yield plateau.

In solving linear elastic and nonlinear problems with the use of Eqs. (7), it was assumed that conditions (8) and (9) are satisfied.

We have obtained numerical values (see the tables below) of displacements (u, v, w), strains $(e_{ij}, \varepsilon_{ij})$, and stresses $(\sigma_{ij},\ i, j = r, \theta)$ at the nodal points of the domain Σ (Fig. 1) and at three points ($\xi = z / h = -0.5$; 0; 0.5) throughout the thickness (h = const) for pressure $p_0 = 2$. The process of loading was divided into 30 steps to solve the geometrically nonlinear (GNP) and physically and geometrically nonlinear (PGNP) problems and into 10 steps to solve the physically nonlinear problem (PNP).

TABLE 3

θ, deg	ξ	$-e_r \cdot 10^2$			
		LP	PNP	GNP	PGNP
0	0.5 –0.5	0.1459 0.2411	0.7011 1.0910	0.1598 0.1575	0.4185 0.3720
45	0.5 –0.5	0.1764 0.1214	0.8785 0.2284	0.1368 0.1094	0.2837 0.1947
81	0.5 –0.5	0.1355 –0.0898	0.4016 0.3026	0.1004 –0.0029	0.1381 –0.0125
90	0.5 –0.5	0.1216 –0.1215	0.3137 –0.3592	0.0955 –0.0259	0.1255 –0.0327
99	0.5 –0.5	0.1137 –0.1301	0.2732 –0.3649	0.0926 –0.0344	0.1183 –0.0402
135	0.5 –0.5	0.1400 0.0670	0.6820 –0.0934	0.1135 0.0769	0.2127 0.0712
180	0.5 –0.5	0.1165 0.2607	0.5028 1.1170	0.1498 0.1692	0.3922 0.3979

TABLE 4

θ, deg	ξ	$e_\theta \cdot 10^2$			
		LP	PNP	GNP	PGNP
0	0.5 –0.5	0.4548 0.7991	1.3520 2.5800	0.5128 0.5167	0.9556 0.8714
45	0.5 –0.5	0.5695 0.4040	2.0230 0.5106	0.4385 0.3668	0.6714 0.4862
81	0.5 –0.5	0.4585 –0.3081	0.9431 –0.7115	0.3205 –0.0231	0.3657 –0.3161
90	0.5 –0.5	0.4182 –0.4154	0.7853 –0.8989	0.3068 –0.1325	0.3438 –0.1086
99	0.5 –0.5	0.3968 –0.4451	0.7070 –0.9360	0.3008 –0.1325	0.3331 –0.1394
135	0.5 –0.5	0.4835 0.2145	1.5020 –0.0817	0.3715 0.2462	0.5240 0.2256
180	0.5 –0.5	0.3438 0.8823	0.7470 2.7940	0.4778 0.5669	0.8789 0.9641

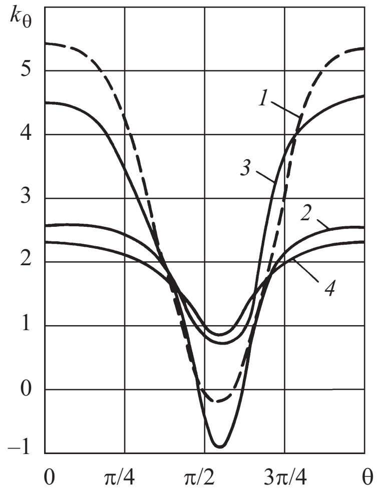

Fig. 3

For reasons of geometrical and force symmetry, the problems were solved in the domain Σ (quarter the shell). It was partitioned into 378 finite elements (FEs), and the boundary of the hole into 20 FEs.

Some of the numerical data obtained are presented in the figures and tables above.

Tables 1–4 show the distribution of deflections ($w^* = w/h$), maximum stresses ($\sigma_{\theta\theta} = \sigma^*_{\theta\theta} \cdot 10^5$ Pa) and strains ($e_{rr}, e_{\theta\theta}$) along the hole boundary ($0 \le \theta \le \pi$) on different surfaces of the shell ($\xi = \pm 0.5$). These data have been obtained by solving the linear problem (LP), the geometrically nonlinear problem (GNP), the physically nonlinear problem (PNP), and the physically and geometrically nonlinear problem (PGNP).

Figure 3 shows how the maximum hoop stress concentration factors ($K_\theta = \sigma_{\theta\theta} h / pR_0$) vary along the boundary of the hole (curves *1*, *2*, *3*, and *4* represent the solutions of the LP, PNP, GNP, and PGNP, respectively).

An analysis of the results indicates that when a conical shell is subjected to internal pressure of prescribed intensity (p = const), the maximum deflections occur at the point $r = r_0$, $\theta = 90°$, and the maximum strains ($e_{\theta\theta}$) and stresses ($\sigma_{\theta\theta}$) at the point $r = r_0$, $\theta = 180°$ of the hole boundary, on the inside surface ($\xi = -0.5$).

Nonlinear factors have a significant effect on the stress and strain fields in the concentration areas. For example, considering that the shell deforms beyond the elastic limit (PNP) decreases the maximum stresses by 63% and the maximum deflections and strains by 139 and 217%, respectively, compared with the LP. Additionally allowing for finite deflections (PGNP) at the elastoplastic stage of deformation decreases the maximum stresses, deflections, and strains by 17, 71, and 65%, respectively, compared with the PNP and by 70, 32, and 9% compared with the LP.

In summary, it should be pointed out that the method proposed here to solve nonlinear problems was also used to analyze the stress concentration around circular and elliptic holes in a conical shell subjected to axial forces.

REFERENCES

1. A. N. Guz, P. Z. Lugovoi, and N. A. Shul'ga, *Conical Shells with Holes* [in Russian], Naukova Dumka, Kyiv (1976).
2. A. N. Guz, A. S. Kosmodamianski, V. P. Shevchenko, et al., *Stress Concentration*, Vol. 7 of the 12-volume series *Mechanics of Composite Materials* [in Russian], A.S.K., Kyiv (1998).
3. A. A. Malinin, "Vibrations of shells of revolution with holes," *Izv. VUZov*, *Mashinostr*., No. 7, 22–27 (1971).
4. B. N. Nemirovskii and I. N. Preobrazhenskii, "Dynamic problems in the theory of plates and shells with holes," in: *Design of Buildings and Other Structures* [in Russian], Nauka, Moscow (1973), pp. 61–71.
5. I. N. Preobrazhenskii, "Stability of a truncated conical shell with holes under axial compression," *Gidromekh. Teor. Uprug*., **18**, 128–135 (1974).

6. A. N. Guz, I. S. Chernyshenko, V. N. Chekhov, et al., *Theory of Thin Shells Weakened by Holes*, Vol. 1 of the five-volume series *Methods of Shell Design* [in Russian], Naukova Dumka, Kyiv (1980).
7. E. N. Troyak, E. A. Storozhuk, and I. S. Chernyshenko, “Elastoplastic state of a conical shell with a circular hole on the lateral surface,” *Int. Appl. Mech*., **24**, No. 1, 65–70 (1988).
8. I. S. Chernyshenko and E. A. Storozhuk, “Inelastic deformation of flexible cylindrical shells with a curvilinear hole,” *Int. Appl. Mech*., **42**, No. 12, 1414–1420 (2006).
9. A. N. Guz, I. S. Chernyshenko, and K. I. Shnerenko, “Stress concentration near openings in composite shells,” *Int. Appl. Mech*., **37**, No. 2, 139–181 (2001).
10. A. N. Guz, E. A. Storozhuk, and I. S. Chernyshenko, “Physically and geometrically nonlinear static problems for thin-walled multiply connected shells,” *Int. Appl. Mech*., **39**, No. 6, 679–687 (2003).
11. A. N. Guz, E. A. Storozhuk, and I. S. Chernyshenko, “Inelastic deformation of flexible spherical shells with two circular openings,” *Int. Appl. Mech*., **40**, No. 6, 672–678 (2004).
12. A. N. Guz, E. A. Storozhuk, and I. S. Chernyshenko, “Elastoplastic state of flexible cylindrical shells with two circular holes,” *Int. Appl. Mech*., **40**, No. 10, 1152–1156 (2004).
13. E. A. Storozhuk and I. S. Chernyshenko, “Elastoplastic deformation of flexible cylindrical shells with two circular holes under axial tension,” *Int. Appl. Mech*., **41**, No. 5, 506–511 (2005).
14. E. A. Storozhuk and I. S. Chernyshenko, “Stress distribution in physically and geometrically nonlinear thin cylindrical shells with two holes,” *Int. Appl. Mech*., **41**, No. 11, 1280–1287 (2005).
15. E. A. Storozhuk and I. S. Chernyshenko, “Physically and geometrically nonlinear deformation of spherical shells with an elliptic hole,” *Int. Appl. Mech*., **41**, No. 6, 666–674 (2005).
16. A. I. Zirka and D. I. Chernopiskii, “On the stress state of a truncated conic shell of medium thickness with square holes,” *Int. Appl. Mech*., **38**, No. 12, 1528–1531 (2002).