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LINEAR RELATIONSHIP BETWEEN THE FIRST INVARIANTS OF THE STRESS AND STRAIN TENSORS IN THEORIES OF PLASTICITY WITH STRAIN HARDENING

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Energy-coupled stress and strain measures are defined in Euler coordinates. They are used to analyze the relationship between the first invariants of the stress and strain tensors for linearity and to determine strains at which the plastic component of the first strain invariant can be neglected. It is established that this relationship remains linear within an engineering plastic-strain tolerance of 0.2% irrespective of the value of strain intensity, which depends on the type of material and its stress state

Keywords: solid body, Eulerian and Lagrangian coordinates, stress and strain tensors, first invariants, linear relationship

Introduction. The modern theories of plasticity with strain hardening [11–15], which refer to Bridgman's study [2], postulated the generalized Hooke's law in both elastic and plastic strain ranges; i.e., it is supposed that the volume of a solid does not change during elastoplastic deformation. In this case, the first invariant of the stress tensor is used as hydrostatic pressure and the first invariant of the strain tensor as volume strain; i.e., all modern theories of plasticity assume that the first invariants of the stress and strain tensors are in a linear relationship. However, the first strain invariant can define volume strain only approximately.

In this connection, we will discuss the values of the strain components at which the plastic component of the first strain invariant can be neglected, which, in fact, is done in each modern theory of plasticity. We will use the principles of solid mechanics based on Cauchy's continuum hypothesis. Unlike Lagrange's approach, this hypothesis suggests using the method of sections to determine the stresses at an arbitrary point of a body on an area element somehow oriented in space and going through this point. According to this method, a solid subjected to specified external loads is partitioned by this plane, and one of the parts is rejected. The equilibrium condition for the remaining part is used to determine the principal vector and the principal moment exerted by the rejected part onto the remaining part in the specified section going through the specified point. Dividing the main vector and principal moment by the area of the section and letting it tend to zero at the point, we obtain that the limit of the ratio of the principal moment to this area tends to zero and the limit of the ratio of the principal vector to this area tends to the value of the stress vector acting on this plane at the specified point [3]. Projecting the vector onto the axes of an orthogonal coordinate system fixed at one point to the solid before deformation and having constant directions during deformation (Eulerian coordinate system), we obtain the stress components in a plane passing through the point of interest.

1. In a deformed solid, we select an elementary rectangular parallelepiped with sizes dx_i in the Eulerian coordinate system x_i, apply the stresses obtained above to each of its sides, and set up differential equilibrium equations in the following form [5]:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + K_i = 0, \qquad \langle i = 1, 2, 3 \rangle \qquad (j = 1, 2, 3), \tag{1.1}$$

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where σ_{ij} are the components of the stress tensor; K_i are the projections of the body forces per unit volume of the parallelepiped.

Here and later on, summation is over repeated indices from 1 to 3 ($j = 1, 2, 3$) in monomial expressions and no summation is carried out over nonrepeated indices, their limits being indicated in angular brackets $\langle i = 1, 2, 3 \rangle$.

Unlike the generalized stresses [5], the stress components thus obtained are independent of the strain components in both magnitude and direction. In specimens with a homogeneous stress state, these stresses can be determined by dividing external loads by the cross-sectional areas of specimens before deformation.

The components of the finite-strain tensor $æ_{ij}$ in the Eulerian coordinate system can be determined from a geometrical analysis of two orthogonal intervals of the solid after deformation. These strains in an orthogonal rectilinear coordinate system are related to the displacement components U_i as follows (Cauchy relations) [5]:

$$æ_{ij} = \varepsilon_{ij} + \frac{1}{2}\frac{\partial U_k}{\partial x_i}\frac{\partial U_k}{\partial x_j}, \tag{1.2}$$

where the first term ε_{ij} is the linear component of the finite-strain tensor $æ_{ij}$,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \tag{1.3}$$

and the second term is its nonlinear component. It is assumed that the coordinates of a point x_i in the undeformed body are associated with the coordinates $x_i + U_i$ in the deformed body, and fibers change their orientation during deformation. If l_{jN} are the direction cosines of a fiber *MN* before deformation and l_{iN_1} are its direction cosines after deformation, then [5]

$$l_{iN_1} = \frac{\delta_{ij} + \dfrac{\partial U_i}{\partial x_j}}{1 + \varepsilon_{MN}} l_{jN}, \tag{1.4}$$

where δ_{ij} is the Kronecker delta and ε_{MN} is the relative elongation of the fiber *MN*. If this fiber is aligned with the x_1-axis before deformation, then $\varepsilon_{MN} = \varepsilon_{11}$ is the relative elongation of the fiber along this axis.

Using formulas for transforming the displacement components and their derivatives upon a rotation of the coordinate axes, it can be shown that the linear components ε_{ij} are transformed according to formulas quadratic with respect to the direction cosines of both coordinate systems; i.e., the linear part ε_{ij} of the strain tensor is a tensor of the second rank. It can similarly be shown that the nonlinear part of the strain tensor is a tensor of the second rank too; i.e., formulas quadratic with respect to the direction cosines are also used to transform its components upon a rotation of the coordinate axes. In the Eulerian coordinate system, the stress components can be transformed similarly using formulas (1.4).

We will write the constitutive equations in terms of the stress and strain measures defined above, no matter strains are small or large. These measures must be such that the sum of products of stresses and strain variations is equal to the work done (the sum of products of forces and displacement variations), i.e., these measures must be energy-coupled.

Let us establish which of the measures meet this requirement. Following [5], we select, in the deformed body, an elementary parallelepiped $dx_1 dx_2 dx_3$ with edges parallel to the Eulerian coordinate axes x_i. Denote by $\vec{\sigma}_3$ the stress vector on an area element perpendicular to the x_3-axis and by $\vec{U}$ the displacement vector of this area element ($x_3 = h/2$, h is the height of the parallelepiped), assuming that the origin of coordinates is at the center of the parallelepiped. Then, $\vec{\sigma}_3 dx_1 dx_2$ is the force acting on this face. The work done by this force to displace the face by $\delta\vec{U}$ is given by $(\vec{\sigma}_3 \cdot \delta\vec{U})_{x_3=h/2} dx_1 dx_2$. The work done by the force on the bottom face ($x_3 = -h/2$) is $(-\vec{\sigma}_3 \cdot \delta\vec{U})_{x_3=-h/2} dx_1 dx_2$. The total work on these two faces is determined as $[(\vec{\sigma}_3 \cdot \delta\vec{U})_{x_3=h/2} - (\vec{\sigma}_3 \cdot \delta\vec{U})_{x_3=-h/2}] dx_1 dx_2$. Multiplying and dividing this expression by dx_3 and passing to the limit, we obtain $\dfrac{\partial(\vec{\sigma}_3 \cdot \delta\vec{U})}{\partial x_3} dx_1 dx_2 dx_3$. The specific work on these two area elements is given by $\dfrac{\partial(\vec{\sigma}_3 \cdot \delta\vec{U})}{\partial x_3}$. Similarly, calculating the specific work on the area elements perpendicular to x_1- and x_2-axes, summing these works, and adding the work done by the body forces, $\vec{K}\cdot\delta\vec{U}$, we obtain $\delta A = \frac{\partial(\vec{\sigma}_i \cdot \delta\vec{U})}{\partial x_j}\delta_{ij} + \vec{K}\cdot\delta\vec{U}$. Expanding the derivative of a product of two functions, we get $\delta A = \frac{\partial(\vec{\sigma}_i)}{\partial x_j}\cdot\delta\vec{U}\delta_{ij} + \vec{\sigma}_i \cdot \frac{\partial\delta\vec{U}}{\partial x_j}\delta_{ij} + \vec{K}\cdot\delta\vec{U}$. Transforming the equilibrium equations (1.1) to obtain $\delta A = \vec{\sigma}_i \cdot \frac{\partial\delta\vec{U}}{\partial x_j}\delta_{ij}$ and introducing the derivative under the variation sign, we find $\delta A = \vec{\sigma}_i \cdot \delta\frac{\partial\vec{U}}{\partial x_j}\delta_{ij}$. Replacing the vectors $\vec{\sigma}_i$ and $\vec{U}$ by their projections onto the coordinate axes and using notation (1.3), we get

$$\delta A = \sigma_{ij}\delta\varepsilon_{ij}. \tag{1.5}$$

Thus, the specific work is the sum of products of the components of the stress tensor and the variation of the linear part of the strain tensor in the Eulerian coordinate system.

Therefore, in deriving constitutive equations, we will establish the relationship between the stress components σ_{ij} and the linear part of the strain tensor ε_{ij}. This Cauchy's approach to the determination of the stress and strain components does not consider that the shape of the elementary parallelepiped changes during loading, i.e., it is assumed that a change in the shape has a weak effect on the value and direction of the stress components. This effect is accounted for in Lagrange's approach so that the stress components are expressed in terms of strains. The projections of the strain components do not generally coincide with the stress components. The material characteristics determined in tension and torsion tests can be used in the constitutive equations only if the angles between the Eulerian and Lagrangian coordinate axes are small, i.e., the direction cosines l_{iN_1} of a fiber after deformation differ insignificantly from the direction cosines l_{jN} before deformation (1.4) and this difference can be neglected.

2. In plasticity theories, the stress components and the linear part of strain are represented by the sum of the corresponding deviatoric, s_{ij} and e_{ij}, and spherical, σ_0 and ε_0, components:

$$\sigma_{ij} = s_{ij} + \sigma_0\delta_{ij}, \tag{2.1}$$

$$\varepsilon_{ij} = e_{ij} + \varepsilon_0\delta_{ij}, \tag{2.2}$$

$$\sigma_0 = \sigma_{ij}\delta_{ij} / 3, \tag{2.3}$$

$$\varepsilon_0 = \varepsilon_{ij}\delta_{ij} / 3. \tag{2.4}$$

The spherical components σ_0 and ε_0 characterize changes in the volume, and the deviator components s_{ij} and e_{ij} the shear properties of the material.

The spherical parts of the stress tensor σ_0 and the strain tensor ε_0 are known [4] to be equal to their first (linear) invariants. Modern theories of plasticity assume a linear relationship between the first invariants of the stress and strain tensors, as mentioned at the beginning of the paper.

Let us find the maximum strain at which this relationship is still linear. We will use published experimental data and data obtained by the authors of this paper.

Columns 1–3 of Table 1 contain results (borrowed from [6]) of uniaxial tension tests on cylindrical specimens made of chromium-nickel steel (their heat treatment conditions not indicated). Here $\sigma_{zz} = 4P/(\pi d^2)$, where P is the tensile force, d is the diameter of a specimen; σ_{zz} are the axial stresses; $\varepsilon_{zz} = \Delta l / l$, where l is the gauge length for measurement of longitudinal displacements; Δl is the increment of this length; ε_{zz} are the axial strains; $\varepsilon_{rr} = \Delta d / d$, where Δd is the increment of the specimen diameter due to deformation; and ε_{rr} are the measured transverse strains. By symmetry, $\varepsilon_{\varphi\varphi}$ are equal to ε_{rr}. The longitudinal, ε_{zz}, and transverse, ε_{rr}, strains were measured with a mechanical extensometer with dial indicators. In the tests: $d = l = 30$ mm.

The first row of this table was used to calculate the elastic modulus $E = \sigma_{zz} / \varepsilon_{zz} = 2.04\cdot 10^5$ MPa and Poisson's ratio $\nu = |\varepsilon_{rr}| / \varepsilon_{zz} = 0.31$. Columns 4 and 5 contain the calculated values of $\sigma_0 = \sigma_{zz}/3$ (2.3) and $\varepsilon_0 = (\varepsilon_{zz} + 2\varepsilon_{rr})/3$ (2.4). The plastic strain components $\varepsilon_{ij}^{(p)}$ were determined from $\varepsilon_{ij}^{(p)} = \varepsilon_{ij} - \varepsilon_{ij}^{(e)}$, where $\varepsilon_{ij}^{(e)} = [\sigma_{ij}(1+\nu) - 3\nu\sigma_0\delta_{ij}] / [2G(1+\nu)]$, whence it follows that $\varepsilon_0^{(e)} = [\sigma_0(1-2\nu)] / [2G(1+\nu)]$, where $G = E / [2(1+\nu)]$ is the shear modulus. It is assumed here that, as in all theories

TABLE 1

σ_{zz}, MPa	$\varepsilon_{zz}\cdot 10^3$	$\varepsilon_{\varphi\varphi}=\varepsilon_{rr}\cdot 10^3$	σ_0, MPa	$\varepsilon_0\cdot 10^4$	$\varepsilon_0^{(p)}\cdot 10^4$	$\Gamma\cdot 10^3$	$\theta\cdot 10^3$	ν^*
1	2	3	4	5	6	7	8	9
249	1.22	–0.38	83.0	1.5	0	0.92	0.46	0.31
286	1.54	–0.53	95.3	1.6	0	1.19	0.48	0.34(0.32)
300	1.68	–0.59	100.0	1.7	0	1.31	0.50	0.35(0.33)
329	2.14	–0.82	109.7	1.7	0	1.71	0.50	0.38(0.36)
343	2.59	–1.00	114.3	2.0	0	2.08	0.59	0.38(0.38)
358	3.03	–1.16	119.3	2.4	0	2.42	0.70	0.38(0.39)
372	3.51	–1.38	124.0	2.5	0	2.82	0.74	0.39(0.40)
387	4.07	–1.67	129.0	2.4	0	3.31	0.72	0.41(0.41)
401	4.75	–1.98	133.7	2.6	0	3.88	0.78	0.41(0.42)
416	5.54	–2.39	138.7	2.5	0	4.58	0.74	0.43(0.43)
431	6.61	–2.90	143.7	2.7	0	5.50	0.78	0.43(0.44)
446	7.92	–3.50	148.7	3.1	0	6.65	0.88	0.44(0.45)
461	9.76	–4.49	153.7	2.6	0	8.23	0.71	0.46(0.45)
476	11.96	–5.56	158.7	2.8	0	10.12	0.74	0.47(0.46)
492	14.85	–6.96	164.0	3.1	0	12.60	0.77	0.47(0.47)
508	18.36	–8.66	169.3	3.5	0	15.60	0.80	0.47(0.47)
525	22.51	–10.61	175.0	4.3	1.1	19.13	0.93	0.47(0.48)
533	24.85	–11.70	177.7	4.8	1.5	21.11	10.10	0.47(0.48)
542	27.45	–12.99	180.7	4.9	1.6	23.34	9.30	0.47(0.48)
564	35.07	–16.51	188.0	6.8	3.3	29.78	11.70	0.47(0.48)

of plasticity, the elastic modulus and Poisson's ratio do not depend on the level of plastic strain. Column 6 collects the values of $\varepsilon_0^{(p)}$ calculated from the above values of E and ν. The shear-strain intensity Γ calculated by the formula

$$\Gamma = \left(\frac{1}{2} e_{ij} e_{ij}\right)^{1/2} \tag{2.5}$$

is in column 7. The relative change of volume θ calculated by the formula

TABLE 2

σ_{zz}, MPa	$\varepsilon_{zz}\cdot 10^3$	$\varepsilon_{\varphi\varphi}=\varepsilon_{rr}\cdot 10^3$	σ_0, MPa	$\varepsilon_0\cdot 10^4$	$\varepsilon_0^{(p)}\cdot 10^4$	$\Gamma\cdot 10^3$	$\theta\cdot 10^3$	ν^*	S, MPa
1	2	3	4	5	6	7	8	9	10
192.3	1.0	–0.27	64.1	1.5	0	0.73	0.46	0.27(0.27)	111.0
251.0	5.5	–2.06	83.7	4.6	0	4.37	1.36	0.37(0.45)	144.9
336.6	25	–10.85	112.2	11.0	8.3	20.7	2.87	0.43(0.48)	194.3
358.7	35	–16.12	119.6	9.2	6.4	29.51	1.91	0.46(0.49)	207.1
381.8	45	–20.94	127.3	10.4	7.3	38.07	1.69	0.47(0.49)	220.4
397.2	55	–25.49	132.4	13.4	10.2	46.47	1.91	0.46(0.49)	229.3
417.4	65	–30.46	139.1	13.6	10.3	55.11	1.11	0.47(0.49)	241.0
434.7	75	–34.55	144.9	19.7	16.2	63.25	2.01	0.46(0.49)	251.0
447.2	85	–39.86	149.1	17.6	14.0	72.09	0.23	0.47(0.49)	258.2
474.1	105	–48.40	158.0	27.3	23.6	88.57	0.63	0.46	273.7
523.2	165	–72.86	174.4	64.2	60.1	137.33	1.41	0.44	302.1
536.6	185	–79.58	178.9	86.1	81.9	152.75	3.90	0.43	309.8
543.4	200	–85.97	181.1	93.6	89.2	165.10	2.55	0.43	313.7
553.9	250	–103.52	184.6	143.2	138.8	204.10	4.60	0.41	319.8
565.5	310	–123.57	188.5	209.5	205.0	250.32	6.24	0.40	326.5

$$\theta=(1+\varepsilon_{zz})(1+\varepsilon_{rr})(1+\varepsilon_{\varphi\varphi})-1 \tag{2.6}$$

is in column 8. The last column contains the values of Poisson's ratio $\nu^*=|\varepsilon_{rr}|/\varepsilon_{zz}$.

From Table 1 it follows that the relationship between σ_0 and ε_0 is linear up to $\Gamma=1.56\%$ and can be considered linear to within a tolerance of 0.033% in the range $1.56\le\Gamma\le 2.98\%$. In this case, the relative change in volume θ (column 8) is not zero. Hence, in view of plastic incompressibility, Poisson's ratio ν^* can be determined from the following formula [4, 8]:

$$\nu^*=\frac{1}{2}-\frac{1-2\nu}{2E}\cdot\frac{\sigma}{\varepsilon}. \tag{2.7}$$

The values of Poisson's ratio ν^* calculated by this formula are presented in parentheses in column 9, whence it is seen that the calculated values of ν^* are in good agreement with its experimental values.

The data from uniaxial tension tests on tubular specimens made of Kh18N10T steel are reported on in [10]. The specimens were first heated to a temperature of 1070 °C, then held at this temperature for two hours, and, finally, let cool down in air. These results are collected in columns 1–3 of Table 2. Here, as in Table 1, $\sigma_{zz}=P/(\pi Dh)$, where P is the tensile force; D is the outer diameter of the tubular specimen; h is its wall thickness; σ_{zz} are the axial stresses; $\varepsilon_{zz}=\Delta l/l$ are the axial strains;

TABLE 3

σ_{zz}, MPa	$\sigma_{\varphi z}$, MPa	$\varepsilon_{zz}\cdot 10^3$	$\varepsilon_{rr}=\varepsilon_{\varphi\varphi}\cdot 10^3$	$\varepsilon_{\varphi z}\cdot 10^3$	σ_0, MPa	$\varepsilon_0\cdot 10^4$	$\varepsilon_0^{(p)}\cdot 10^4$	$\Gamma\cdot 10^3$	$\theta\cdot 10^3$	ν^*
1	2	3	4	5	6	7	8	9	10	11
559.6	0	2.86	–0.75	0	186.5	0.45	0	2.09	1.35	0.26
612.4	0	6.24	–1.94	0	204.1	0.78	0	4.72	2.33	0.31
650.3	11.2	13.41	–3.79	0.05	216.8	1.94	14.2	9.93	5.74	0.28
650.3	82.4	14.77	–4.25	0.69	216.8	2.09	15.6	11.01	6.15	0.29
650.3	161.6	19.77	–6.30	4.22	216.8	2.39	18.6	15.63	7.00	0.32
650.3	176.0	22.95	–7.36	6.07	216.8	2.75	22.2	18.52	7.94	0.32
650.3	196.6	25.98	–8.42	7.70	216.8	3.05	25.2	21.30	8.77	0.32
650.3	202.0	29.00	–9.54	9.45	216.8	3.31	27.9	24.17	9.47	0.33
650.3	228.5	41.17	–15.27	16.22	216.8	3.54	30.2	36.40	9.61	0.37
650.3	231.7	44.05	–16.39	17.89	216.8	3.76	32.3	39.21	10.11	0.37

$\varepsilon_{\varphi\varphi}=\Delta D/D$, where ΔD is the increment of the outer diameter due to deformation; $\varepsilon_{\varphi\varphi}$ are the measured hoop strains; and $\varepsilon_{rr}=\varepsilon_{\varphi\varphi}$ by symmetry. In these tests: $D=30.17$ mm, $h=1.16$ mm, $l=20$ mm. The longitudinal, ε_{zz}, and circumferential, $\varepsilon_{\varphi\varphi}$, strains were measured with a strain gauge described in [1]. The data in the first row of Table 2 were used to calculate the elastic modulus $E=\sigma_{zz}/\varepsilon_{zz}=1.92\cdot 10^5$ MPa and Poisson's ratio $\nu=|\varepsilon_{\varphi\varphi}|/\varepsilon_{zz}=0.27$. Columns 4–9 contain the values of σ_0, ε_0, $\varepsilon_0^{(p)}$, Γ, θ, and $\nu^*=|\varepsilon_{\varphi\varphi}|/\varepsilon_{zz}$ calculated using the formulas presented above, after Table 1, and the values of E and ν for Kh18N10T steel. Column 10 has the values of tangential-stress intensity $S=\sigma_{zz}/\sqrt{3}$.

Analyzing Table 2, we conclude that the relationship between σ_0 and ε_0 is linear when $\Gamma=0.4\%$ and can be considered linear within an engineering plastic-strain tolerance of 0.2% in the range $0.4\le\Gamma\le 8\%$. Hence, in the case of uniaxial tension, the function $\sigma_0(\varepsilon_0)$ is linear and formula (2.7) can be used to calculate Poisson's ratio ν^* when $\Gamma\approx 8\%$ with a plastic-strain tolerance of 0.2%. These results are in parentheses in column 9 of Table 2.

As Γ increases, the relationship between σ_0 and ε_0 ceases to be linear, and when $\Gamma=25\%$, the plastic component of the first strain invariant $\varepsilon_0^{(p)}$ reaches 2%. In this case, the relative change in volume θ is also nonzero, which means that the condition $\theta=0$ should not be used to determine ε_{rr} [9].

Based on results of uniaxial tension tests (Tables 1 and 2), we have studied the relationship between the first invariants σ_0 and ε_0. Let us now examine dependence of σ_0 on ε_0 for tubular specimens with a combined (plane) stress state induced by tensile force P and torque M or by tensile force P and internal pressure p.

The book [7] reports on results on compound deformation of tubular ÉI 437 alloy specimens. They were first stretched by a force P to a stress $\sigma_{zz}=689.5$ MPa and then, at this level of tension, additionally twisted by a torque M. Heat-treatment conditions: austenitization for 8 h at 1080 °C, cooling in air, ageing for 16 h at 700 °C, and, again, cooling in air. These results are in columns 1–5 of Table 3. Here $\sigma_{zz}=P/(\pi Dh)$ are the axial stresses, $\sigma_{\varphi z}=2M/(\pi D^2 h)$ are the tangential stresses, $\varepsilon_{zz}=\Delta l/l$ are the axial strains, and $\varepsilon_{\varphi\varphi}=\Delta D/D$ are the hoop strains. Again by symmetry, the strains ε_{rr} are equal to $\varepsilon_{\varphi\varphi}$; $\varepsilon_{\varphi z}=\gamma_{\varphi z}/2=(D\Delta\varphi)/(4l)$ are the shear strains, where $\Delta\varphi$ is the angle between two sections spaced by a distance l; and $\gamma_{\varphi z}$ is the shear angle. The strains in the tubular specimens were measured with an electromechanical strain gauge [7], which is capable

TABLE 4

σ_{zz}, MPa	$\sigma_{\varphi\varphi}$, MPa	$\varepsilon_{zz}\cdot 10^3$	$\varepsilon_{\varphi\varphi}\cdot 10^3$	$\varepsilon_{rr}\cdot 10^3$	S, MPa	ν^*	σ_0, MPa	$\varepsilon_0\cdot 10^4$	$\varepsilon_0^{(p)}\cdot 10^4$	$\Gamma\cdot 10^3$	$\theta\cdot 10^3$
1	2	3	4	5	6	7	8	9	10	11	12
233.6	239.7	2.03	–0.50	–0.63	136.7	0.42	157.8	3	0.0	1.00	0.89
252.1	254.3	2.75	0.00	–1.21	146.2	0.44	168.8	5	1.1	2.03	1.53
278.8	281.7	4.50	0.50	–2.31	161.8	0.46	186.8	9	4.5	3.42	2.68
304.2	313.2	8.10	5.02	–6.09	178.3	0.46	205.8	23	18.5	7.46	6.99
326.6	335.0	10.80	8.78	–9.14	191.0	0.47	220.5	35	29.5	10.98	10.36
366.4	398.3	17.19	18.85	–16.67	221.3	0.46	254.9	65	58.5	20.04	19.08
401.4	428.8	25.85	29.01	–25.46	240.0	0.46	276.7	98	91.4	30.58	28.73
418.4	439.7	30.28	34.86	–30.45	247.9	0.47	286.0	116	108.8	36.46	33.73
436.0	454.9	35.40	40.56	–35.20	257.4	0.46	297.0	136	128.8	42.33	39.47
449.5	467.2	39.27	45.34	–39.33	264.8	0.46	305.5	151	143.6	47.23	43.66
469.8	483.4	46.21	54.13	–46.72	275.3	0.47	317.7	179	171.2	56.08	51.32
481.5	496.0	50.27	59.48	–50.72	282.3	0.46	325.8	197	189.0	61.14	56.31
492.2	505.2	54.25	64.31	–54.63	288.0	0.46	332.5	213	205.1	65.96	60.75
507.4	518.5	64.31	74.68	–64.00	296.2	0.46	342.0	250	241.8	77.25	70.59
520.6	525.8	68.39	78.83	–67.77	302.1	0.46	348.8	265	256.5	81.79	74.50
535.0	533.0	73.07	83.48	–71.08	308.3	0.45	356.0	285	276.4	86.39	80.01
550.7	543.2	81.73	93.48	–76.95	315.8	0.44	364.6	328	318.8	95.19	91.84
555.1	548.0	84.18	95.50	–78.23	318.5	0.44	367.7	338	329.4	97.20	94.81
561.9	554.7	87.93	100.05	–80.86	322.4	0.43	372.2	357	348.2	101.13	100.01
565.4	558.6	90.52	102.45	–82.97	324.5	0.43	374.7	367	357.7	103.78	102.49

of recording longitudinal, ε_{zz}, circumferential, $\varepsilon_{\varphi\varphi}$, and shear, $\varepsilon_{\varphi z}$, strains simultaneously. The parameters had the following values: $D=$ 17.6 mm, $h=1$ mm, $l=20$ mm. The first row of Table 3 contains the values of the elastic modulus $E=\sigma_{zz}/\varepsilon_{zz}=1.96\cdot 10^5$ MPa and Poisson's ratio $\nu=|\varepsilon_{\varphi\varphi}|/\varepsilon_{zz}=0.26$. Columns 6 and 7 include the values of σ_0 (2.3) and ε_0 (2.4) calculated by the same expressions as in the case of pure tension. The values of $\varepsilon_0^{(p)}$, Γ, and θ calculated from the same formulas as in the previous case but with the values of E and ν for ÉI 437 alloy are in columns 8–10. The last column contains the values of $\nu^*=|\varepsilon_{\varphi\varphi}|/\varepsilon_{zz}$.

TABLE 5

σ_{zz}, MPa	$\sigma_{\varphi\varphi}$, MPa	$\varepsilon_{zz}\cdot 10^3$	$\varepsilon_{\varphi\varphi}\cdot 10^3$	$\varepsilon_{rr}\cdot 10^3$	S, MPa	ν^*	σ_0, MPa	$\varepsilon_0\cdot 10^4$	$\varepsilon_0^{(p)}\cdot 10^4$	$\Gamma\cdot 10^3$	$\theta\cdot 10^3$
1	2	3	4	5	6	7	8	9	10	11	12
167.0	334	0.2	3.60	–1.50	167.0	0.39	167.0	8	0	26.0	2.30
193.6	387.2	0.4	11.12	–4.89	193.6	0.42	193.6	22	17	81.6	6.58
215.6	431.1	0.1	26.37	–12.19	215.6	0.46	215.6	48	42	197.0	13.96
232.9	465.8	–0.1	46.76	–21.68	232.9	0.46	232.9	83	78	349.9	23.97
233.1	466.2	–0.3	53.14	–24.55	233.1	0.46	233.1	94	89	397.5	26.98
237.5	475	–0.3	58.59	–27.02	237.5	0.46	237.5	104	99	438.0	29.67
240.6	481.2	–0.4	66.80	–30.83	240.6	0.46	240.6	119	113	499.6	33.50
246.9	493.8	–0.7	76.17	–35.25	246.9	0.47	246.9	134	128	570.3	37.51
249.7	499.4	–0.8	80.45	–37.29	249.7	0.47	249.7	141	135	602.7	39.33
252.9	505.7	–0.9	86.87	–40.13	252.9	0.47	252.9	153	147	650.3	42.31
256.8	513.6	–0.9	95.38	–43.82	256.8	0.46	256.8	169	163	712.9	46.44
259.6	519.2	–1.0	100.29	–45.84	259.6	0.46	259.6	178	172	748.6	48.80
262.8	525.7	–1.1	106.08	–48.57	262.8	0.46	262.8	188	182	792.2	51.20
264.4	528.9	–1.2	108.51	–49.84	264.4	0.46	264.4	192	185	811.1	52.00
267.8	535.6	–1.5	118.82	–54.93	267.8	0.47	267.8	208	202	889.9	55.78
272.1	544.3	–1.8	126.66	–58.33	272.1	0.47	272.1	222	215	948.0	59.04

Table 3 suggests that the relationship between σ_0 and ε_0 for ÉI 437 is linear to strain $\Gamma = 0.47\%$ and can be considered linear with an engineering plastic-strain tolerance of 0.2% in the range $0.47 \le \Gamma \le 1.8\%$. When $\Gamma > 1.8\%$, the relationship is no longer linear, and when $\Gamma \approx 3.9\%$, the plastic component of the first strain invariant $\varepsilon_0^{(p)}$ becomes ~0.32%, which is out of the tolerance. Poisson's ratio v^* changes from $v = 0.26$ to $v^* = 0.37$. According to column 10, the relative change of volume θ is nonzero, as in the previous cases.

3. Let us now discuss the results from tests on Kh18N10T steel tubular specimens deformed by a tensile force P and an internal pressure p along a path slightly different from that at which $\sigma_{zz} = \sigma_{\varphi\varphi}$. The ratio $\sigma_{zz} / \sigma_{\varphi\varphi}$ varied from 0.9 to 1.01. The results are in columns 1–4 of Table 4. Here

$$\sigma_{zz} = \frac{P}{\pi D h} + \frac{pD}{4h} \tag{3.1}$$

are the axial stresses,

TABLE 6

σ_{zz}, MPa	$\sigma_{\varphi\varphi}$, MPa	$\varepsilon_{zz}\cdot 10^3$	$\varepsilon_{\varphi\varphi}\cdot 10^3$	$\varepsilon_{rr}\cdot 10^3$	S, MPa	ν^*	σ_0, MPa	$\varepsilon_0\cdot 10^4$	$\varepsilon_0^{(p)}\cdot 10^4$	$\Gamma\cdot 10^3$	$\theta\cdot 10^3$
1	2	3	4	5	6	7	8	9	10	11	12
333.2	121.5	12.07	–2.86	–3.64	168.6	0.40	151.6	19	14.9	8.86	5.50
385.7	151.5	26.58	–5.67	–8.89	194.3	0.43	179.1	40	35.8	19.62	11.69
439.5	174.5	53.14	–10.98	–19.50	221.3	0.46	204.7	76	70.6	39.71	21.27
470.8	188.7	74.22	–13.51	–28.15	236.9	0.46	219.8	109	103.3	55.36	29.87
495.1	195.1	91.59	–16.23	–35.27	249.4	0.47	230.1	134	128.1	68.41	35.99
526.8	212.9	115.48	–19.89	–44.46	265.0	0.47	246.6	170	164.5	86.13	44.69
551.9	217.9	142.4	–23.53	–55.18	278.0	0.46	256.6	212	206.1	106.12	53.96
565.5	228.1	157.34	–25.53	–60.77	284.5	0.46	264.5	237	230.5	117.09	59.27
578.9	239.2	172.64	–27.51	–66.86	290.9	0.46	272.7	261	254.4	128.43	64.13
594.2	245.6	194.09	–30.25	–75.43	298.6	0.46	279.9	295	288.0	144.34	70.64
605.2	250.7	217.38	–33.24	–84.76	304.1	0.46	285.3	331	324.4	161.64	77.16
617.9	255.4	249.95	–36.68	–95.83	310.5	0.45	291.1	391	384.5	184.94	88.71
638.6	261.9	282.59	–38.26	–105.50	321.0	0.43	300.2	463	455.6	207.4	103.38
653.0	263.3	307.23	–39.82	–113.38	328.5	0.42	305.4	513	506.1	224.64	112.86
656.3	263.6	331.65	–42.58	–121.34	330.3	0.42	306.6	559	551.8	242.02	120.24
661.4	266.0	372.89	–47.04	–134.76	332.8	0.41	309.1	637	629.6	271.34	132.01

$$\sigma_{\varphi\varphi} = \frac{pD}{2h} \tag{3.2}$$

are the hoop stresses, $\varepsilon_{zz} = \Delta l / l$ are the axial strains; and $\varepsilon_{\varphi\varphi} = \Delta D / D$ are the hoop strains. Here $D = 30.17$ mm, $h = 1.16$ mm, $l = 20$ mm. The strains ε_{zz} and $\varepsilon_{\varphi\varphi}$ were measured with a strain gauge described in [1].

In these tests, the maximum value of stress σ_{rr} is equal to $-p$. This stress is much less than the stresses $\sigma_{\varphi\varphi}$ and σ_{zz}. Therefore, the stresses σ_{rr} can be neglected compared with $\sigma_{\varphi\varphi}$ and σ_{zz}. Also we will neglect the part of the strain ε_{rr} corresponding to the stress σ_{rr}. The other part of ε_{rr} will be determined using the property of the material to contract in the transverse direction when stretched in the longitudinal direction. This property is evidenced by data obtained in tests on tubular specimens subject to simple tension at equal stress intensities, assuming that Poisson's ratio is independent of the type of loading. In this connection, we can use the following formula (different from that in [9]) to determine the strain ε_{rr} in tests on cylindrical specimens under combined loading:

$$\varepsilon_{rr} = -\nu^* (\varepsilon_{zz} + \varepsilon_{\varphi\varphi}), \tag{3.3}$$

TABLE 7

σ_{zz}, MPa	$\sigma_{\varphi\varphi}$, MPa	$\varepsilon_{zz}\cdot 10^3$	$\varepsilon_{\varphi\varphi}\cdot 10^3$	$\varepsilon_{rr}\cdot 10^3$	S, MPa	ν^*	σ_0, MPa	$\varepsilon_0\cdot 10^4$	$\varepsilon_0^{(p)}\cdot 10^4$	$\Gamma\cdot 10^3$	$\theta\cdot 10^3$
1	2	3	4	5	6	7	8	9	10	11	12
148.9	295.9	0.1	2.77	–0.93	148.0	0.32	148.3	6	2.9	1.91	1.94
176.9	353.8	0.2	6.24	–2.51	176.9	0.39	176.9	13	8.9	4.48	3.92
199.9	399.1	0.3	9.59	–4.09	199.5	0.41	199.7	19	14.6	6.99	5.76
229.7	457.7	0	29.96	–13.56	228.8	0.45	229.1	55	49.2	22.27	16.00
234.7	467.3	–0.1	50.71	–23.31	233.6	0.46	234.0	91	85.4	37.86	26.11
243.6	485.3	–0.2	57.80	–26.70	242.7	0.46	243.0	103	97.2	43.22	29.35
249.3	498.2	–0.3	60.60	–28.05	249.1	0.47	249.2	107	101.5	45.35	30.54
354.1	490.0	0.8	70.10	–32.92	253.0	0.46	281.4	127	119.9	52.52	35.70
380.6	480.1	1.8	72.27	–34.37	253.4	0.46	286.9	132	125.5	54.23	37.28
469.6	478.0	4.32	78.60	–38.39	273.6	0.46	315.9	148	140.9	59.20	41.67
494.5	477.5	7.35	83.95	–42.54	280.7	0.47	324.0	163	154.8	63.71	45.46
521.6	478.1	12.35	90.05	–47.67	289.4	0.47	333.2	182	174.5	69.05	50.91
547.2	478.2	18.74	97.05	–53.38	298.0	0.46	341.8	208	199.9	75.24	57.96
581.5	477.8	27.32	104.26	–60.58	310.2	0.46	353.1	237	228.2	82.48	65.71
606.9	478.4	40.13	112.05	–68.97	319.8	0.45	361.8	277	268.7	91.14	76.90
619.2	478.7	51.48	118.82	–75.25	324.6	0.44	366.0	317	308.1	98.53	87.89

where v^* is Poisson's ratio determined from tests on a tubular specimen stretched along the z-axis (column 9 in Table 2). The value of v^* depends on the level of stress state characterized by the tangential-stress intensity S:

$$S = (s_{ij}s_{ij} / 2)^{1/2}, \tag{3.4}$$

which is given by $S = \sqrt{[(\sigma_{zz} - \sigma_{\varphi\varphi})^2 + (\sigma_{zz} - \sigma_{rr})^2 + (\sigma_{\varphi\varphi} - \sigma_{rr})^2]/6}$ here.

Neglecting σ_{rr} compared with $\sigma_{\varphi\varphi}$ and σ_{zz}, we obtain $S = \sqrt{(\sigma_{zz}^2 + \sigma_{\varphi\varphi}^2 - \sigma_{zz}\sigma_{\varphi\varphi})/3}$. Its values are in column 6 of Table 4. Now, using these values of S and Table 2, we can find the necessary values of v^* by linear interpolation (column 7 in Table 4). Next, we calculate the strains ε_{rr} (3.3) (column 5). Columns 8 and 9 contain the values of $\sigma_0 = (\sigma_{zz} + \sigma_{\varphi\varphi})/3$ (2.3) and $\varepsilon_0 = (\varepsilon_{zz} + \varepsilon_{rr} + \varepsilon_{\varphi\varphi})/3$ (2.4). The values of $\varepsilon_0^{(p)}$, Γ (2.5), and θ (2.6) are in columns 10, 11, and 12, respectively. Analyzing the table, we conclude that the first invariants σ_0 and ε_0 are in linear relationship up to $\Gamma \approx 0.5\%$ with a plastic-strain tolerance of 0.2%. As Γ increases, this linearity is distorted; when $\Gamma \approx 10\%$, the plastic component of the first strain invariant $\varepsilon_0^{(p)}$ is equal to 3.6% approximately.

Table 5 summarizes the results from tests on tubular Kh18N10T steel specimens proportionally loaded by an internal pressure p alone, i.e., for $\sigma_{zz} = pD/4h$, $\sigma_{\varphi\varphi} = pD/(2h)$. The table is similar in structure to Table 4. The values of ε_{rr} are found by the same procedure as in Table 4 and the values of v^* are taken from Table 2. Analyzing Table 5, we conclude that the relationship between the first invariants σ_0 and ε_0 are linear up to $\Gamma \approx 0.26\%$ and can be considered linear with a plastic-strain tolerance of 0.2% up to $\Gamma = 0.86\%$. As Γ increases, the linearity is distorted; when $\Gamma \approx 9.5\%$, the plastic component of the first strain invariant $\varepsilon_0^{(p)}$ is equal to 2.2%.

Table 6 collects the results of tests on tubular Kh18N10T steel specimens loaded by a tensile force P and internal pressure p so that $\sigma_{zz} \approx 2.5\sigma_{\varphi\varphi}$ (the true values of $\sigma_{zz}/\sigma_{\varphi\varphi}$ are from 2.4 to 2.7). The table is similar in structure to Tables 4 and 5. The stresses and strains are calculated in the same way as in Tables 4 and 5; the values of v^* are taken from Table 2. Analyzing Table 6, we conclude that the relationship between the first invariants σ_0 and ε_0 can be considered linear with a plastic-strain tolerance of 0.2% up to $\Gamma \approx 1\%$. As Γ increases, the linearity is distorted; when $\Gamma \approx 27\%$, the plastic component of the first strain invariant $\varepsilon_0^{(p)}$ is equal to 6.3%.

Table 7 contains the results of tests on tubular Kh18N10T steel specimens subject to a tensile force P and internal pressure p varying in a prescribed manner [9]. The heat-treatment for the specimens are the same as in the previous tests on this steel. The specimens were subjected first to internal pressure that increased to $p = 38.2$ MPa and then, maintaining this level of pressure, to the tensile force P. These results are in columns 1–4. Here σ_{zz} (3.1) and $\sigma_{\varphi\varphi}$ (3.2) are the axial and hoop stresses, respectively; $\varepsilon_{zz} = \Delta l/l$ and $\varepsilon_{\varphi\varphi} = \Delta D/D$ are the axial and hoop strains, respectively. As in the previous tests on this steel, $D =$ 30.17 mm, $h = 1.16$ mm, and $l = 20$ mm. The strains ε_{zz} and $\varepsilon_{\varphi\varphi}$ were measured with a strain gauge described in [1]. The values of S (3.4) are in column 6. These values of S and Table 2 are then used to find, by linear interpolation, the values of v^* (column 7). These values of v^* are substituted into formula (3.3) to calculate ε_{rr} (its values are in column 5).

Columns 8 to 12 contain the values of σ_0 (1.8), ε_0 (1.9), $\varepsilon_0^{(p)}$, Γ (2.5), and θ (2.6) calculated by the formulas given above.

Analyzing Table 7, we conclude that the relationship between the first invariants σ_0 and ε_0 is linear up to $\Gamma \approx 0.2\%$ and can be considered linear with an engineering plastic-strain tolerance of 0.2% up to $\Gamma \approx 0.91\%$. As Γ increases, this linearity is distorted; when $\Gamma \approx 9.9\%$, the plastic component of the first strain invariant $\varepsilon_0^{(p)}$ is equal to 3% approximately.

Conclusions. Analyzing Tables 1–7, we draw the following conclusions.

The relationship between the first invariants of the stress and strain tensors σ_0 and ε_0 is linear with an engineering plastic-strain tolerance of 0.2% for different levels of strain intensity Γ. This level depends on the type of material and its stress state—it is much greater under uniaxial tension than under combined loading. For example, for Kh18N10T steel, this linearity remains up to $\Gamma = 8\%$ (Table 2) under uniaxial tension and up to $\Gamma = 1\%$ (Table 6) under combined loading.

Whether the relationship between the first invariants of the stress and strain tensors σ_0 and ε_0 is linear depends not only on the strain intensity Γ, but also on the material properties and the stress state of an element of a body—a tubular specimen (Tables 1–7).

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