International Applied Mechanics, Vol. 43, No. 3, 2007

THE AXISYMMETRIC LAMB'S PROBLEM FOR A FINITE PRESTRAINED HALF-SPACE COVERED WITH A FINITE PRESTRETCHED LAYER

S. D. Akbarov UDK 539.3

The piecewise-homogeneous body model and the three-dimensional linearized theory of elastic waves in prestressed bodies are used to solve the axisymmetric time-harmonic Lamb's problem for a finite prestrained half-space covered with a finite prestretched layer. It is assumed that the half-space and layer are incompressible and their deformation is described by the Treloar potential. The normal stress at the interface is calculated

Keywords: Lamb's problem, finite initial strains, half-space, Treloar potential, time-harmonic stress field

1. Introduction. Lamb's problem, a fundamental problem in the analysis of elastic waves generated by external forces in solid bodies bounded by plane surfaces, has been studied extensively since Lamb [18]. These studies are referred to in many papers such as [22, 24, etc.]. However, such investigations have been carried out within the framework of the classical linear theory of elastic waves for homogeneous, isotropic or anisotropic half-spaces. Interesting and urgent elastodynamic problems, which cannot be solved by the classical linear theory of elastic waves, are elastodynamic problems for initially stressed bodies. Up to now, a great number of theoretical and experimental investigations have been carried out in this field. Systematic consideration and analysis of relevant results obtained before 1986 can be found in the monographs [10, 11]. Recent studies are reviewed in the papers [1, 4–6, 13, 16, 19, 21, etc.]. It follows from these reviews that almost all these investigations were performed within the framework of the three-dimensional linearized theory of elastic waves in initially stressed bodies (TLTEWISB) and many of these studies refer to waves propagating in layered composite materials with homogeneous initial stresses.

It is evident that studies of the influence of initial strains (or stresses) on the dynamic stress field in homogeneous and layered materials are also of great theoretical and practical importance. Of fundamental significance for the analysis of such problems is the solution of Lamb's problem for a prestrained homogenous and layered (inhomogeneous) half-space. However, the number of investigations in this field is not enough. Here we briefly consider those of them that are related to the subject of the present paper.

Lamb's problem for a homogeneously prestrained half-plane and half-space was investigated in [15, 17]. Lamb's problem for a half-plane covered by a prestretched layer under a harmonic linear load perpendicular to the free layer surface was studied in [2, 3, 9]. Moreover, an attempt was made in [7] to investigate the three-dimensional Lamb's problem for a half space covered by a biaxially prestretched layer. Note that in these investigations, the initial strain state was determined using the classical linear theory of elasticity. But the perturbed state caused by normal linear or concentrated harmonic forces is determined by the TLTEWISB. Moreover, in [2, 3, 7, 9], the interfacial stresses were calculated for some values of the frequency of the external force and the influence of the initial stretching of the layer on the stress amplitude was analyzed.

To investigate the influence of initial strains on the interfacial stresses, it is reasonable to consider finite initial strains. Moreover, the assumption of small initial strains is not generally applicable to rubber-like materials. For these and many other reasons, the present paper develops the above-cited investigations for a system consisting of an initially finite prestretched layer and an initially finite prestrained (prestretched or precompressed) half-space. It is assumed that a harmonically varying normal

Yildiz Technical University, Istanbul, Turkey. Published in Prikladnaya Mekhanika, Vol. 43, No. 3, pp. 132–143, March 2007. Original article submitted February 3, 2005.

concentrated force is applied to the free surface of the layer and the stress state is axisymmetric. We also assume that the layer and half-space are incompressible and finite prestrained radially and axisymmetricaly and that their stress–strain relations are expressed in terms of the Treloar potential. The investigations are carried out using the piecewise-homogeneous body model and the TLTEWISB. The stresses at the interface are calculated. Emphasis is on the dependences between the stresses and the frequency of the external forces and on the influence of the initial strains on these dependences.

2. Formulation of the Problem. Let us consider a half-space covered with a layer of thickness h. We use a Cartesian coordinate system $Oy_1y_2y_3$ and a Lagrangian cylindrical coordinate system $Or\theta y_3$ to describe the layer and half-space in the natural state. Before coupling, the layer and half-space are each prestrained in the radial direction, which results in a homogeneous axisymmetric initial finite strain state. Let the initial state of the layer and half-space be associated with the Lagrangian cylindrical coordinate system $O'r'\theta'y'_3$ and the Cartesian coordinate system $O'y'_1y'_2y'_3$. Assume that the layer and half-space are incompressible. We use the superscripts "(1)" and "(2)" to refer to the layer and half-space, respectively, and the superscript "0" to refer to the initial state. Thus, the initial state in the layer and half-space can be expressed as follows:

$$u_{\underline{m}}^{(k),0} = (\lambda_{\underline{m}}^{(k)} - 1)y_{\underline{m}}, \quad \lambda_1^{(k)} = \lambda_2^{(k)} \neq \lambda_3^{(k)}, \quad \lambda_m^{(k)} = \text{const}, \quad \lambda_1^{(\underline{k})}\lambda_2^{(\underline{k})}\lambda_3^{(\underline{k})} = 1,$$

$$m = 1, 2, 3, \quad k = 1, 2, \tag{1}$$

where $u_m^{(k),0}$ is the displacement and $\lambda_m^{(k)}$ is the elongation along the Oy_m-axis.

We introduce the following notation:

$$\lambda_1^{(k)} = \lambda_2^{(k)} = \lambda^{(k)}, \quad \lambda_3^{(k)} = (\lambda^{(k)})^{-2}. \tag{2}$$

Let us now investigate the stress state of the system subject to a normal harmonic concentrated force on the free surface of the layer. We will use the coordinates r' and y'_3 within the framework of the TLTEWISB.

It follows from (1) that

$$y'_i = \lambda_{\underline{i}}^{(k)} y_i, \quad h' = (\lambda^{(1)})^{-2} h. \tag{3}$$

Below, the prime refers to the system of coordinates associated with the initial state, i.e., with $O'y'_1y'_2y'_3$.

Thus, according to [10, 11, 14], we write the basic relations of the TLTEWISB for an incompressible body in an axisymmetrical state. These relations are satisfied within the layer and half-space separately because we use the piecewise-homogeneous body model.

The equations of motion are

$$\frac{\partial}{\partial r'} Q'^{(k)}_{r'r'} + \frac{\partial}{\partial y'_3} Q'^{(k)}_{r'3} + \frac{1}{r'}\left(Q'^{(k)}_{r'r'} - Q'^{(k)}_{\theta'\theta'}\right) = \rho'^{(\underline{k})} \frac{\partial^2}{\partial t^2} u'^{(\underline{k})}_{r'},$$

$$\frac{\partial}{\partial r'} Q'^{(k)}_{3r'} + \frac{\partial}{\partial y'_3} Q'^{(k)}_{33} + \frac{1}{r'} Q'^{(k)}_{3r'} = \rho'^{(\underline{k})} \frac{\partial^2}{\partial t^2} u'^{(\underline{k})}_{3}. \tag{4}$$

The mechanical relations are

$$Q'^{(k)}_{r'r'} = \chi'^{(\underline{k})}_{1111} \frac{\partial u'^{(\underline{k})}_{r'}}{\partial r'} + \chi'^{(\underline{k})}_{1122} \frac{u'^{(\underline{k})}_{r'}}{r'} + \chi'^{(\underline{k})}_{1133} \frac{\partial u'^{(\underline{k})}_{3}}{\partial y'_3} + p'^{(k)},$$

$$Q'^{(k)}_{\theta'\theta'} = \chi'^{(\underline{k})}_{2211} \frac{\partial u'^{(\underline{k})}_{r'}}{\partial r'} + \chi'^{(\underline{k})}_{2222} \frac{u'^{(\underline{k})}_{r'}}{r'} + \chi'^{(\underline{k})}_{2233} \frac{\partial u'^{(\underline{k})}_{3}}{\partial y'_3} + p'^{(k)},$$

$$Q'^{(k)}_{33} = \chi'^{(\underline{k})}_{3311} \frac{\partial u'^{(\underline{k})}_{r'}}{\partial r'} + \chi'^{(\underline{k})}_{3322} \frac{u'^{(\underline{k})}_{r'}}{r'} + \chi'^{(\underline{k})}_{3333} \frac{\partial u'^{(\underline{k})}_{3}}{\partial y'_3} + p'^{(k)},$$

$$Q'^{(k)}_{r'3} = \chi'^{(\underline{k})}_{1313} \frac{\partial u'^{(\underline{k})}_{r'}}{\partial y'_3} + \chi'^{(\underline{k})}_{1331} \frac{\partial u'^{(\underline{k})}_{3}}{\partial r'} ,$$

$$Q'^{(k)}_{3r'} = \chi'^{(\underline{k})}_{3113} \frac{\partial u'^{(\underline{k})}_{r'}}{\partial y'_3} + \chi'^{(\underline{k})}_{3131} \frac{\partial u'^{(\underline{k})}_{3}}{\partial r'} , \tag{5}$$

where the $Q'^{(k)}_{r'r'}$, …, $Q'^{(k)}_{3r'}$ are the perturbations of the components of the Kirchhoff stress tensor; $u'^{(k)}_{r'}$ and $u'^{(k)}_{3}$ are the perturbations of the displacement components; and $p'^{(k)} = p'^{(k)}(r', y'_3, t)$ is an unknown function (Lagrangian multiplier). The constants $\chi'^{(k)}_{1111}$, …, $\chi'^{(k)}_{3333}$ in (4) and (5) are determined in terms of the mechanical constants of the layer and half-space and the initial stress state, $\rho'^{(k)}$ is the density of the kth material. Note that the constants $\chi'^{(k)}_{1111}, \ldots, \chi'^{(k)}_{3333}, \rho'^{(k)}$ are expressed in terms of those in the coordinate system Ory_3 (denoted by $\chi^{(k)}_{1111}, \ldots, \chi^{(k)}_{3333}, \rho^{(k)}$) as follows:

$$\chi'^{(k)}_{1111} = (\lambda^{(\underline{k})})^2 \chi^{(\underline{k})}_{1111}, \quad \chi'^{(k)}_{1122} = (\lambda^{(\underline{k})})^2 \chi^{(\underline{k})}_{1122}, \quad \chi'^{(k)}_{1133} = (\lambda^{(\underline{k})})^{-1} \chi^{(\underline{k})}_{1133},$$

$$\chi'^{(k)}_{2222} = (\lambda^{(\underline{k})})^2 \chi^{(\underline{k})}_{2222}, \quad \chi'^{(k)}_{1221} = (\lambda^{(\underline{k})})^2 \chi^{(\underline{k})}_{1221}, \quad \chi'^{(k)}_{1313} = (\lambda^{(\underline{k})})^{-1} \chi^{(\underline{k})}_{1313},$$

$$\chi'^{(k)}_{1331} = (\lambda^{(\underline{k})})^2 \chi^{(\underline{k})}_{1331}, \quad \chi'^{(k)}_{3131} = \chi'^{(k)}_{1313}, \quad \chi'^{(k)}_{2211} = \chi'^{(\underline{k})}_{1122}, \quad \chi'^{(k)}_{2233} = \chi'^{(\underline{k})}_{1133},$$

$$\chi'^{(k)}_{3311} = \chi'^{(\underline{k})}_{1133} = \chi'^{(\underline{k})}_{3322} = \chi'^{(\underline{k})}_{2233}, \quad \chi'^{(k)}_{3113} = (\lambda^{(\underline{k})})^2 \chi^{(\underline{k})}_{3113}, \quad \chi'^{(k)}_{3333} = (\lambda^{(\underline{k})})^{-4} \chi^{(\underline{k})}_{3333},$$

$$\rho'^{(k)} = \rho^{(k)}. \tag{6}$$

We assume that the elasticity relations for the layer and half-space are expressed in terms of a Neo-Hookean-type (Treloar) potential. This potential is given by

$$\Phi = C_{10}(I_1 - 3), \quad I_1 = 3 + 2A_1, \quad A_1 = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{33}, \tag{7}$$

where C_{10} is an elastic constant; A_1 is the first algebraic invariant of Green's strain tensor; and ε_{rr}, $\varepsilon_{\theta\theta}$, and ε_{33} are the components of this tensor. In the axisymmetric case under consideration, the components of Green's strain tensor are expressed in terms of the displacement components as follows:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right), \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{2}\left(\frac{u_r}{r}\right)^2,$$

$$\varepsilon_{r3} = \frac{1}{2}\left(\frac{\partial u_3}{\partial r} + \frac{\partial u_r}{\partial y_3} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial y_3} + \frac{\partial u_3}{\partial r}\frac{\partial u_3}{\partial y_3}\right), \quad \varepsilon_{33} = \frac{\partial u_3}{\partial y_3} + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial y_3}\right)^2. \tag{8}$$

The components S_{ij} of the Lagrange stress tensor are determined as follows:

$$S_{rr} = \frac{\partial \Phi}{\partial \varepsilon_{rr}} + p g^*_{rr}, \quad S_{\theta\theta} = \frac{\partial \Phi}{\partial \varepsilon_{\theta\theta}} + p g^*_{\theta\theta}, \quad S_{33} = \frac{\partial \Phi}{\partial \varepsilon_{33}} + p g^*_{33}, \quad S_{r3} = \frac{\partial \Phi}{\partial \varepsilon_{r3}}, \quad S_{r3} = S_{3r},$$

$$g^*_{rr} = 1 + 2\frac{\partial u_r}{\partial r} + \left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_3}{\partial r}\right)^2, \quad g^*_{33} = 1 + 2\frac{\partial u_3}{\partial y_3} + \left(\frac{\partial u_3}{\partial y_3}\right)^2 + \left(\frac{\partial u_r}{\partial y_3}\right)^2,$$

$$g^*_{\theta\theta} = 1 + 2\frac{u_r}{r} + \frac{1}{r^2}\left(\frac{u_r}{r}\right)^2. \tag{9}$$

Note that expressions (7)–(9) are written in an arbitrary system of cylindrical coordinates associated with neither natural nor initial state of the layer and half-space.

In this case, the perturbation of the Kirchhoff stress tensor and the perturbation of the components of the Lagrangian stress tensor are related as follows:

$$Q'^{(k)}_{r'r'} = \lambda^{(k)} S^{(k)}_{r'r'} + S^{0(k)}_{rr} \frac{\partial u'^{(k)}_{r'}}{\partial r'}, \quad Q'^{(k)}_{\theta'\theta'} = \lambda^{(k)} S^{(k)}_{\theta'\theta'} + S^{0(k)}_{rr} \frac{u'^{(k)}_{r'}}{r'}, \quad Q'^{(k)}_{33} = (\lambda^{(k)})^{-2} S^{(k)}_{33},$$

$$Q'^{(k)}_{r'3} = \lambda^{(k)} S^{(k)}_{r'3} + S^{0(k)}_{rr} \frac{\partial u'^{(k)}_{3}}{\partial r'}, \quad Q'^{(k)}_{3r'} = (\lambda^{(k)})^{-2} S^{(k)}_{3r'}. \tag{10}$$

Linearizing Eq. (9) and taking (1), (2), and (10) into account, we obtain the following expressions for the constants $\chi^{(k)}_{1111}, \ldots, \chi^{(k)}_{3333}$ in (6):

$$\chi^{(k)}_{1111} = \chi^{(k)}_{2222} = 2C^{(k)}_{10} (\lambda^{(k)})^{-2} \left((\lambda^{(k)})^{2} + (\lambda^{(k)})^{-4} \right),$$

$$\chi^{(k)}_{1122} = \chi^{(k)}_{1133} = \chi^{(k)}_{2233} = \chi^{(k)}_{3311} = \chi^{(k)}_{2211} = \chi^{(k)}_{3322} = 0,$$

$$\chi^{(k)}_{1331} = 2C^{(k)}_{10}, \quad \chi^{(k)}_{1221} = 2C^{(k)}_{10}, \quad \chi^{(k)}_{3333} = 4C^{(k)}_{10},$$

$$\chi^{(k)}_{1313} = \chi^{(k)}_{3131} = 2C^{(k)}_{10} (\lambda^{(k)})^{-3}, \quad \chi^{(k)}_{3113} = 2C^{(k)}_{10}. \tag{11}$$

These equations should be supplemented with the incompressibility condition for the layer and half-space. This condition can be written as follows:

$$\frac{1}{\lambda^{(k)}} \left(\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \frac{u'^{(k)}_{r'}}{r'} \right) + (\lambda^{(k)})^{2} \frac{\partial u'^{(k)}_{3}}{\partial y'_3} = 0. \tag{12}$$

Thus, we use Eqs. (4)–(12) to analyze the stress state of the system under consideration. We assume that the following boundary and contact conditions are satisfied:

$$Q'^{(1)}_{33}\Big|_{y'_3=0} = -P_0 \delta(r') e^{i\omega t} \frac{1}{(\lambda^{(1)})^{2}}, \quad Q'^{(1)}_{3r'}\Big|_{y'_3=0} = 0,$$

$$Q'^{(1)}_{33}\Big|_{y'_3=-h/(\lambda^{(1)})^2} = Q'^{(2)}_{33}\Big|_{y'_3=-h/(\lambda^{(1)})^2}, \quad Q'^{(1)}_{3r'}\Big|_{y'_3=-h/(\lambda^{(1)})^2} = Q'^{(2)}_{3r'}\Big|_{y'_3=-h/(\lambda^{(1)})^2},$$

$$u'^{(1)}_{r'}\Big|_{y'_3=-h/(\lambda^{(1)})^2} = u'^{(2)}_{r'}\Big|_{y'_3=-h/(\lambda^{(1)})^2}, \quad u'^{(1)}_{3}\Big|_{y'_3=-h/(\lambda^{(1)})^2} = u'^{(2)}_{3}\Big|_{y'_3=-h/(\lambda^{(1)})^2},$$

$$\left\{ \left|Q'^{(2)}_{33}\right|, \left|Q'^{(2)}_{3r'}\right|, \left|Q'^{(2)}_{r'r'}\right|, \left|Q'^{(2)}_{\theta'\theta'}\right|, \left|Q'^{(2)}_{r'3}\right|, \left|u'^{(2)}_{3}\right|, \left|u'^{(2)}_{r'}\right| \right\} < M = \text{const} \quad \text{as} \quad y'_3 \to -\infty. \tag{13}$$

It should be noted that if $\lambda^{(k)} = 1 (k = 1, 2)$, then Eqs. (4)–(6), (10)–(12) and conditions (13) transform into those in the classical linear theory of elasticity for incompressible body.

3. Solution Method. Substituting (5) into (4), we obtain the following equations of motion in terms of displacements:

$$\chi'^{(k)}_{1111} \frac{\partial^2 u'^{(k)}_{r'}}{\partial r'^2} + \chi'^{(k)}_{1122} \frac{\partial}{\partial r'} \left(\frac{u'^{(k)}_{r'}}{r'} \right) + \left(\chi'^{(k)}_{1133} + \chi'^{(k)}_{1331} \right) \frac{\partial^2 u'^{(k)}_{3}}{\partial r' \partial y'_3} + \chi'^{(k)}_{1313} \frac{\partial^2 u'^{(k)}_{r'}}{\partial y'^2_3}$$

$$+ \frac{1}{r'} \left(\chi'^{(k)}_{1111} - \chi'^{(k)}_{2211} \right) \frac{\partial u'^{(k)}_{r'}}{\partial r'} + \left(\chi'^{(k)}_{1122} - \chi'^{(k)}_{2222} \right) \frac{u'^{(k)}_{r'}}{r'^2} + \left(\chi'^{(k)}_{1133} - \chi'^{(k)}_{2233} \right) \frac{1}{r'} \frac{\partial u'^{(k)}_{3}}{\partial y'_3} = \rho'^{(k)} \frac{\partial^2 u'^{(k)}_{r'}}{\partial t^2} - \frac{\partial p'^{(k)}}{\partial r'},$$

$$\chi'^{(\underline{k})}_{3322}\frac{\partial^2 u'^{(\underline{k})}_{r'}}{\partial r'\partial y'_3}+\chi'^{(\underline{k})}_{3131}\frac{\partial^2 u'^{(\underline{k})}_3}{\partial r'^2}+\frac{1}{r'}\chi'^{(\underline{k})}_{3113}\frac{\partial u'^{(\underline{k})}_{r'}}{\partial y'_3}+\frac{1}{r'}\chi'^{(\underline{k})}_{3131}\frac{\partial u'^{(\underline{k})}_3}{\partial r'}+\chi'^{(\underline{k})}_{3311}\frac{\partial^2 u'^{(\underline{k})}_{r'}}{\partial y'_3 dr'}$$

$$+\chi'^{(\underline{k})}_{3322}\frac{1}{r'}\frac{\partial u'^{(\underline{k})}_{r'}}{\partial y'_3}+\chi'^{(\underline{k})}_{3333}\frac{\partial^2 u'^{(3)}_3}{\partial y'^2_3}=\rho'^{(\underline{k})}\frac{\partial^2 u'^{(\underline{k})}_3}{\partial t^2}-\frac{\partial p'^{(\underline{k})}}{\partial y'_3}. \tag{14}$$

Equations (12) and (14) compose a complete system of equations for the unknown functions $u'^{(k)}_{r'}$, $u'^{(k)}_3$, and $p'^{(k)}$. According to [10, 11], the displacement and unknown function $p'^{(k)}$ are represented as

$$u'^{(k)}_{r'}=-\frac{\partial^2}{\partial r'\partial y'_3}\mathrm{X}'^{(k)},\qquad u'^{(k)}_3=\Delta'_1\mathrm{X}'^{(k)},$$

$$p'^{(k)}=\left[\left(\chi'^{(k)}_{1111}-\chi'^{(k)}_{1133}-\chi'^{(k)}_{1313}\right)\Delta'_1+\chi'^{(\underline{k})}_{3113}\frac{\partial^2}{\partial y'^2_3}-\rho'^{(\underline{k})}\frac{\partial^2}{\partial t^2}\right]\frac{\partial}{\partial y'_3}\mathrm{X}'^{(\underline{k})}, \tag{15}$$

where

$$\Delta'_1=\frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial r'}, \tag{16}$$

and the function $\mathrm{X}'^{(k)}$ satisfies the equation

$$\left[\left(\Delta'_1+\left(\xi'^{(\underline{k})}_2\right)^2\frac{\partial^2}{\partial y'^2_3}\right)\left(\Delta'_1+\left(\xi'^{(\underline{k})}_3\right)^2\frac{\partial^2}{\partial y'^2_3}\right)-\frac{\rho'^{(\underline{k})}}{\chi'^{(\underline{k})}_{1331}}\left(\Delta'_1+\frac{\partial^2}{\partial y'^2_3}\right)\frac{\partial^2}{\partial t^2}\right]\mathrm{X}'^{(\underline{k})}=0, \tag{17}$$

where

$$\left(\xi'^{(k)}_2\right)^2=1,\qquad \left(\xi'^{(k)}_3\right)^2=\left(\lambda^{(k)}\right)^{-6}. \tag{18}$$

According to the problem statement, all the dependent variables become harmonic in time and can be represented as

$$\left\{Q'^{(\underline{k})}_{r'r'},\ldots,Q'^{(\underline{k})}_{33},u'^{(\underline{k})}_{r'},u'^{(\underline{k})}_3,p'^{(k)},\mathrm{X}'^{(\underline{k})}\right\}$$

$$=\left\{\overline{Q'}^{(\underline{k})}_{r'r'},\ldots,\overline{Q'}^{(\underline{k})}_{33},\bar{u}'^{(\underline{k})}_{r'},\bar{u}'^{(\underline{k})}_3,\bar{p}'^{(k)},\overline{\mathrm{X}}'^{(\underline{k})}\right\}e^{i\omega t}, \tag{19}$$

where the overbar denotes the amplitude of the corresponding quantity. We will omit this overbar below.

With (19), we can replace the operator $\partial^2/\partial t^2$ with $-\omega^2$ in (14)–(17) to obtain the same equations and conditions for the amplitude of the sought quantities. Consequently, introducing the dimensionless coordinates $r'\to r'/h$ and $y'_3\to y'_3/h$ and the dimensionless frequency

$$\Omega^2=\frac{(\omega h)^2\rho'^{(2)}}{2C^{(2)}_{10}}, \tag{20}$$

we obtain the following equation for the potential $\mathrm{X}'^{(k)}$:

$$\left[\left(\Delta'_1+\left(\xi'^{(\underline{k})}_2\right)^2\frac{\partial^2}{\partial y'^2_3}\right)\left(\Delta'_1+\left(\xi'^{(\underline{k})}_3\right)^2\frac{\partial^2}{\partial y'^2_3}\right)-\frac{\Omega^2}{(\lambda^{(\underline{k})})^2}\left(\Delta'_1+\frac{\partial^2}{\partial y'^2_3}\right)\frac{C^{(2)}_{10}\rho'^{(\underline{k})}}{C^{(\underline{k})}_{10}\rho'^{(2)}}\right]\mathrm{X}'^{(\underline{k})}=0. \tag{21}$$

To solve Eq. (21), we apply the Hankel transform to the function $\mathrm{X}'^{(k)}$:

$$\mathrm{X}'^{(k)} = \int_0^\infty F_1^{(\underline{k})} e^{\gamma^{(\underline{k})} y_3'} J_0(sr')sds, \tag{22}$$

where $J_0(sr')$ is the zeroth-order Bessel function.

Substituting (22) into (21), we obtain the following algebraic equation for $\gamma^{(k)}$:

$$A^{(\underline{k})}(\gamma^{(\underline{k})})^4 + B^{(\underline{k})}(\gamma^{(\underline{k})})^2 + C^{(\underline{k})} = 0, \tag{23}$$

where

$$A^{(\underline{k})} = \left(\xi_2'^{(\underline{k})}\right)^2 \left(\xi_3'^{(\underline{k})}\right)^2, \quad B^{(k)} = \frac{1}{(\lambda^{(\underline{k})})^2} \frac{C_{10}^{(2)}}{C_{10}^{(\underline{k})}} \frac{\rho'^{(\underline{k})}}{\rho'^{(2)}} \Omega^2 - \left(\left(\xi_2'^{(\underline{k})}\right)^2 + \left(\xi_3'^{(\underline{k})}\right)^2\right) s^2,$$

$$C^{(k)} = s^4 - s^2 \frac{C_{10}^{(2)}}{C_{10}^{(\underline{k})}} \frac{\rho'^{(\underline{k})}}{\rho'^{(2)}} \frac{1}{(\lambda^{(\underline{k})})^2}. \tag{24}$$

We obtain from (23) that

$$(\gamma^{(\underline{k})})^2 = \frac{-B^{(\underline{k})} \pm \sqrt{(B^{(\underline{k})})^2 - 4A^{(\underline{k})}C^{(\underline{k})}}}{2A^{(\underline{k})}}. \tag{25}$$

It can be proved by direct verification that

$$(\gamma^{(\underline{k})})^2 = \frac{-B^{(\underline{k})} + \sqrt{(B^{(\underline{k})})^2 - 4A^{(\underline{k})}C^{(\underline{k})}}}{2A^{(\underline{k})}} > 0. \tag{26}$$

For $(\gamma^{(\underline{k})})^2 = \left(-B^{(\underline{k})} - \sqrt{(B^{(\underline{k})})^2 - 4A^{(\underline{k})}C^{(\underline{k})}}\right) \Big/ 2A^{(\underline{k})}$, however, there may be the following two cases:

$$(\gamma^{(\underline{k})})^2 = \frac{-B^{(\underline{k})} - \sqrt{(B^{(\underline{k})})^2 - 4A^{(\underline{k})}C^{(\underline{k})}}}{2A^{(\underline{k})}} > 0, \tag{27}$$

$$(\gamma^{(\underline{k})})^2 = \frac{-B^{(\underline{k})} - \sqrt{(B^{(\underline{k})})^2 - 4A^{(\underline{k})}C^{(\underline{k})}}}{2A^{(\underline{k})}} < 0. \tag{28}$$

In case 1, Eq. (21) has the solution

$$\mathrm{X}'^{(1)} = \int_0^\infty \left[F_1^{(1)} e^{\gamma_1^{(1)} y_3'} + F_2^{(1)} e^{-\gamma_1^{(1)} y_3'} + F_3^{(1)} e^{\gamma_2^{(1)} y_3'} + F_4^{(1)} e^{-\gamma_2^{(1)} y_3'} \right] J_0(sr')sds, \quad \text{for} \quad -h' < y_3' < 0,$$

$$\mathrm{X}'^{(2)} = \int_0^\infty \left[F_1^{(2)} e^{\gamma_1^{12)} y_3'} + F_3^{(2)} e^{\gamma_2^{12)} y_3'} \right] J_0(sr')sds, \quad \text{for} \quad -\infty < y_3' < -h', \tag{29}$$

where

$$\gamma_1^{(k)} = \sqrt{\frac{-B^{(\underline{k})} + \sqrt{(B^{(\underline{k})})^2 - 4A^{(\underline{k})}C^{(\underline{k})}}}{2A^{(\underline{k})}}} > 0,$$

$$\gamma_2^{(k)} = \sqrt{\frac{-B^{(\underline{k})} - \sqrt{(B^{(\underline{k})})^2 - 4A^{(\underline{k})}C^{(\underline{k})}}}{2A^{(\underline{k})}}} > 0. \tag{30}$$

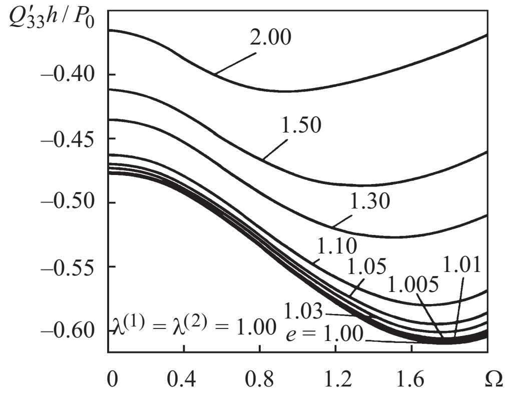

Fig. 1

In case 2, this solution is

$$X'^{(1)} = \int_0^{\infty} \left[F_1^{(1)} e^{\gamma_1^{(1)} y'_3} + F_2^{(1)} e^{-\gamma_1^{(1)} y'_3} + F_3^{(1)} e^{i\gamma_2^{(1)} y'_3} + F_4^{(1)} e^{-i\gamma_2^{(1)} y'_3} \right] J_0(sr') s ds \quad \text{for} \quad -h' < y'_3 < 0,$$

$$X'^{(2)} = \int_0^{\infty} \left[F_1^{(2)} e^{\gamma_1^{12)} y'_3} + F_3^{(2)} e^{i\gamma_2^{12)} y'_3} \right] J_0(sr') s ds \quad \text{for} \quad -\infty < y'_3 < -h', \tag{31}$$

where

$$\gamma_2^{(k)} = \sqrt{\frac{B^{(\underline{k})} + \sqrt{(B^{(\underline{k})})^2 - 4A^{(\underline{k})} C^{(\underline{k})}}}{2A^{(\underline{k})}}} > 0. \tag{32}$$

Using Eqs. (5), (15), (31), and (32), we obtain integral expressions for stresses and displacements, which are similar to (29) and (31). These expressions are omitted here as very cumbersome.

To find the unknowns $F_1^{(k)}(s), \ldots, F_4^{(k)}(s)$, we use the boundary and contact conditions (13). For this purpose, we take the Hankel transform of the right-hand side of the first condition in (13). Using the equality $P_0 \delta(r') = \lim_{r' \to 0} (P_0 / \pi r'^2)$, we obtain $P_0 / 2\pi$ for the Hankel transform $P_0 \delta(r')$ from $\lim_{\varepsilon \to 0} \int_0^{\varepsilon} P_0 r' J_0(sr') dr' / (\pi \varepsilon^2)$. Thus, we derive the following equations for the above-listed unknowns:

$$F_j^{(1)}(s) \alpha_{ij}^{(1)}(s) = P_0 / (2\pi (\lambda^{(1)})^2) \delta_i^1, \qquad i = 1, 2, \qquad j = 1, 2, 3, 4, \qquad \delta_1^1 = 1, \qquad \delta_2^1 = 0,$$

$$F_j^{(1)}(s) \alpha_{ij}^{(1)}(s) + F_n^{(2)}(s) \alpha_{in}^{(2)}(s) = 0, \qquad i = 3, 4, 5, 6, \qquad n = 1, 3, \tag{33}$$

where summation is over the repeated indices j and n. The coefficients of the unknowns in (33) are determined from the expressions for the stresses and displacements. Thus, the unknowns $F_1^{(k)}(s), \ldots, F_4^{(k)}(s)$ are determined from Eq. (33), and then the stresses and displacements can be calculated from the corresponding integral expressions. The integrals are evaluated by the algorithm proposed in [2, 3, 7, 9].

We will now discuss some numerical results obtained using the above solution procedure and related to the influence of the prestretching of the layer and the prestraining of the half-space on the distribution of the stress Q'_{33} at the interface between them.

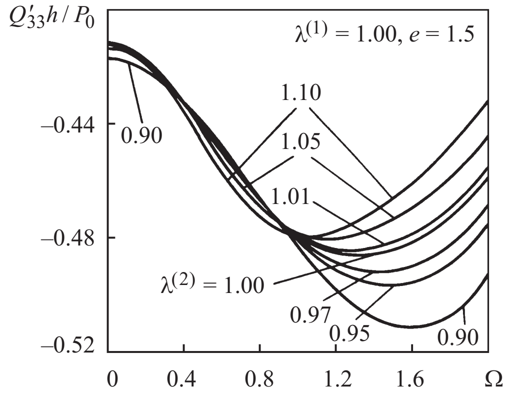

Fig. 2

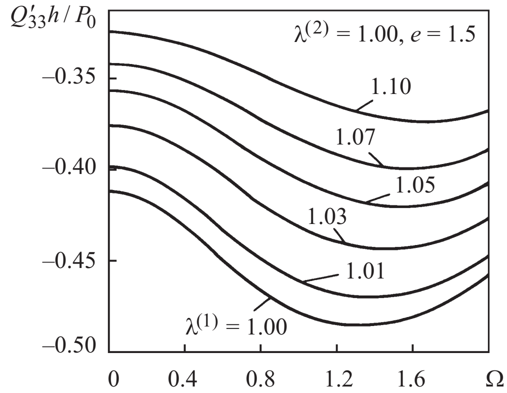

Fig. 3

4. Numerical Results and Discussion. We introduce the parameter $e = C_{10}^{(1)} / C_{10}^{(2)}$ and assume that $e \geq 1$. We also assume that $0 \leq \Omega \leq 2$ and $C_{10}^{(1)} / \rho'^{(1)} = C_{10}^{(2)} / \rho'^{(2)}$.

To test the algorithm and its software implementation, we set $e = 1$ and $\lambda^{(1)} = \lambda^{(2)} = 1$. According to the mechanical considerations, as $\Omega \to 0$, the values of Q'_{33} must approach the values obtained by solving the corresponding Boussinesq problem [23]. Moreover, according to [8, 18, 20, etc.], the behavior of the half-space under forced vibration is similar to the forced vibration of a system consisting of a mass and parallel-connected spring and dashpot. Consequently, the absolute values of Q'_{33} must have maximum at certain Ω, which will be called the resonance value of Ω. These two statements are confirmed by the graph for $e = 1$ in Fig. 1, which shows the dependences of $Q'_{33}h/P_0$ (at $r'/h = 0$) on Ω for different values of e and for $\lambda^{(1)} = \lambda^{(2)} = 1$. According to these graphs, as $e \to 1$, the values of $Q'_{33}h/P_0$ approach the corresponding values of $Q'_{33}h/P_0$ obtained for $e = 1$. Also, Fig. 1 shows that the resonance values of Ω and the absolute values of $Q'_{33}h/P_0$ decrease with *e*. These results agree with the well-known mechanical considerations. Consequently, the results given in the Fig. 1 confirm the validity and reliability of the algorithm and software.

Now we consider the influence of the initial strain of the half-space on the dependence of $Q'_{33}h/P_0$ on Ω for $\lambda^{(1)} = 1$ and $e = 1.5$. This dependence is shown in Fig. 2. It follows from this figure that when the half-space is precompressed, the absolute maximum values of $Q'_{33}h/P_0$ and the resonance values of Ω increase with decreasing $\lambda^{(2)}$. However, if the half-space is prestretched, i.e., $\lambda^{(2)} > 1$, the values of Q'_{33} and the resonance values of Ω decrease monotonically with $\lambda^{(2)}$.

Figure 3 shows the dependence of $Q'_{33}h/P_0$ on Ω for different $\lambda^{(1)} > 1$ and $\lambda^{(2)} = 1$, $e = 1.5$. It follows from this figure that when the layer is prestretched, i.e., $\lambda^{(1)}$, the absolute maximum values of $Q'_{33}h/P_0$ decrease, but the resonance values of Ω increase.

We have examined only the case where the layer is prestretched because the initial strains occur before the layer is coupled with the half-space. Consequently, initial compression of an infinite layer has no physical meaning due to loss of stability. Moreover, for a precompressed half-space, it is assumed that $\lambda^{(2)} > \lambda_{cr}^{(2)}$, where $\lambda_{cr}^{(2)}$ corresponds to near-surface instability. According to [12], we have $\lambda_{cr}^{(2)} \approx 0.667$.

We have considered the case $C_{10}^{(1)} / C_{10}^{(2)} = \rho'^{(2)} / \rho'^{(1)}$. Numerical analyses show that the above results are also valid, in a quantitative sense, for $C_{10}^{(1)} / C_{10}^{(2)} \neq \rho'^{(2)} / \rho'^{(1)} = 1$. However, the resonance values of Ω determined for the latter case are smaller than the values for $C_{10}^{(1)} / C_{10}^{(2)} = \rho'^{(2)} / \rho'^{(1)}$. This conclusion is confirmed by Fig. 4, where the dashed graphs represent the case $C_{10}^{(1)} / C_{10}^{(2)} = \rho'^{(2)} / \rho'^{(1)}$. A comparison of the graphs in Fig. 4 shows that the absolute maximum values of $Q'_{33}h/P_0$ at the resonance values of Ω for $C_{10}^{(1)} / C_{10}^{(2)} \neq \rho'^{(2)} / \rho'^{(1)} = 1$ are greater than those for $C_{10}^{(1)} / C_{10}^{(2)} = \rho'^{(2)} / \rho'^{(1)}$. Note that the above results are valid, in a qualitative sense, for the other stresses and displacements.

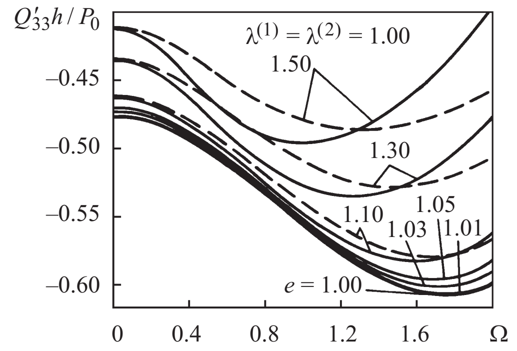

Fig. 4

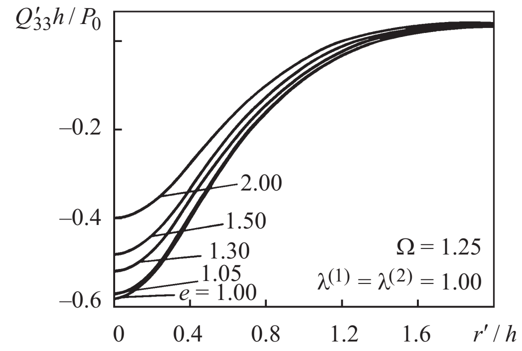

Fig. 5

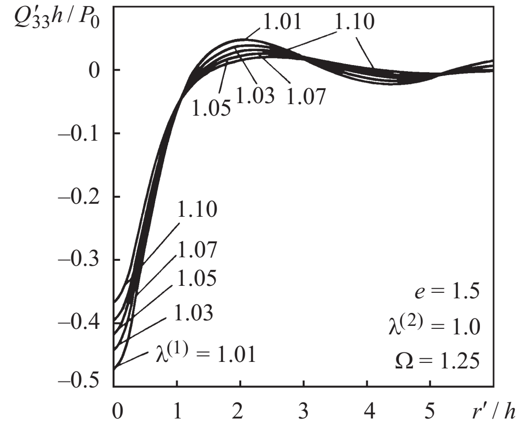

Fig. 6

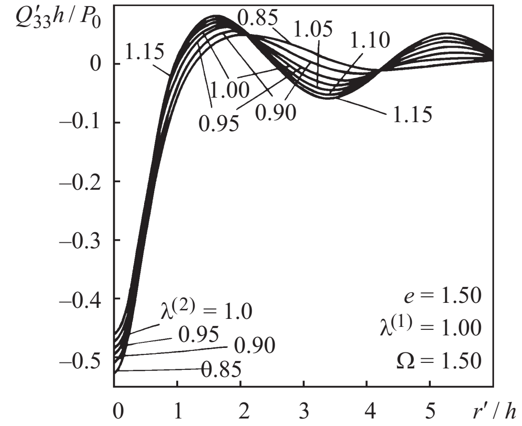

Fig. 7

We return to the discussion of the numerical result for $C_{10}^{(1)} / C_{10}^{(2)} = \rho'^{(2)} / \rho'^{(1)}$ and consider the distribution of the stress Q'_{33} versus r' / h for selected values of Ω and e. Figure 5 shows these distributions for $\lambda^{(1)} = \lambda^{(2)} = 1, \Omega = 1.25$ and different values of e. It follows from this figure that the absolute maximum values for Q'_{33} are attained at $r' / h = 0$ and decrease with e. The influence of the prestretching of the layer on this distribution is illustrated by Fig. 6. It can be seen that the absolute values of Q'_{33} decrease with $\lambda^{(1)}$.

Figure 7 shows the influence of the initial strain of the half-space on this distribution. It can be seen the character of the influence of the initial strains on the stress Q'_{33} established above remains at all interfacial points.

5. Conclusions. The dynamic (time-harmonic) axisymmetric stress field in an initially finite strained half-space covered with an initially finite stretched layer has been investigated using the piecewise-homogeneous body model and the TLTEWISB. It was assumed that the layer and the half-space are incompressible and their elastic relations include the Treloar potential. The dependences of the normal stresses at the interface between the layer and half-space on the frequency of the external force have been determined numerically. It was assumed that $C_{10}^{(1)} > C_{10}^{(2)}$, where $C_{10}^{(1)}$ and $C_{10}^{(2)}$ are the material constants for the layer and half-space, respectively, which appear in the expression of the Treloar potential.

The numerical analyses done lead to the following conclusions: the mechanical behavior of the half-space covered with the layer is similar to the forced vibration of a system consisting of a mass, a spring, and a dashpot; the resonance values of the frequency of the external force and the absolute maximum values of the interfacial normal stresses decrease with $C_{10}^{(1)} / C_{10}^{(2)}$; as a result of the prestretching of the layer, the absolute values of the stress decrease and the resonance values of the frequency increase; the resonance values of the frequency and the absolute values of the stress increase (decrease) with initial compression (tension) of the half-space; the influence of the initial strains on the stress distribution and on the resonance values of the frequency is significant in a quantitative sense and must be taken into account.

REFERENCES

1. S. D. Akbarov and A. N. Guz, "Axisymmetric longitudinal wave propagation in prestressed compound circular cylinders," *Int. J. Eng. Sci*., **42**, 769–791 (2004).
2. S. D. Akbarov and O. Ozaydin, "The effect of initial stresses on harmonic stress field within a stratified half-plane," *Eur. J. Mech. A/Solids*, **20**, 385–396 (2001).
3. S. D. Akbarov and O. Ozaydin, "Lamb's problem for an initially stressed stratified half-plane," *Int. Appl. Mech*., **37**, No. 10, 1363–1367 (2001).
4. S. D. Akbarov and M. Ozisik, "The influence of the third order elastic constants on the generalized Rayleigh wave dispersion in a pre-stressed stratified half-plane," *Int. J. Eng. Sci*., **41**, 2047–2061 (2003).
5. S. D. Akbarov and M. Ozisik, "Dynamic interaction of a prestressed nonlinear elastic layer and a half-plane," *Int. App. Mech*., **40**, No. 9, 1056–1063 (2004).
6. P. Chadwick, "Interfacial waves in pre-strained isotropic elastic media," *Z. Angew. Math. Phys*., **46**, S51–S71 (1995).
7. I. Emiroglu, F. Tasci, and S. D. Akbarov, "Lamb's problem for a half-space covered with a two-axially pre-stretched layer," *Mech. Comp. Mater*., **40**, No. 3, 227–236 (2004).
8. G. M. L. Gladwell, "The calculation of mechanical impedances related with the surface of a semi-infinite elastic body," *J. Sound Vibr*., **8**, 215–219 (1968).
9. C. Guler and S. D. Akbarov, "Dynamic (harmonic) interfacial stress field in a half-plane covered with a pre-stretched soft layer," *Mech. Comp. Mater*., **40**, No. 5, 379–388 (2004).
10. A. N. Guz, *Elastic Waves in Bodies with Initial* (*Residual*) *Stresses* [in Russian], A.S.K., Kyiv (2004).
11. A. N. Guz and I. A. Guz, "On models in the theory of stability of multiwalled carbon nanotubes in matrix," *Int. Appl. Mech*., **42**, No. 6, 617–628 (2006).
12. A. N. Guz, *Fundamentals of the Three-Dimensional Theory of Stability of Deformable Bodies*, Springer-Verlag, Berlin (1999).
13. A. N. Guz, "On two-scale model of fracture mesomechanics of composites with cracks under compression," *Int. Appl. Mech*., **41**, No. 6, 582–585 (2005).
14. A. N. Guz, A. A. Rodger, and I. A. Guz, "Developing a compressive failure theory for nanocomposites," *Int. Appl. Mech*., **41**, No. 3, 233–255 (2005).
15. A. N. Guz and V. N. Chekhov, "On the determination of loads upon the ground support in underground mine workings," *Int. Appl. Mech*., **41**, No. 11, 1239–1245 (2005).
16. W. Hussain and R. W. Ogden, "The effect of pre-strain on the reflection and transmission of plane waves at an elastic interface," *Int. J. Eng. Sci*., **39**, No. 8, 925–950 (2001).
17. V. P. Koshman, "Lamb's plane problem for a compressible half-space with initial stresses," *Int. Appl. Mech*., **16**, No. 10, 912–917 (1980).
18. H. Lamb, "On the propagation of tremors over the surface of an elastic solid," *Philos. Trans. of the Royal Society*, **A203**, 1–42 (1904).
19. M. Ozisik and S. D. Akbarov, "Generalized Rayleigh wave dispersion in a pre-stressed layer under complete and incomplete interfacial contact," *Mech. Comp. Mater*., **39**, No. 2, 177–182 (2003).
20. I. A. Robertson, "Forced vertical vibration of a rigid circular disc on a semi-infinite solid," *Proc. Camb. Philos. Soc*., **62**, 547–557 (1966).
21. G. A. Rogerson and K. J. Sandiford, "The effect of finite primary deformations on harmonic waves in layered elastic media," *Int. J. Solids Struct*., **37**, 2059–2087 (2000).
22. A. Tadeu, J. Antonio, and L. Godinho, "Green's function for two-and-a-half dimensional elastodynamic problems in a half-space," *Comp. Mech*., **27**, 484–491 (2001).
23. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York (1975).
24. C. Y. Wang and J. D. Achenbach, "Lamb's problem for solids of general anisotropy," *Wave Motion*, **24**, 227–242 (1996).