PRINCIPLES OF THE MICROMECHANICS OF MATERIAL DAMAGE. 1. LONG-TERM DAMAGE*

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The principles of the theory of long-term damage based on the mechanics of stochastically inhomogeneous media are set out. The process of damage is modeled as randomly dispersed micropores resulting from the destruction of microvolumes. A failure criterion for a single microvolume is associated with its long-term strength dependent on the relationship of the time to brittle failure and the difference between the equivalent stress and the Huber–von Mises failure stress, which is assumed to be a random function of coordinates. The stochastic elasticity equations for porous media are used to determine the effective moduli and the stress–strain state of microdamaged materials. The porosity balance equation is derived in finite-time and differential-time forms for given macrostresses or macrostrains and arbitrary time using the properties of the distribution function and the ergodicity of the random field of short-term strength as well as the dependence of the time to brittle failure on the stress state and the short-term strength. The macrostress–macrostrain relationship and the porosity balance equation describe the coupled processes of deformation and long-term damage

Keywords: short-term damage, long-term damage, microstrength distribution, stochastically inhomogeneous media, destroyed microvolumes, porosity, porosity balance, effective moduli

Introduction. Abrupt failure of structural members under long-term loads that are less than the ultimate strength is usually associated with the occurrence and development of dispersed microdamages [5], which may produce main cracks. Physically, microdamages are considered as vacancies, microcracks, and microvoids, which reduce the effective or load-bearing portion of the material.

The available mathematical models describing the damage of materials can be split into three groups. The models of the first group are based on a certain idea about the microstructure of the material and microdamages in it represented by damaged structural elements such as microcracks or micropores [2, 8–11, 15, 17, 27]. The governing equations here are based on the equations of mechanics and certain failure mechanisms for individual structural elements. The models of the second group employ a formal damage parameter, whose physical meaning is not always indicated, and postulate an evolution equation to relate the damage rate and stresses [4–7]. The models of the third group assume that the measure of damage is defined by some thermodynamic parameters that, together with stresses and strains, satisfy the fundamental thermodynamic relations. This makes it possible to establish relationships among stresses, strains, and damage parameters [1, 3, 13, 14].

The informal first group, which is based on structural models and damage mechanisms, appears most adequate to real processes of microdamage. The nonuniformity of microstrength inherent in real materials in the form of probability distributions

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[8, 15, 17] enables explanation and description of short-term (instantaneous) damage occurring under high loads. Modeling microdamages by pores with the aid of the mechanics of stochastically inhomogeneous media made it possible to theoretically describe the coupled processes of deformation and short-term damage of homogeneous and composite materials [15–18] and to study them over a wide range of mechanical properties, including physical nonlinearity [19–24].

The objective of the present study is to construct a theory of long-term damage on the basis of models and methods of the mechanics of stochastically inhomogeneous media underlying the theory of short-term damage [15]. The damage process is modeled as destruction of microvolumes and occurrence of randomly dispersed micropores in their place. A failure criterion for a single microvolume is associated with its long-term strength dependent on the relationship of the time to brittle failure and the difference between the equivalent stress [5] and the short-term strength defined by the Huber–von Mises failure criterion. The short-term strength is assumed to be a random function of coordinates with power-law or Weibull one-point distribution. The effective elastic properties and stress–strain state of a material with randomly dispersed microdamages are determined using the stochastic equations of elasticity of porous media [12]. The damage balance equation is derived in finite-time and differential-time forms for given macrostresses or macrostrains and arbitrary time, using the properties of distribution functions and the ergodicity of the random field of short-term microstrength as well as the dependence of the time to brittle failure for a microvolume on its stress state and short-term microstrength. The macrostress–macrostrain relationship for a porous material and the porosity balance equation describe the coupled processes of deformation and long-term damage, which lead to an increase in macrostrains at given macrostresses and to a decrease in macrostresses at given macrostrains, and to nonlinear isochronic stress–strain curves.

1. Preliminaries. Fracture mechanics distinguishes short-term and long-term strengths. The former is characterized by a limiting homogeneous stress state in which the specimen fails instantaneously. If the stress state is not limiting, the specimen may collapse after a time essentially dependent on how close this stress state is to its limit. This dependence characterizes long-term strength. It is a more general characteristic than short-term strength, which is a limit point on the long-term strength curve corresponding to zero time to failure.

Long-term strength is usually interpreted as the result of accumulation of dispersed microdamages in the form of micropores, microcracks, and vacancies in atomic/molecular structure, etc. A macrospecimen fails if damage reaches some critical level.

The dimensions and shapes of dispersed submicrocracks in polymer materials and their dependence on loading conditions have been much studied [11]. The basic damage mechanisms in a number of polymers are the following. The dimensions of submicrocracks are almost independent of strain, stress, and time of loading. The ratio of the longitudinal (relative to the tension direction) dimension of a submicrocrack to the transverse one in different polymers varies from 0.4 to 1.3. A certain tensile strain corresponds to a certain relative fraction of submicrocracks, which increases with the strain. Submicrocracks begin to form only after some level of strain. Given tensile stress, submicrocracks accumulate with time, first quickly and then slower and slower. The rate of accumulation of submicrocracks and their relative fraction increase with time and with applied stress. If the stresses are small and not greater than half the breaking stress, submicrocracks do not accumulate for a long time.

These experimental patterns can be explained using statistical notions. At the microscopic level, the strength of a material is nonuniform, i.e., the ultimate short-term strength and long-term strength curves for a microvolume are random functions of coordinates with certain distribution density or cumulative distribution. When a macrospecimen is subject to a constant tensile stress, some microvolumes whose ultimate strength is less than this stress will be destroyed, i.e., microcracks or microcavities will form in their place. Microvolumes where the stress is less than, yet close to the ultimate strength will fail after some lapse of time, which depends on how close the applied stress is to the ultimate microstrength. After such microvolumes are exhausted, damages no longer build up. As the stress increases, microvolumes with higher microstrength are involved in microdamage, the process becoming more intensive as microstresses redistribute because the destroyed microvolumes have lost their load-bearing capacity.

Failure of microvolumes after a time is indicative of damage accumulation, but now at the atomic/molecular level (occurrence of vacancies or dislocations).

2. Structural Model of Long-Term Damage in Finite-Time Form. The structural theory of damage [15, 25] is based on the idea that microstrength is nonuniform, with the result that under loading damage occurs in dispersed microvolumes where ultimate microstrength is less than some combination of stresses defined by a failure criterion. Modeling the destroyed microvolumes by quasispherical pores, empty or filled with particles of a destroyed material, we obtain a porous material of stochastic structure whose damage is described as increasing porosity p. If the homogeneous stress state of a porous material is characterized by macrostresses $\langle\sigma_{ij}\rangle$, then the mean stresses $\langle\sigma^1_{ij}\rangle$ in the undestroyed portion of the material are defined by the following formula [12]:

$$\langle\sigma^1_{ij}\rangle = \frac{1}{1-p}\langle\sigma_{ij}\rangle. \tag{2.1}$$

The simplest way to describe the short-term damage of a microvolume in an isotropic material is the Huber–von Mises failure criterion

$$I^1_\sigma = \left(\langle\sigma^1_{ij}\rangle'\langle\sigma^1_{ij}\rangle'\right)^{1/2} = k, \tag{2.2}$$

where $\langle\sigma^1_{ij}\rangle'$ is the mean stress deviator in the undestroyed portion of the material, and k is the limiting value of the invariant I^1_σ, which is a random function of coordinates.

If the invariant I^1_σ does not achieve its limiting value k in some microvolume, then, according to the long-term failure criterion, failure will occur after some time τ_k, which depends on how close I^1_σ is to k. In the general case, this dependence can be represented by

$$\tau_k = \varphi(I^1_\sigma, k), \tag{2.3}$$

where $\varphi(k,k)=0$ and $\varphi(0,k)=\infty$, according to (2.2).

The one-point cumulative distribution function $F(k)$ of the ultimate strength k in a microvolume of the undamaged portion of the material can be approximated by a power-law function on some interval

$$F(k)=\begin{cases} 0, & k<k_0, \\ \left(\dfrac{k-k_0}{k_1-k_0}\right)^n, & k_0 \le k \le k_1, \\ 1, & k<k_0, \end{cases} \tag{2.4}$$

or by the Weibull function

$$F(k)=\begin{cases} 0, & k<k_0, \\ 1-\exp\left[-m(k-k_0)^n\right], & k \ge k_0, \end{cases} \tag{2.5}$$

where k_0 is the minimum value of the ultimate microstrength, and k_1, m, and n are constants determined by fitting experimental microstrength scatter curves or microdamage curves.

Assume that the random field of ultimate microstrength k is statistically homogeneous, which is typical of real materials, and the size of single microdamages and distance between them are negligible compared with the size of the macrovolume. Then there is ergodicity—the cumulative distribution function $F(k)$ defines the relative fraction of the undestroyed material in which the ultimate microstrength is less than k. Therefore, with nonzero stresses $\langle\sigma^1_{ij}\rangle$, the function $F(I^1_\sigma)$ defines, according to (2.2), (2.4), and (2.5), the relative fraction of instantaneously damaged microvolumes. Since destroyed microvolumes are modeled by pores, we can express the balance of destroyed microvolumes or porosity under short-term damage as follows:

$$p = p_0 + (1-p_0)F(I^1_\sigma), \tag{2.6}$$

where p_0 is the initial porosity. With (2.1), this equations can be rearranged as

$$p = p_0 + (1-p_0)\,F\left(\frac{1}{1-p}I_\sigma\right), \tag{2.7}$$

where $I_\sigma = (\langle\sigma_{ij}\rangle'\langle\sigma_{ij}\rangle')^{1/2}$ is the second invariant of the macrostress deviator $\langle\sigma_{ij}\rangle'$.

If the stresses $\langle\sigma^1_{ij}\rangle$ act for some time t, then, according to (2.3), microvolumes with the following values of k will fail in this time:

$$t \ge \tau_k = \varphi(I^1_\sigma, k). \tag{2.8}$$

The time to brittle failure τ_k for real materials at low temperatures is finite only starting with some value of $I^1_\sigma > 0$. In this case, the function $\varphi(I^1_\sigma, k)$ may be assumed to obey, for example, a fractional power law:

$$\varphi(I^1_\sigma, k) = \tau_0\left(\frac{k - I^1_\sigma}{I^1_\sigma - \gamma k}\right)^{n_1} \qquad \left(\gamma k \le I^1_\sigma \le k, \gamma < 1\right), \tag{2.9}$$

where τ_0, n_1, and γ are determined by fitting experimental long-term strength curves.

Substituting (2.9) into (2.8), we arrive at the inequality

$$k \le I^1_\sigma \frac{1 + \bar{t}^{1/n_1}}{1 + \gamma\bar{t}^{1/n_1}} \qquad \left(\bar{t} = \frac{t}{\tau_0}\right). \tag{2.10}$$

Considering the definition of the cumulative distribution function of ultimate microstrength $F(k)$, we conclude that the function $F[I^1_\sigma \psi(\bar{t})]$, where

$$\psi(\bar{t}) = \frac{1 + \bar{t}^{1/n_1}}{1 + \gamma\bar{t}^{1/n_1}}, \tag{2.11}$$

defines the relative fraction of destroyed microvolumes in the portion of material that is undamaged prior to loading at the time $\bar{t}$. Then given macrostresses $\langle\sigma_{ij}\rangle$, the equation of balance of destroyed microvolumes or porosity under long-term damage can be represented, in view of (2.1), as

$$p = p_0 + (1 - p_0) F\left[\frac{I_\sigma}{1 - p}\psi(\bar{t})\right], \tag{2.12}$$

where the porosity p is a function of the dimensionless time $\bar{t}$.

In fact, Eq. (2.12) is approximate, since the time to brittle failure τ_k is determined, according to (2.3), for constant stresses $\langle\sigma^1_{ij}\rangle$. If constant macrostress $\langle\sigma_{ij}\rangle$ are given, then the stresses $\langle\sigma^1_{ij}\rangle$ increase as damage, i.e., porosity p, builds up; therefore, the real time to failure for a microvolume will be a little longer than τ_k defined from (2.3) for constant stresses $\langle\sigma^1_{ij}\rangle$ at the time $\bar{t}$. Hence, we can assume that Eq. (2.12) overstates the porosity p for each time $\bar{t} > 0$.

If the time τ_k is finite for arbitrary I^1_σ, which is observed at high temperatures, the long-term strength function can be assumed to obey an exponential power law:

$$\varphi\left(I^1_\sigma, k\right) = \tau_0\left\{\exp m_1\left[\left(\frac{k}{I^1_\sigma}\right)^{n_1} - 1\right] - 1\right\}^{n_2}, \tag{2.13}$$

which has a sufficient number of constants τ_0, m_1, n_1, and n_2 to fit experimental curves. Substituting (2.13) into (2.8), we arrive at the inequality

$$k \le I^1_\sigma\left[1 + \frac{1}{m_1}\ln\left(1 + \bar{t}^{1/n_2}\right)\right]^{1/n_1}. \tag{2.14}$$

Then if the macrostresses $\langle\sigma_{ij}\rangle$ are given, the equation of porosity balance under long-term damage reads as (2.12), where

$$\psi(\bar{t})=\left[1+\frac{1}{m_1}\ln\left(1+\bar{t}^{1/n_2}\right)\right]^{1/n_1}. \tag{2.15}$$

For an isotropic material with damage characterized by porosity p, the macrostresses $\langle\sigma_{ij}\rangle$ and macrostrains $\langle\varepsilon_{ij}\rangle$ are in the following relationship:

$$\langle\sigma_{ij}\rangle=\left(K^*-\frac{2}{3}\mu^*\right)\langle\varepsilon_{rr}\rangle\delta_{ij}+2\mu^*\langle\varepsilon_{ij}\rangle, \tag{2.16}$$

where the effective bulk, K^*, and shear, μ^*, moduli are expressed in terms of the moduli K and μ of the undestroyed material and the porosity p, according to the theory of porous materials [12]:

$$K^*=\frac{4K\mu\,(1-p)^2}{4\mu+(3K-4\mu)},\qquad \mu^*=\frac{(9K+8\mu)\,\mu\,(1-p)^2}{9K+8\mu-(3K-4\mu)\,p}. \tag{2.17}$$

Then in view of the dependence $I_\sigma=2\mu^* I_\varepsilon$, where $I_\varepsilon=(\langle\varepsilon_{ij}\rangle'\langle\varepsilon_{ij}\rangle')^{1/2}$ is the second invariant of the macrostrain deviator $\langle\varepsilon_{ij}\rangle'$, the porosity balance equation (2.12) for given macrostrains $\langle\varepsilon_{ij}\rangle$ can be rearranged as

$$p=p_0+(1-p_0)\,F\left[\frac{2\mu^* I_\varepsilon}{1-p}\psi(\bar{t})\right]. \tag{2.18}$$

It follows from (2.1), (2.17), and (2.18) that if the macrostrains $\langle\varepsilon_{ij}\rangle$ are given, the stresses $\langle\sigma_{ij}^1\rangle$ decrease with increasing porosity; therefore, the time to brittle failure for a microvolume will be a little shorter than τ_k determined from (2.3) for constant stresses $\langle\sigma_{ij}^1\rangle$ at time $\bar{t}$. Hence, we can assume that Eq. (2.18) understates the porosity for each $\bar{t}>0$.

The porosity balance equations (2.12) and (2.18) can be corrected by introducing an additional constant α determined experimentally. To this end, when the invariant I_σ^1 changes with time from 0 to $\bar{t}$, its average value is substituted into (2.9) and (2.13), i.e., $I_\sigma^1(\alpha\bar{t})\,(0\le\alpha\le1)$ instead of $I_\sigma^1(\bar{t})$. The porosity balance equations (2.12) and (2.18) can now be written as

$$p(\bar{t})=p_0+(1-p_0)\,F\left[\frac{I_\sigma}{1-p(\alpha\bar{t})}\psi(\bar{t})\right] \tag{2.19}$$

if the macrostresses $\langle\sigma_{ij}\rangle$ are given and

$$p(\bar{t})=p_0+(1-p_0)F\left[\frac{2\mu^*(p(\alpha\bar{t}))I_\varepsilon}{1-p(\alpha\bar{t})}\psi(\bar{t})\right] \tag{2.20}$$

if the macrostrains $\langle\varepsilon_{ij}\rangle$ are given.

At the initial time $\bar{t}=0$, the porosity balance equations (2.12), (2.18)–(2.20) define the short-term (instantaneous) damage of a material. With increasing time, Eqs. (2.12), (2.18)–(2.20) define long-term damage that consists of short-term damage and additional damage developing with time.

3. Structural Model of Long-Term Damage in Differential-Time Form. As indicated above, the porosity balance equations (2.12) and (2.18) for given macrostresses $\langle\sigma_{ij}\rangle$ and macrostrains $\langle\varepsilon_{ij}\rangle$, respectively, are, in fact, approximate. This is because the relationship between the time to brittle failure and the acting stresses (2.3) is experimentally established for constant loads, whereas the invariant I_σ^1 changes with increasing damage. Though such relationships for real materials range wide in stresses and especially in time [5], which justifies the applicability of Eqs. (2.12) and (2.18), it is nevertheless of interest to derive more rigorous porosity balance equations that would be free from this shortcoming. This can be done by changing over to the differential-time form of the relationship between damage and macrostresses or macrostrains.

We will use the following relationship between the time to brittle failure τ_k for some microvolume with ultimate microstrength k and the constant invariant I_σ^1:

$$\tau_k = \tau_0 \frac{k - I_\sigma^1}{I_\sigma^1 - \gamma k} \qquad \left(\gamma k \le I_\sigma^1 \le k, \gamma < 1\right), \tag{3.1}$$

which is equivalent to specifying how quickly a microvolume loses its load-bearing capacity:

$$\frac{dr_k}{dt} = \frac{1}{\tau_0} \frac{I_\sigma^1 - \gamma k}{k - I_\sigma^1}. \tag{3.2}$$

The load-bearing capacity is completely lost at $r_k = 1$. The integration of (3.2) over time from 0 to τ_k at constant I_σ^1 yields formula (3.1).

In contrast to (3.1), expression (3.2) is valid even if the invariant I_σ^1 is time-dependent, which is the instantaneous rate of exhaustion of a microvolume's load-bearing capacity. Integrating (3.2) from zero to an infinitesimal time increment Δt and using the mean-value theorem, we obtain

$$\Delta r_k = \frac{I_\sigma^1\left(\beta \Delta \bar{t}\right) - \gamma k}{k - I_\sigma^1\left(\beta \Delta \bar{t}\right)} \Delta \bar{t} \qquad \left(\Delta \bar{t} = \frac{\Delta t}{\tau_0},\ 0 \le \beta \le 1\right). \tag{3.3}$$

If $\Delta r_k = 1$, a microvolume will be damaged in time $\Delta \bar{t}$. Therefore, from (3.3) we derive the following inequality accurate to $\Delta \bar{t}$:

$$k \le I_\sigma^1(0) + \left[\left(1 - \gamma\right) I_\sigma^1(0) + \beta \frac{dI_\sigma^1(0)}{d\bar{t}}\right] \Delta \bar{t}, \tag{3.4}$$

which is a condition whereby microvolumes with the corresponding ultimate microstrength will be destroyed. Then the porosity balance equation can be represented as

$$p + \frac{dp}{d\bar{t}} \Delta \bar{t} = p_0 + \left(1 - p_0\right)\left\{ F(I_\sigma^1) + F'(I_\sigma^1)\left[\left(1 - \gamma\right) I_\sigma^1 + \beta \frac{dI_\sigma^1}{d\bar{t}}\right] \Delta \bar{t} \right\}. \tag{3.5}$$

Since given macrostresses $\langle \sigma_{ij} \rangle$ or macrostrains $\langle \varepsilon_{ij} \rangle$, short-term or instantaneous damage defined by (2.6) occurs at the initial time, the differential-time porosity balance equation follows from (3.5) for small values of $\bar{t}$:

$$\frac{dp}{d\bar{t}} = (1 - p_0) f(I_\sigma^1) \left[\left(1 - \gamma\right) I_\sigma^1 + \beta \frac{dI_\sigma^1}{d\bar{t}}\right], \tag{3.6}$$

where $f(k) = F'(k)$ is the one-point microstrength density function. The short-term damage $p(0) = p_k$ should be set as the initial condition. The constants α and β, appearing in (3.6), allow the experimental long-term damage curves to be fitted as close as possible.

If the macrostresses $\langle \sigma_{ij} \rangle$ are given, then, according to (2.1), Eq. (3.6) takes the form

$$\frac{dp}{d\bar{t}} = \left(1 - p_0\right) f\left(\frac{I_\sigma}{1 - p}\right) \left\{ \left[\frac{1 - \gamma}{1 - p} + \frac{\beta}{(1 - p)^2} \frac{dp}{d\bar{t}}\right] I_\sigma + \frac{\beta}{1 - p} \frac{dI_\sigma}{d\bar{t}} \right\}. \tag{3.7}$$

If the macrostrains $\langle \varepsilon_{ij} \rangle$ are given, then, according to (2.16), Eq. (3.7) yields

$$\frac{dp}{d\bar{t}} = (1 - p_0) f\left[\kappa(p) I_\varepsilon\right] \left\{ \left[(1-\gamma)\kappa(p) + \beta\kappa'(p)\frac{dp}{d\bar{t}} \right] I_\varepsilon + \beta\kappa(p)\frac{dI_\varepsilon}{d\bar{t}} \right\}$$

$$\left(\kappa(p) = \frac{2\mu(9K + 8\mu)(1-p)}{9K + 8\mu - (3K - 4\mu)p} \right). \tag{3.8}$$

For a material with Poisson's ratio $\nu = 0.2$ in the undamaged portion, Eq. (3.8) takes the simplest form

$$\frac{dp}{d\bar{t}} = 2(1 - p_0) f\left[2\mu(1-p) I_\varepsilon\right] \left\{ \left[(1-\gamma)(1-p) - \beta\frac{dp}{d\bar{t}} \right] I_\varepsilon + \beta(1-p)\frac{dI_\varepsilon}{d\bar{t}} \right\}. \tag{3.9}$$

4. Uniform Microstrength Density Function. According to the finite-time model, the development of damage with time can be described by solving Eq. (2.12) for the porosity p if the macrostresses $\langle \sigma_{ij} \rangle$ are given and Eq. (2.18) if the macrostrains $\langle \varepsilon_{ij} \rangle$ are given. In the general case, these equations can be solved numerically. Its analytic solution can be found only in the special case where the microstrength has a uniform density function on some interval, which corresponds to $n = 1$ in (2.4). In this case, the porosity balance equation (2.12) takes the form

$$p = \begin{cases} p_0, & \dfrac{I_\sigma \psi(\bar{t})}{1-p} < k_0, \\ p_0 + \dfrac{1-p_0}{k_1 - k_0}\left[\dfrac{I_\sigma \psi(\bar{t})}{1-p} - k_0\right], & k_0 \le \dfrac{I_\sigma \psi(\bar{t})}{1-p} \le k_1, \\ 1, & \dfrac{I_\sigma \psi(\bar{t})}{1-p} > k_1, \end{cases} \tag{4.1}$$

i.e., we arrive at the quadratic equation

$$(k_1 - k_0)q^2 - k_1 q_0 q + q_0 I_\sigma \psi(\bar{t}) = 0 \quad (q = 1 - p,\ q_0 = 1 - p_0). \tag{4.2}$$

It has two roots of which one has a physical meaning:

$$q = \frac{1}{2(k_1 - k_0)}\left(k_1 q_0 + \sqrt{k_1^2 q_0^2 - 4(k_1 - k_0) q_0 I_\sigma \psi(\bar{t})} \right). \tag{4.3}$$

Hence, the range of macrostresses $\langle \sigma_{ij} \rangle$ for which formula (4.3) holds is defined by the inequalities

$$k_0 q_0 \le I_\sigma \psi(\bar{t}) \le \frac{k_1^2 q_0}{4(k_1 - k_0)}. \tag{4.4}$$

If the macrostrains $\langle \varepsilon_{ij} \rangle$ are given, the porosity balance equation (2.18) for $n = 1$ in (2.4) takes the form

$$p = \begin{cases} p_0, & \dfrac{2\mu I_\varepsilon \psi(\bar{t})}{1-p} < k_0, \\ p_0 + \dfrac{1-p_0}{k_1 - k_0}\left[\dfrac{2\mu^* I_\varepsilon \psi(\bar{t})}{1-p} - k_0\right], & k_0 \le \dfrac{2\mu^* I_\varepsilon \psi(\bar{t})}{1-p} \le k_1, \\ 1, & \dfrac{2\mu^* I_\varepsilon \psi(\bar{t})}{1-p} > k_1. \end{cases} \tag{4.5}$$

In view of (2.17), we obtain the quadratic equation

$$(k_1-k_0)\nu_2 q^2+\left[(k_1-k_0)\nu_1-k_1q_0\nu_2+2\mu q_0 I_\varepsilon\psi(\bar t)\right]q-k_1q_0\nu_1=0$$

$$\left(\nu_1=\frac{2(4-5\nu)}{7-5\nu},\quad \nu_2=\frac{5\nu-1}{7-5\nu}\right), \tag{4.6}$$

where ν is Poisson's ratio in the undamaged portion of the material.

Only one root of Eq. (4.6) has a physical meaning:

$$q=\frac{1}{2(k_1-k_0)\nu_2}\left\{k_1q_0\nu_2-(k_1-k_0)\nu_1-2\mu q_0I_\varepsilon\psi(\bar t)\right.$$

$$\left.+\left[(k_1-k_0)^2\nu_1^2+q_0^2\left(2\mu I_\varepsilon\psi(\bar t)-k_1\nu_2\right)^2+2(k_1-k_0)q_0\nu_1\left(2\mu I_\varepsilon\psi(\bar t)+k_1\nu_2\right)\right]^2\right\}, \tag{4.7}$$

where the range of macrostrains $\langle\varepsilon_{ij}\rangle$ is defined by the inequalities

$$\frac{k_0(\nu_1+\nu_2q_0)}{2\mu q_0}\le I_\varepsilon\psi(\bar t)<\infty. \tag{4.8}$$

For a material with Poisson's ratio $\nu = 0.2$ in the undamaged portion, expressions (4.7) and (4.8) become simpler

$$q=\frac{k_1q_0}{k_1-k_0+2\mu q_0I_\varepsilon\psi(\bar t)}\qquad\left(\frac{k_0}{2\mu q_0}\le I_\varepsilon\psi(\bar t)<\infty\right). \tag{4.9}$$

According to the differential-time long-term damage model, the porosity balance equation (3.7) has the following form for uniform microstrength density function k on the interval (k_0, k_1) and given constant macrostresses $\langle\sigma_{ij}\rangle$:

$$\frac{dp}{dt}=\begin{cases}0, & \dfrac{I_\sigma}{1-p}<k_0,\\ \dfrac{1-p_0}{k_1-k_0}\left[\dfrac{1-\gamma}{1-p}+\dfrac{\beta}{(1-p)^2}-\dfrac{dp}{d\bar t}\right]I_\sigma, & k_0\le\dfrac{I_\sigma}{1-p}\le k_1,\\ 1, & \dfrac{I_\sigma}{1-p}>k_1.\end{cases} \tag{4.10}$$

Introducing the notation $q=1-p$, $q_0=1-p_0$, $q_k=1-p_k$, we find the solution

$$\frac{1}{2}\left(q_k^2-q^2\right)-\frac{\beta q_0I_\sigma}{k_1-k_0}\ln\frac{q_k}{q}=\frac{(1-\gamma)I_\sigma}{k_1-k_0}\bar t, \tag{4.11}$$

which holds if

$$\frac{I_\sigma}{k_1}\le q\le\frac{I_\sigma}{k_0}. \tag{4.12}$$

For a small difference q_k-q, expanding $\ln(q_k/q)$ into a series and retaining two terms, we arrive at the quadratic equation

$$\left[\beta I_\sigma+(k_1-k_0)\right]\frac{q^2}{q_k^2}-4\beta I_\sigma\frac{q}{q_k}+3\beta I_\sigma-(k_1-k_0)q_k^2+2(1-\gamma)I_\sigma\bar t=0. \tag{4.13}$$

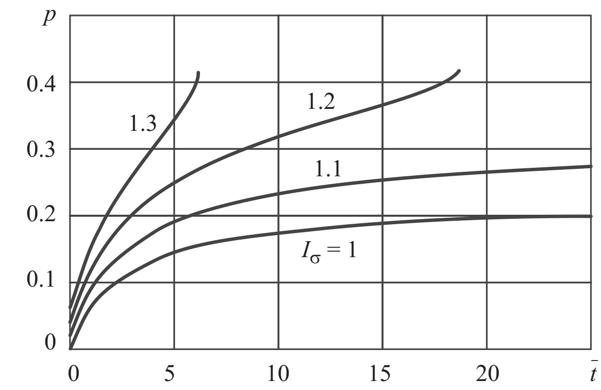

Fig. 1

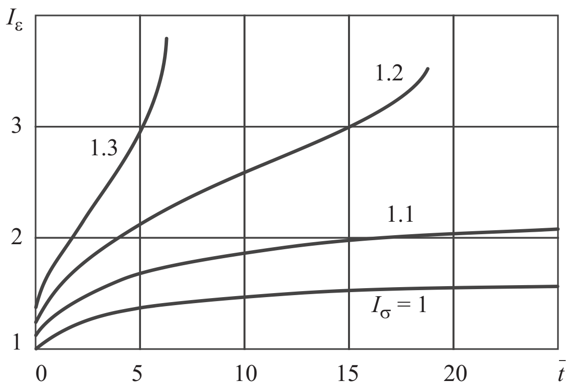

Fig. 2

One of its roots has a physical meaning:

$$q = \frac{q_k}{\beta I_\sigma + (k_1 - k_0)\, q_k^2} \left\{ 2\beta I_\sigma + \sqrt{\left[(k_1 - k_0) q_k^2 - \beta I_\sigma\right]^2 - 2(1-\gamma) I_\sigma \left[\beta I_\sigma + (k_1 - k_0) q_k^2\right] \bar{t}} \right\}. \tag{4.14}$$

For given constant macrostrains $\langle \varepsilon_{ij} \rangle$ and uniform microstrength density function k on the interval (k_0, k_1), the porosity balance equation (3.9) takes the form

$$\frac{dp}{dt} = \begin{cases} 0, & 2\mu(1-p) I_\varepsilon < k_0, \\ \dfrac{2(1-p_0)\mu}{k_1 - k_0} \left[(1-\gamma)(1-p) - \beta \dfrac{dp}{d\bar{t}}\right] I_\varepsilon, & k_0 \le 2\mu(1-p) I_\varepsilon \le k_1, \\ 0, & 2\mu(1-p) I_\varepsilon > k_1. \end{cases} \tag{4.15}$$

Its solution is

$$q = q_k \exp\left(- \frac{2 q_0 (1-\gamma) \mu I_\varepsilon}{k_1 - k_0 + 2 q_0 \beta \mu I_\varepsilon} \bar{t} \right), \tag{4.16}$$

which holds if

$$k_0 \le 2\mu q I_\varepsilon \le k_1. \tag{4.17}$$

It follows from (4.16), (4.17) that damage has a limiting value:

$$p_n = 1 - \frac{k_0}{2\mu I_\varepsilon}, \tag{4.18}$$

which is reached in a finite time defined by

$$\bar{t}_n = \frac{k_1 - k_0 + 2 q_0 \beta \mu I_\varepsilon}{2 q_0 \alpha \mu I_\varepsilon} \ln \frac{2 q_k \mu I_\varepsilon}{k_0}. \tag{4.19}$$

Figures 1 and 2 show (solid lines) the porosity (damage) p and invariant $\bar{I}_\varepsilon = 2\bar{\mu} I_\varepsilon = \bar{I}_\sigma / (1-p)^2$ ($\bar{I}_\sigma = I_\sigma / k_0$, $\bar{\mu} = \mu / k_0$) as functions of time for given macrostresses $\langle \sigma_{ij} \rangle$, calculated by (2.11), (2.17), (4.3) for $p_0 = 0, \nu = 0.2, n_1 = 1, \gamma = 0.5$, $\bar{k}_1 = k_1 / k_0 = 8, \bar{I}_\sigma = 1$, 1.1, 1.2, 1.3. As is seen, for $\bar{I}_\sigma < 8/7$, damage has a horizontal asymptote, as on the experimental curves for polymers [11]. For $\bar{I}_\sigma \ge 8/7$ and times at which the radicand in (4.3) vanishes, damage reaches a critical level.

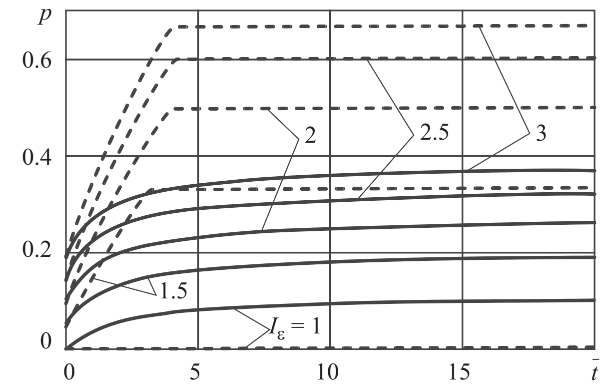

Fig. 3

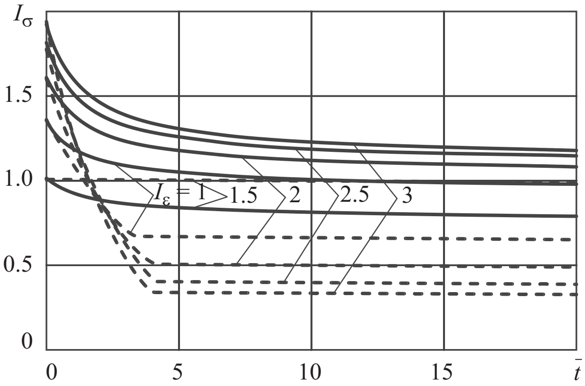

Fig. 4

Figures 3 and 4 show the porosity (damage) p and the invariant $\bar{I}_\sigma$ as functions of time for given macrostrains $\langle \varepsilon_{ij} \rangle$, calculated by (2.11), (2.17), and (4.8) for $p_0 = 0, \nu = 0.2, n_1 = 1, \gamma = 0.5, \bar{k}_1 = k_1 / k_0 = 8, \bar{I}_\varepsilon = 1$, 1.5, 2, 2.5, 3. It can be seen that here damage also builds up with time, whereas experiments on polymers [11] showed no noticeable changes in damage under constant strains. This can be attributed to both stress relaxation in polymers due to creep, which is neglected here, and the approximateness of the finite-time damage model.

According to the differential-time long-term damage model, for given macrostresses $\langle \sigma_{ij} \rangle$, the time dependence of damage (4.11), at least for a small difference $q_k - q$, leads to curves similar to those in Figs. 1 and 2 without horizontal asymptote. It is beyond reason to speak about the existence of a horizontal asymptote because Eq. (3.6) may fail for long times $\bar{t}$.

Figures 3 and 4 show (dashed lines) the porosity (damage) p and the invariant $\bar{I}_\sigma$ as functions of time for given macrostrains $\langle \varepsilon_{ij} \rangle$, calculated by (4.16) for $p_0 = 0, \nu = 0.2, \gamma = 0.5, \bar{k}_1 = k_1 / k_0 = 8, \bar{I}_\varepsilon = 1$, 1.5, 2, 2.5, 3. As is seen, damage changes in rather short finite times $\bar{t}_n$, according to (4.19). This is to a degree consistent with the experimental results for polymers [11], where damage does not noticeably change under constant strains.

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