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STRESS STATE OF COMPOUND SOLIDS OF REVOLUTION MADE OF DAMAGED ORTHOTROPIC MATERIALS WITH DIFFERENT TENSILE AND COMPRESSIVE MODULI

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A technique is proposed to allow for damages and different tensile and compressive moduli of orthotropic materials in stress–strain analysis of compound bodies of revolution under nonaxisymmetric loading and heating. The technique combines the semi-analytic finite-element method and the method of successive approximations

Keywords: nonaxisymmetric thermostressed state, solid of revolution, orthotropic materials with different tensile and compressive moduli, microdamage

Introduction. Many important engineering problems involve thermostress analysis of various structural members to infer their reliability, performance, and endurance. Efficient methods were developed in [3, 7, 11, etc.] to determine the thermoelastoplastic stress–strain state of bodies of various shapes. The accumulation of damages in viscoplastic isotropic materials is described by introducing a scalar parameter [2, 4, 10, 12, 14, 15, etc.]. This parameter is determined from a kinematic equation that relates its time derivative with some equivalent stress. Such an approach reduces the process of damage to loosening of a microvolume. Another approach to the description of damage is to use a structural model [9, 17, 18, etc.] based on stochastic equations for microinhomogeneous materials. Here, dispersed microdamages are modeled by quasispherical micropores that may be filled with particles of destroyed materials, and the accumulation of microdamages during deformation is modeled as increased porosity. Both approaches assume that all area elements associated with one point are equally damaged. Actually, the distribution of damages over a deformed element under combined stress was experimentally shown to be anisotropic. This can drastically affect the behavior of the material. Therefore, only one measure of damage accumulation may appear insufficient in many cases to correctly identify the onset of failure if for no other reason than different failure mechanisms provided by tangential and normal stresses. In this connection, the papers [5, 19] propose to describe damages in orthotropic materials by introducing six damage parameters, one per each area element. These parameters account for changes in the initial structure of a material; nucleation, development, and coalescence of pores; and formation of microdefects during deformation, which decreases the effective area of sections over which the stress components are distributed. Also, these parameters help to explain the nonlinearity of tension, compression, torsion, and shear curves. A literature survey indicates that the numerical analysis of deformational damage in composite materials has received inadequate development.

Although the curve of interatomic forces passes through zero smoothly, modern composite materials may have, at the macrolevel, different elastic moduli in tension and compression because of the presence of internal pores, inclusions, and other defects [1, 8, 13, 16, etc.]. In some fibrous and particulate materials, Young's moduli in tension (E^+) and compression (E^-) may differ by even 50%. The tensile and compression stress–strain curves of such materials may also differ significantly.

Methods to analyze the stress–strain state of structural members made of materials with different tensile and compressive moduli are outlined in [1, 6, 20]. Studies on damage of such materials are unavailable.

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1. Here we will outline a technique for analyzing the strain state and microdamage of compound solids of revolution made of elastic cylindrically orthotropic materials with different tensile and compressive moduli and subject to nonaxisymmetric loading and heating. Let us consider, using cylindrical coordinates z, r, φ, an elastic orthotropic compound solid of revolution subject to volume ($\vec{K}(K_z, K_r, K_\varphi)$) and surface ($\vec{t}_n(t_{nz}, t_{nr}, t_{n\varphi})$) forces and nonstationary heating. By a compound body is meant a discretely homogeneous solid of revolution whose component parts are solids of revolution too. For both the whole body and its individual components, there is a common axis of revolution aligned with the z-axis. The component parts are fastened together without interference at a temperature T_0 so as to provide perfect mechanical and thermal contact at the interfaces. The materials of the bodies resist tension and compression differently. The principal axes of anisotropy are aligned with the cylindrical coordinate axes. Thus, we will examine cylindrically orthotropic materials. The level of loading is assumed such that the materials do not display rheological properties, though their mechanical characteristics are dependent on temperature.

The process of loading and heating is divided into short time intervals so that their ends concur, whenever possible, with the moments the damage parameters stop increasing. The thermostressed state of such solids is determined by solving the heat-conduction problem to find the temperature T and the thermoelastic problem to find the displacements u_i, strains ε_{ij}, and stresses σ_{ij} $(i, j = z, r, \varphi)$ at chosen points in time.

The behavior of a structural member made of a composite material is usually described neglecting its heterogeneous structure and using anisotropic elasticity theory.

For an anisotropic material with equal elastic moduli in tension and compression, the existence condition for a positive definite potential-energy function leads to the relation $\nu_{ij} / E_i = \nu_{ji} / E_j$ and a symmetric compliance matrix. If the tensile and compressive moduli are different, this relation between Poisson's ratios and Young's moduli is invalid and the compliance matrix remains asymmetric [1]. It can be made symmetric by assigning relations between the tensile and compressive material constants such that the well-known transformations of anisotropic elasticity theory hold. In the present paper, the matrix is symmetrized by summing the tensile and compressive entries of the compliance matrix in proportion to the respective tensile and compressive stresses and using some weighting coefficients to correct for the sign of the normal stresses in two perpendicular directions. Though this approach has not been theoretically justified, it allows us to symmetrize the compliance matrix and to use anisotropic elasticity theory to solve the problem posed. With such an approach [20], the compliance matrix has the following elements, depending on the sign of stresses:

$$\frac{1}{E_z} = \begin{cases} \dfrac{1}{E_z^-}, & \sigma_{zz} < 0, \\ \dfrac{1}{E_z^+}, & \sigma_{zz} > 0, \end{cases}$$

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$$\frac{\nu_{zr}}{E_z} = \frac{\nu_{rz}}{E_r} = \begin{cases} \dfrac{\nu_{zr}^+}{E_z^+}, & \sigma_{zz} > 0,\ \sigma_{rr} > 0, \\ \dfrac{\nu_{zr}^-}{E_z^-}, & \sigma_{zz} < 0,\ \sigma_{rr} < 0, \end{cases}$$

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$$\frac{\nu_{zr}}{E_z} = \begin{cases} \dfrac{\nu_{rz}}{E_z} = \dfrac{|\sigma_{zz}|}{|\sigma_{zz}| + |\sigma_{rr}|} \dfrac{\nu_{zr}^+}{E_z^+} + \dfrac{|\sigma_{rr}|}{|\sigma_{zz}| + |\sigma_{rr}|} \dfrac{\nu_{rz}^-}{E_r^-}, & \sigma_{zz} > 0,\ \sigma_{rr} < 0, \\ \dfrac{\nu_{rz}}{E_r} = \dfrac{|\sigma_{zz}|}{|\sigma_{zz}| + |\sigma_{rr}|} \dfrac{\nu_{zr}^-}{E_z^-} + \dfrac{|\sigma_{rr}|}{|\sigma_{zz}| + |\sigma_{rr}|} \dfrac{\nu_{rz}^+}{E_r^+}, & \sigma_{zz} < 0,\ \sigma_{rr} > 0, \end{cases}$$

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$$\frac{1}{G_{zr}} = \begin{cases} \dfrac{1}{G_{zr}^{+}} = \dfrac{1}{E_{zr}^{45+}} - \left(\dfrac{1}{E_z^{+}} + \dfrac{1}{E_r^{+}} - \dfrac{\nu_{zr}^{+}}{E_z^{+}} \right), & \sigma_{zr} > 0, \\ \dfrac{1}{G_{zr}^{-}} = \dfrac{1}{E_{zr}^{45-}} - \left(\dfrac{1}{E_z^{-}} + \dfrac{1}{E_r^{-}} - \dfrac{\nu_{zr}^{-}}{E_z^{-}} \right), & \sigma_{zr} < 0. \end{cases} \tag{1}$$

Resolving the stress–strain relations with a symmetric compliance matrix for the stress components, we obtain

$$\sigma_{ij} = A_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^{\mathrm{T}}). \tag{2}$$

The coefficients $A_{ijkl} \, (i, j, k, l = z, r, \varphi)$ are expressed as follows [3, 21]:

$$A_{zzzz} = \Delta_{11} / \Delta, \quad A_{zzrr} = A_{rrzz} = \Delta_{12} / \Delta, \quad A_{zz\varphi\varphi} = A_{\varphi\varphi zz} = \Delta_{13} / \Delta,$$

$$A_{rrrr} = \Delta_{22} / \Delta, \quad A_{rr\varphi\varphi} = A_{\varphi\varphi rr} = \Delta_{23} / \Delta, \quad A_{\varphi\varphi\varphi\varphi} = \Delta_{33} / \Delta,$$

$$A_{zrzr} = G_{zr}, \quad A_{z\varphi z\varphi} = G_{z\varphi}, \quad A_{r\varphi r\varphi} = G_{r\varphi},$$

$$A_{zzzr} = A_{zzz\varphi} = A_{zzr\varphi} = A_{rrzr} = A_{rrz\varphi} = A_{rrr\varphi} = A_{\varphi\varphi zr} = A_{\varphi\varphi z\varphi} = A_{\varphi\varphi r\varphi} = A_{zrzz} = A_{zrrr} = A_{zr\varphi\varphi}$$

$$= A_{zrz\varphi} = A_{zrr\varphi} = A_{z\varphi zz} = A_{z\varphi rr} A_{z\varphi\varphi\varphi} = A_{z\varphi zr} = A_{z\varphi r\varphi} = A_{r\varphi zz} = A_{r\varphi rr} = A_{r\varphi\varphi\varphi} = A_{r\varphi zr} = A_{r\varphi z\varphi} = 0, \tag{3}$$

where

$$\Delta_{11} = \left(\frac{1}{E_\varphi} - \frac{\nu_{r\varphi}^2}{E_r} \right) \Big/ E_r, \quad \Delta_{12} = \left(\frac{\nu_{z\varphi}\nu_{r\varphi}}{E_r} + \frac{\nu_{zr}}{E_\varphi} \right) \Big/ E_z, \quad \Delta_{13} = \frac{\nu_{zr}\nu_{r\varphi} + \nu_{z\varphi}}{E_z E_r},$$

$$\Delta_{22} = \left(\frac{1}{E_\varphi} - \frac{\nu_{z\varphi}^2}{E_z} \right) \Big/ E_z, \quad \Delta_{23} = \left(\frac{\nu_{zr}\nu_{z\varphi}}{E_z} + \frac{\nu_{r\varphi}}{E_r} \right) \Big/ E_z, \quad \Delta_{33} = \left(\frac{1}{E_r} - \frac{\nu_{zr}^2}{E_z} \right) \Big/ E_z,$$

$$\Delta = \left(\Delta_{11} - \nu_{zr}\Delta_{12} - \nu_{z\varphi}\Delta_{13} \right) \Big/ E_z,$$

$$\varepsilon_{zz}^{\mathrm{T}} = \alpha_{zz}^{\mathrm{T}} (T - T_0), \quad \varepsilon_{rr}^{\mathrm{T}} = \alpha_{rr}^{\mathrm{T}} (T - T_0), \quad \varepsilon_{\varphi\varphi}^{\mathrm{T}} = \alpha_{\varphi\varphi}^{\mathrm{T}} (T - T_0), \quad \varepsilon_{zr}^{\mathrm{T}} = \varepsilon_{z\varphi}^{\mathrm{T}} = \varepsilon_{r\varphi}^{\mathrm{T}} = 0, \tag{4}$$

where E_i are Young's moduli along the principal axes of anisotropy aligned with the coordinate axes z, r, φ; G_{ij} are the shear moduli in the coordinate planes; ν_{ij} are Poisson's ratios characterizing compression along the X_j-axis after tension along the X_i-axis; and α_{ii}^{T} are the thermal-expansion coefficients along the principal axes of anisotropy.

In what follows, we will allow for the effect of damage on the deformation processes at each step of loading by introducing six damage parameters to decrease the effective areas of sections on which the associated stress components act and by using true stresses instead of engineering stresses [5, 19]

$$\tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{1 - \omega_{ij}^{\mathrm{D}}}, \tag{5}$$

where σ_{ij} are the engineering stress components, i.e., stresses divided by the original, undamaged elementary areas; and ω_{ij}^{D} are the damage parameters.

In view of (5), the stress–strain relations (2) can be written in the form of Hooke's law for a homogeneous material. To this end, the coefficients A_{ijkl} are represented as $A_{ijkl} = A_{ijkl}^0 (1 - \omega_{ijkl})$, where A_{ijkl}^0 are the coefficients A_{ijkl} averaged over φ, and $A_{ijkl}^0 \omega_{ijkl}$ are functions indicating the deviation of A_{ijkl} from the average values. Then the stress–strain relations become

$$\begin{Bmatrix} \sigma_{zz} \\ \sigma_{rr} \\ \sigma_{\varphi\varphi} \\ \sigma_{zr} \\ \sigma_{z\varphi} \\ \sigma_{r\varphi} \end{Bmatrix} = \begin{bmatrix} A^0_{zzzz} & A^0_{zzrr} & A^0_{zz\varphi\varphi} & 0 & 0 & 0 \\ A^0_{zzrr} & A^0_{rrrr} & A^0_{rr\varphi\varphi} & 0 & 0 & 0 \\ A^0_{zz\varphi\varphi} & A^0_{rr\varphi\varphi} & A^0_{\varphi\varphi\varphi\varphi} & 0 & 0 & 0 \\ 0 & 0 & 0 & A^0_{zrzr} & 0 & 0 \\ 0 & 0 & 0 & 0 & A^0_{z\varphi z\varphi} & 0 \\ 0 & 0 & 0 & 0 & 0 & A^0_{r\varphi r\varphi} \end{bmatrix} \times \begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{rr} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{zr} \\ \varepsilon_{z\varphi} \\ \varepsilon_{r\varphi} \end{Bmatrix} - \begin{Bmatrix} \sigma^*_{zz} \\ \sigma^*_{rr} \\ \sigma^*_{\varphi\varphi} \\ \sigma^*_{zr} \\ \sigma^*_{z\varphi} \\ \sigma^*_{r\varphi} \end{Bmatrix}, \tag{6}$$

where

$$\sigma^*_{zz} = A_{zzzz}\varepsilon^{\mathrm{T}}_{zz} + A_{zzrr}\varepsilon^{\mathrm{T}}_{rr} + A_{zz\varphi\varphi}\varepsilon^{\mathrm{T}}_{\varphi\varphi} + A^0_{zzzz}\omega_{zzzz}\varepsilon_{zz} + A^0_{zzrr}\omega_{zzrr}\varepsilon_{zr} + A^0_{zz\varphi\varphi}\omega_{zz\varphi\varphi}\varepsilon_{\varphi\varphi}$$

$$+\omega^{\mathrm{D}}_{zz}[A_{zzzz}(\varepsilon_{zz} - \varepsilon^{\mathrm{T}}_{zz}) + A_{zzrr}(\varepsilon_{rr} - \varepsilon^{\mathrm{T}}_{rr}) + A_{zz\varphi\varphi}(\varepsilon_{\varphi\varphi} - \varepsilon^{\mathrm{T}}_{\varphi\varphi})],$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sigma^*_{r\varphi} = 2(A^0_{r\varphi r\varphi}\omega_{r\varphi r\varphi} + A_{r\varphi r\varphi}\omega^{\mathrm{D}}_{r\varphi})\varepsilon_{r\varphi}. \tag{7}$$

These relations are nonlinear. Their nonlinearity can be explained as follows: since the material properties depend on the stress state, the material characteristics involved in the stress–strain analysis are unknown, and the damage parameters ω^{D}_{ij} and functions $A^0_{ijkl}\omega_{ijkl}$, on which the stresses depend, are in turn dependent on the stress state. They will be linearized by the method of successive approximations. The elements of the compliance matrix, the damage parameters ω^{D}_{ij}, and the functions $A^0_{ijkl}\omega_{ijkl}$ will be calculated from the previous approximation, using six instantaneous thermomechanical surfaces $\sigma_{ij} = F_{ij}(\varepsilon_{ij}, T)$ obtained in tension (compression) or torsion (shear) tests.

The following algorithm may be followed to calculate the damage parameters for a cylindrically orthotropic material with different tensile and compressive moduli. In each approximation, calculate the mechanical characteristics of the orthotropic material in tension ($E^+, 2G^+, \nu^+_{ij}$) and compression ($E^-, 2G^-, \nu^-_{ij}$) at the temperature of the current step of loading. Depending on the sign and magnitude of the stress components, determine the coefficients A_{ijkl} and, hence, the functions $A^0_{ijkl}\omega_{ijkl}$ using formulas (1), (3)–(6). Set the parameters ω^{D}_{ij} at zero in the first approximation of the first step of loading. In the first approximation of any subsequent step, put these parameters equal to their values in the last approximation of the previous step. In each approximation of a step, the damage parameters take their values from the previous approximation and cannot be less than their values in the last approximation of the previous step, i.e., damage is assumed not to decrease. Solve the elastic problem with additional stresses in the first approximation of the *k*th step of loading to obtain the distribution of stresses σ_{ij} and strains ε_{ij} in each element of the body and calculate the strains $\varepsilon^*_{ij} = \sigma_{ij}/[E_{ij}(1-\omega^{\mathrm{D}(K-1)}_{ij})]$ corresponding to stresses σ_{ij} under uniaxial loading. Here $E_{ij} = E_i$ if $i = j$ and $E_{ij} = 2G_{ij}$ if $i \neq j$, and $\omega^{\mathrm{D}(K-1)}_{ij}$ are the values of the damage parameters at the previous step of loading or at the end of the previous approximation. Next, use the curves $\sigma_{ij} = F_{ij}(\varepsilon_{ij})$ for the temperature T_K of the *m*th stage (which are plotted after linear interpolation of the instantaneous thermomechanical surfaces $\sigma_{ij} = F_{ij}(\varepsilon_{ij}, T)$ with respect to temperature) to find pairs σ^{C}_{ij}, $\varepsilon^{\mathrm{C}}_{ij}$ such that $(\sigma^{\mathrm{C}}_{ij} \cdot \varepsilon^{\mathrm{C}}_{ij}) = (\sigma_{ij} \cdot \varepsilon^*_{ij})_K$ for each curve. Use these pairs σ^{C}_{ij}, $\varepsilon^{\mathrm{C}}_{ij}$ to determine the damage parameters ω^{D}_{ij} using the formulas $(\omega^{\mathrm{D}}_{ij})_K = 1 - \sigma^{\mathrm{C}}_{ij}/(E_{ij}\varepsilon^{\mathrm{C}}_{ij})$. With these damage parameters, again solve the boundary-value problem to determine the stress and strain components and calculate new values of ω^{D}_{ij}.

The process stops once the values of the strain energy $U = 0.5\sigma_{ij}(\varepsilon_{ij} - \varepsilon^{\mathrm{T}}_{ij})$ calculated for each element of the body in two successive approximations differ by less than a prescribed tolerance.

Assigning the temperature and displacement components to be unknowns, we will seek the solution in the form of trigonometric series in the circumferential coordinate:

$$T(z,r,\varphi,t)=\sum_{m=0}^{\infty}\bar{T}_m(z,r,t)\cos m\varphi+\sum_{m=1}^{\infty}\bar{\bar{T}}_m(z,r,t)\sin m\varphi\,, \tag{8}$$

$$u_z(z,r,\varphi,t)=\sum_{m=0}^{\infty}\bar{u}_z^{(m)}(z,r,t)\cos m\varphi+\sum_{m=1}^{\infty}\bar{\bar{u}}_z^{(m)}(z,r,t)\sin m\varphi\ \ (z,r),$$

$$u_\varphi(z,r,\varphi,t)=\sum_{m=1}^{\infty}\bar{u}_\varphi^{(m)}(z,r,t)\sin m\varphi+\sum_{m=0}^{\infty}\bar{\bar{u}}_\varphi^{(m)}(z,r,t)\cos m\varphi\,, \tag{9}$$

whose coefficients are determined using finite elements in the meridional section of the body.

This approach reduces the three-dimensional problem to a sequence of two-dimensional variational problems for the unknown coefficients of (8) and (9). The finite elements in the meridional section are triangular ones within which the coefficients vary linearly.

Repeating the derivations made in [3], we arrive at the following recurrence formulas to determine the coefficients $\bar{T}_m$ at the nodes (i,j,k) of finite elements whose side ij lies on the surface of the body:

$$\bar{T}_{mi}(t+\Delta t)=\bar{T}_{mi}(t)+\frac{\Delta t}{\sum_{q=1}^{M}\langle c\rho\rangle_q H_i^{(q)}}\sum_{q=1}^{M}\Big[A_{ij}\bar{\theta}_{mi}(t+\Delta t)+B_{ij}\bar{\theta}_{mj}(t+\Delta t)$$

$$-(D_{ii}+m^2N_{ii}+A_{ij})\bar{T}_{mi}(t)-(D_{ij}+m^2N_{ij}+B_{ij})\bar{T}_{mj}(t)-(D_{ik}+m^2N_{ik})\bar{T}_{mk}(t)$$

$$+L_i(\bar{q}_z^{*(m)}(t)-\bar{\bar{q}}_{z\varphi}^{*(m)}(t))_i+P_i(\bar{q}_r^{*(m)}(t)-\bar{\bar{q}}_{r\varphi}^{*(m)}(t))_i-mR_i\bar{q}_{\varphi i}^{*(m)}(t)\Big]_q\qquad(i=1,2,\ldots,N) \tag{10}$$

in the case of an explicit difference scheme of solving the heat-conduction problem (these formulas allow us to calculate $\bar{T}_m$ at the time $t+\Delta t$ from their values at the time t) and

$$\sum_{q=1}^{M}\Big[(D_{ii}+m^2N_{ii}+\frac{1}{\Delta t}\langle c\rho\rangle H_i+A_{ij})\bar{T}_{mi}(t+\Delta t)+(D_{ij}+m^2N_{ij}+B_{ij})\bar{T}_{mj}(t+\Delta t)$$

$$+(D_{ik}+m^2N_{ik})\bar{T}_{mk}(t+\Delta t)\Big]_q=\sum_{q=1}^{M}\Big[\frac{1}{\Delta t}\langle c\rho\rangle H_i\bar{T}_{mi}(t)+A_{ij}\bar{\theta}_{mi}(t+\Delta t)+B_{ij}\bar{\theta}_{mj}(t+\Delta t)$$

$$+L_i(\bar{q}_z^{*(m)}(t)-\bar{\bar{q}}_{z\varphi}^{*(m)}(t))+P_i(\bar{q}_r^{*(m)}(t)-\bar{\bar{q}}_{r\varphi}^{*(m)}(t))-mR_i\bar{q}_{\varphi}^{*(m)}(t)\Big]_q\qquad(i=1,2,\ldots,N) \tag{11}$$

in the case of an implicit difference scheme.

Here, m is the harmonic number; N is the number of nodes; M is the number of triangular elements in the meridional section; q is the triangular element number; and $\bar{\theta}_{mi}$ and $\bar{\bar{\theta}}_{mi}$ are the coefficients of the trigonometric series expansion (similar to (8)) of the ambient temperature.

Since the thermal conductivity coefficients depend on temperature, the associated problem is nonlinear. It will be linearized by the method of successive approximations. However, the step Δt of explicit integration of the heat-conduction equation over time is very short (the maximum step is derived from the computational stability condition); therefore, the amplitudes of the temperature and additional terms on the right-hand side are determined from the previous time step without successive approximations. When an implicit difference scheme is used to calculate the temperature, the time step is specified depending on how the heating conditions change and how strong the temperature dependence of the thermal characteristics is.

The explicit difference scheme (10) is more effective with high temperature gradients (initial heating or abrupt cooling). As the body is further heated or cooled (temperature equalizes), the implicit difference scheme becomes more appropriate.

Using the approach detailed in [3, 7], we obtain a system of $3N$ linear algebraic equations for the determination of the coefficients $\overline{u}_\alpha^{(m)}$ $(\alpha = z, r, \varphi)$ at the nodes $(i,\ j,\ k)$ of the triangular elements q in each approximation:

$$\sum_{q=1}^{M} (B_{zp}^{zi(q)} \overline{u}_{zp} + B_{rp}^{zi(q)} \overline{u}_{rp} + B_{\varphi p}^{zi(q)} \overline{u}_{\varphi p}) = D_{zi},$$

$$\sum_{q=1}^{M} (B_{zp}^{ri(q)} \overline{u}_{zp} + B_{rp}^{ri(q)} \overline{u}_{rp} + B_{\varphi p}^{ri(q)} \overline{u}_{\varphi p}) = D_{ri},$$

$$\sum_{q=1}^{M} (B_{zp}^{\varphi i(q)} \overline{u}_{zp} + B_{rp}^{\varphi i(q)} \overline{u}_{rp} + B_{\varphi p}^{\varphi i(q)} \overline{u}_{\varphi p}) = D_{\varphi i}$$

$$(p = i, j, k),\ < i = 1,\ 2,\ \ldots, N >. \tag{12}$$

Such systems are so many as there are terms in solution (9). The elements of the matrix of these systems are calculated from the coefficients of Eqs. (6) and the nodal coordinates of the finite elements in the meridional plane, and the right-hand side is found from the amplitudes of the additional stresses σ_{ij}^* and volume and surface loads at the respective points.

The expressions for the coefficients in (10)–(12) are omitted as awkward (see [3, 21] for such expressions for a single triangular element).

The coefficients $\overline{\overline{T}}_m$ and $\overline{\overline{u}}_\alpha^{(m)}$ $(\alpha = z, r, \varphi)$ are defined by (10)–(12) where m should be made negative ($-m$) and all overbars should be replaced with double overbars, and vice versa.

After finding the displacement amplitudes from Eqs. (12), we calculate the displacement, strain, and stress components in each approximation at a time point of interest.

In conclusion, it should be pointed out that when $m = 0$, relations (10)–(12) represent the axisymmetric case (the ambient temperature and loading conditions do not change along the circumference of the solid of revolution).

The use of relations (1) to symmetrize the asymmetric compliance matrix was validated by calculating the stresses and strains in a uniformly heated thin-walled hollow cylinder (shell) subject to internal pressure, axial compression, and torsion. The stress state of such a cylinder can also be determined by summing the elementary solutions for all the loads, taking the difference of the tensile and compressive moduli into account without imposing conditions (1). The calculated stresses and strains are in good agreement, with some tensile and compressive moduli differing by more than 50%.

To validate the method (for stress–strain analysis of solids of revolution made of cylindrically orthotropic materials with different tensile and compressive moduli and deformational damages), a tubular specimen was tested under internal pressure, axial tension, and torsion. The experimental tension and torsion curves appear to be in perfect agreement with the solution of the corresponding boundary-value problem.

2. Let us examine, as an example, the nonstationary temperature field in and the stress–strain state of a two-layer cylinder cooling by convection. Its layers differently resist tension and compression. The cylinder, which is initially ($t = 0$) at temperature $T_0 = 20$°C, is then heated on the cylindrical surface by an ambient medium with a temperature changing as $\theta = (320 + 300\cos\varphi)$°C, and at the ends $z = \pm 0.3$ m by a medium of constant temperature $\theta = 300$ °C. The heat-transfer factor α between the ambient medium and the cylinder is assumed to be constant in the circumferential direction.

The inner layer ($0.035 \le r \le 0.04$ m) has the following temperature-independent material characteristics:

$$E_z^+ = E_r^+ = E_\varphi^+ = 9.327 \cdot 10^4 \text{ MPa}, \quad \nu_{zr}^+ = \nu_{z\varphi}^+ = \nu_{r\varphi}^+ = 0.22,$$

$$G_{zr}^+ = G_{z\varphi}^+ = G_{r\varphi}^+ = 3.823 \cdot 10^4 \text{ MPa}, \quad E_z^- = E_r^- = E_\varphi^- = 12.436 \cdot 10^4 \text{ MPa},$$

$$\nu_{zr}^- = \nu_{z\varphi}^- = \nu_{r\varphi}^- = 0.27, \quad G_{zr}^- = G_{z\varphi}^- = G_{r\varphi}^- = 4.896 \cdot 10^4 \text{ MPa}$$

TABLE 1

ε_{ij}		0	0.0005	0.001	0.0015	0.002	0.005	0.01	0.10
σ_{ij}^{+}	$i = j$	0	46.635	93.27	97.0	100.0	107.0	120.0	130.0
	$i \neq j$	0	38.23	42.00	43.5	44.0	47.0	50.0	55.0
σ_{ij}^{-}	$i = j$	0	62.18	124.36	184.0	232.0	390.0	500.0	550.0
	$i \neq j$	0	48.96	97.0	130.0	155.0	189.0	200.0	210.0

TABLE 2

ε_{ij}		0	0.0001	0.001	0.002	0.005	0.01	0.02	0.10
σ_{ij}^{+}	$i = j$	0	0.7	3.0	4.8	8.0	10.0	11.0	12.0
	$i \neq j$	0	0.6	2.5	4.0	6.0	7.0	7.1	7.2
σ_{ij}^{-}	$i = j$	0	1.75	12.0	22.0	38.5	49.0	50.0	51.0
	$i \neq j$	0	1.5	8.5	14.5	25.0	34.0	36.0	37.0

and the following thermal characteristics:

$$c\rho = 4.19 \text{ MJ/(m}^3\text{·K)}, \quad \lambda_{zz} = \lambda_{rr} = \lambda_{\varphi\varphi} = 0.05349 \text{ kW/(m·K)}; \quad \alpha_{zz}^{\mathrm{T}} = \alpha_{rr}^{\mathrm{T}} = \alpha_{\varphi\varphi}^{\mathrm{T}} = 1\cdot 10^{-5} \text{ K}^{-1}.$$

The outer layer ($0.04 \le r \le 0.05$ m) has the following material and thermal characteristics:

$$E_z^{+} = E_r^{+} = E_\varphi^{+} = 0.7\cdot 10^4 \text{ MPa}, \quad \nu_{zz}^{+} = \nu_{z\varphi}^{+} = \nu_{r\varphi}^{+} = 0.17,$$

$$G_{zr}^{+} = G_{z\varphi}^{+} = G_{r\varphi}^{+} = 0.3\cdot 10^4 \text{ MPa}, \quad E_z^{-} = E_r^{-} = E_\varphi^{-} = 1.75\cdot 10^4 \text{ MPa},$$

$$\nu_{zz}^{-} = \nu_{z\varphi}^{-} = \nu_{r\varphi}^{-} = 0.17, \quad G_{zr}^{-} = G_{z\varphi}^{-} = G_{r\varphi}^{-} = 0.750\cdot 10^4 \text{ MPa},$$

$$\alpha_{zz} = \alpha_{xx}^{T} = \alpha_{yy}^{T} = 1\cdot 10^{-5} \text{ K}^{-1}, \quad c\rho = 1.7582 \text{ MJ/(m}^3\text{·K)}, \quad \lambda_{zz} = \lambda_{rr} = \lambda_{\varphi\varphi} = 0.00093 \text{ kW/(m·K)}.$$

Tables 1 and 2 present the function $\sigma_{ij} = F_{ij}(\varepsilon_{ij})$ for both the inner and outer layers, respectively, for tensile and compressive stresses separately.

The next tables show the radial variation of the temperature, damage parameter, and stresses (in MPa) σ_{zz} (Tables 3 and 4) and $\sigma_{\varphi\varphi}$ (Tables 5 and 6) at the 60th second and after the 10th minute of heating. The asterisk refers to the case of no damage. The coordinates $\varphi = 0$, $\varphi = \pi/2$, and $\varphi = \pi$ have been chosen because the stress components are maximum there. However, a comparison shows that nonaxisymmetric loading induces tangential stresses $\sigma_{z\varphi}$ and $\sigma_{r\varphi}$ that are 10 to 15% of the normal stresses. It follows from the tabulated results that microdamage of the outer layer occurs during first seconds of heating, when the temperature gradient is maximum.

For reference, the tables include the normal stress components $\sigma_{zz}^{+*}, \sigma_{\varphi\varphi}^{+*}$ and $\sigma_{zz}^{-*}, \sigma_{\varphi\varphi}^{-*}$ for material properties established in tension (+) or compression (–) tests. An analysis shows that failure to account for the difference of the tensile and compressive moduli may change the result almost twofold.

TABLE 3

r, cm		3.55	3.75	3.95	4.05	4.25	4.45	4.65	4.85	4.95
$\varphi = 0$	T, °C	74	75	76	96	178	271	373	483	539
	$\sigma_{zz}^{\pm *}$	81.2	84.8	87.8	6.2	–1.9	–21.9	–43.0	–65.5	–77.0
	$\sigma_{zz}^{+ *}$	39.8	42.2	43.9	2.3	–4.5	–12.2	–20.6	–29.7	–34.3
	$\sigma_{zz}^{- *}$	88.2	92.3	95.5	10.4	–6.5	–25.5	–46.5	–69.0	–80.6
	σ_{zz}	51.3	54.0	56.1	2.1	–6.4	–17.9	–26.2	–33.4	–37.0
	ω_{zz}^{D}	0	0	0	0.615	0.285	0.358	0.467	0.535	0.560
$\varphi = \frac{\pi}{2}$	T, °C	49	49	50	59	100	146	197	252	280
	$\sigma_{zz}^{\pm *}$	41.4	41.1	40.5	2.8	–2.0	–12.1	–23.0	–34.5	–40.5
	$\sigma_{zz}^{+ *}$	20.1	19.8	19.3	0.9	–2.5	–6.4	–10.7	–15.3	–17.7
	$\sigma_{zz}^{- *}$	44.5	44.1	43.4	4.5	–4.1	–13.8	–24.6	–36.1	–42.0
	σ_{zz}	26.9	26.5	25.9	1.2	–3.9	–10.2	–16.8	–22.1	–24.5
	ω_{zz}^{D}	0	0	0	0.508	0.237	0.309	0.352	0.384	0.429
$\varphi = \pi$	T, °C	24	24	24	23	22	22	21	21	20
	$\sigma_{zz}^{\pm *}$	1.5	–3.1	–8.4	–1.4	–1.8	–2.3	–3.0	–3.6	–3.9
	$\sigma_{zz}^{+ *}$	0.3	–2.5	–5.5	–0.4	–0.5	–0.7	–0.8	–1.0	–1.1
	$\sigma_{zz}^{- *}$	0.7	–4.0	–8.9	–1.4	–1.7	–2.1	–2.6	–3.1	–3.4
	σ_{zz}	4.0	1.0	–2.2	–0.3	–0.7	–1.1	–1.5	–1.5	–1.6
	ω_{zz}^{D}	0	0	0	0	0	0	0.013	0.055	0.021

TABLE 4

r, cm		3.55	3.75	3.95	4.05	4.25	4.45	4.65	4.85	4.95
$\varphi = 0$	T, °C	384	384	385	397	441	485	528	570	591
	$\sigma_{zz}^{\pm *}$	242	318	390	27	–2.0	–10.3	–18.2	–25.8	–29.5
	$\sigma_{zz}^{+ *}$	99	172	240	13	–2.1	–5.3	–8.5	–11.5	–13.0
	$\sigma_{zz}^{- *}$	278	379	474	52	3.1	–11.1	–19.0	–26.5	–30.2
	σ_{zz}	125	196	261	8	–3.4	–8.3	–11.2	–13.2	–14.2
	ω_{zz}^{D}	0	0	0	0.738	0.306	0.358	0.467	0.535	0.560
$\varphi = \frac{\pi}{2}$	T, °C	282	282	282	283	290	297	304	311	315
	$\sigma_{zz}^{\pm *}$	98	99	99	8	0.2	–1.1	2.6	–4.1	–4.9
	$\sigma_{zz}^{+ *}$	31	30	29	1	–0.4	–1.0	–1.6	–2.2	–2.5
	$\sigma_{zz}^{- *}$	68	67	65	7	–0.7	–2.1	–3.7	–5.2	–6.0
	σ_{zz}	64	65	63	2	–0.3	–1.3	–2.3	2.9	–3.1
	ω_{zz}^{D}	0	0	0	0.624	0.275	0.309	0.352	0.384	0.429
$\varphi = \pi$	T, °C	179	179	179	170	138	108	7.9	52	39
	$\sigma_{zz}^{\pm *}$	–73	–176	–279	–27	1.0	3.1	5.1	6.9	7.7
	$\sigma_{zz}^{+ *}$	–37	–114	–189	–10	1.3	3.3	5.3	7.1	8.0
	$\sigma_{zz}^{- *}$	–144	–253	–356	–38	1.8	6.9	11.7	16.2	18.3
	σ_{zz}	0	–95	–190	–13	0.9	1.8	2.4	3.1	2.8
	ω_{zz}^{D}	0	0	0	0.194	0.556	0.598	0.628	0.652	0.662

TABLE 5

r, cm		3.55	3.75	3.95	4.05	4.25	4.45	4.65	4.85	4.95
$\varphi = 0$	T, °C	74	75	76	96	178	271	373	483	599
	$\sigma_{\varphi\varphi}^{\pm *}$	62.4	62.0	61.3	4.2	–6.1	–23.8	–42.5	–62.1	–72.1
	$\sigma_{\varphi\varphi}^{+ *}$	28.7	29.3	29.4	1.2	–5.2	–12.2	–19.7	–27.6	–31.6
	$\sigma_{\varphi\varphi}^{- *}$	67.7	67.8	67.4	6.2	–9.9	–27.4	–46.4	–66.2	–76.1
	$\sigma_{\varphi\varphi}$	36.4	36.2	35.7	1.6	–8.0	–18.0	–25.4	–31.5	–34.7
	$\omega_{\varphi\varphi}^{D}$	0	0	0	0	0.301	0.362	0.464	0.526	0.550
$\varphi = \frac{\pi}{2}$	T, °C	49	49	50	59	100	146	197	252	280
	$\sigma_{\varphi\varphi}^{\pm *}$	34.3	32.3	30.3	2.0	–3.7	–12.9	–22.7	–32.9	–38.0
	$\sigma_{\varphi\varphi}^{+ *}$	16.3	15.2	14.0	0.5	–2.8	–6.4	–10.3	–14.3	–16.4
	$\sigma_{\varphi\varphi}^{- *}$	37.6	35.1	32.7	2.9	–5.4	–14.5	–24.3	–34.5	–39.6
	$\sigma_{\varphi\varphi}$	22.6	21.7	20.6	1.5	–3.8	–9.3	–15.0	–20.0	–22.2
	$\omega_{\varphi\varphi}^{D}$	0	0	0	0	0.244	0.307	0.346	0.368	0.388
$\varphi = \pi$	T, °C	24	24	24	23	22	22	21	21	20
	$\sigma_{\varphi\varphi}^{\pm *}$	6.3	2.7	–0.9	–0.2	–1.1	–2.0	–2.8	–3.6	–3.9
	$\sigma_{\varphi\varphi}^{+ *}$	4.0	1.2	–1.6	–0.2	–0.3	–0.6	–0.8	–1.0	–1.1
	$\sigma_{\varphi\varphi}^{- *}$	6.9	2.4	–2.3	–0.4	–0.9	–1.6	–2.2	–2.8	–3.1
	$\sigma_{\varphi\varphi}$	6.9	3.2	–0.4	–0.2	–1.1	–1.9	–2.5	2.7	–2.8
	$\omega_{\varphi\varphi}^{D}$	0	0	0	0	0	0.031	0.135	0.157	0.168

TABLE 6

r, cm		3.55	3.75	3.95	4.05	4.25	4.45	4.65	4.85	4.95
$\varphi = 0$	T, °C	384	384	385	397	441	485	528	570	591
	$\sigma_{\varphi\varphi}^{\pm *}$	12.7	19.6	25.9	1.7	–3.9	–11.1	–17.8	–24.1	–27.1
	$\sigma_{\varphi\varphi}^{+ *}$	0.3	7.0	13.1	0.4	–2.7	–5.7	–8.4	–11.0	–12.2
	$\sigma_{\varphi\varphi}^{- *}$	11.6	20.4	28.4	2.4	–5.5	–12.9	–19.8	–26.2	–29.3
	$\sigma_{\varphi\varphi}$	2.9	9.4	15.1	0.6	–4.1	–8.2	–10.7	–12.2	–12.8
	$\omega_{\varphi\varphi}^{D}$	0	0	0	0	0.312	0.362	0.464	0.526	0.550
$\varphi = \frac{\pi}{2}$	T, °C	282	282	282	283	290	297	304	311	315
	$\sigma_{\varphi\varphi}^{\pm *}$	7.2	5.7	4.4	0.2	–1.0	–2.6	–4.1	–5.5	–6.3
	$\sigma_{\varphi\varphi}^{+ *}$	2.5	2.3	2.1	0.1	–0.4	–1.0	–1.5	–2.1	–2.3
	$\sigma_{\varphi\varphi}^{- *}$	5.6	5.3	4.9	0.4	–0.9	–2.2	–3.6	–5.0	–5.3
	$\sigma_{\varphi\varphi}$	4.6	3.6	2.9	0.1	–0.8	–1.7	–2.5	–3.2	–3.2
	$\omega_{\varphi\varphi}^{D}$	0	0	0	0	0.274	0.370	0.346	0.368	0.388
$\varphi = \pi$	T, °C	179	179	179	170	138	108	79	52	39
	$\sigma_{\varphi\varphi}^{\pm *}$	5.5	–1.5	–9.7	0	2.1	4.1	5.8	7.3	8.0
	$\sigma_{\varphi\varphi}^{+ *}$	4.9	–2.6	–9.5	–0.3	1.9	3.7	5.4	6.9	7.6
	$\sigma_{\varphi\varphi}^{- *}$	–0.3	–10.4	–19.7	–1.4	3.9	8.6	12.7	16.4	18.1
	$\sigma_{\varphi\varphi}$	7.4	0.6	–7.6	0.1	2.1	3.5	4.3	5.4	3.6
	$\omega_{\varphi\varphi}^{D}$	0	0	0	0.112	0	0.031	0.136	0.157	0.686

Thus, the technique proposed here to describe microdamages in orthotropic materials with different tensile and compressive moduli in stress–strain analysis of compound solids of revolution subject to nonstationary heating greatly improves the solution of the boundary-value problem.

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