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STRESS ANALYSIS OF NONTHIN ELLIPTIC CYLINDRICAL SHELLS IN REFINED AND SPATIAL FORMULATIONS

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A spatial model and a refined model based on the straight-line hypothesis are used to analyze the stress state of nonthin elliptic cylindrical shells with certain end conditions for different thicknesses and aspect ratios. The results obtained are compared, and the validity range of the refined model is established

Keywords: nonthin noncircular shell, refined model, spatial model, stress analysis, numerical methods

Introduction. Many modern structural members have the form of circular and noncircular cylindrical shells. Of special interest is the stress analysis of nonthin noncircular cylindrical shells [13] because the classical theory of shells does not work alone here.

The stress analysis of nonthin shells is usually based on one refined theory of shells or another [1–3, 12]. One of such widely used theories is based on the straight-line hypothesis [2, 3]. Refined theories of shells are usually validated against solutions of three-dimensional problems. Some comparisons for nonthin circular cylindrical shells were made in [1, 2] to validate the refined theory for certain geometrical and material parameters and loading and end conditions. In the case of nonthin noncircular cylindrical shells, it is also necessary to take into account the variation in the cross-sectional curvature because the error of an applied theory increases with the ratio of thickness to radius of curvature. Therefore, it is of interest to compare solutions for nonthin noncircular cylindrical shells produced by refined and spatial theories.

1. Let us consider nonthin isotropic cylindrical shells whose cross section is everywhere elliptical. The cross section of the reference surface is parametrically defined as

$$x = b\cos\theta, \qquad z = a\sin\theta \qquad (\theta \le 0 \le 2\pi), \tag{1}$$

where b and a are the major and minor semiaxes of an ellipse with a perimeter equal to that of a circle of radius R, i.e.,

$$\pi(a+b)f = 2\pi R, \qquad f = 1+\frac{\Delta^2}{4}+\frac{\Delta^4}{64}+\frac{\Delta^6}{256}+\ldots, \qquad \Delta = \frac{b-a}{b+a}. \tag{2}$$

Then

$$a = \frac{R}{f}(1-\Delta), \qquad b = \frac{R}{f}(1+\Delta), \qquad \frac{b}{a} = \frac{1+\Delta}{1-\Delta}. \tag{3}$$

The refined formulation based on the straight-line hypothesis [2, 3, 10, 11] assumes that the displacements vary across the thickness in the following manner:

$$u_s(s,t,\gamma) = u(s,t)+\gamma\psi_s(s,t),$$

$$u_t(s,t,\gamma) = v(s,t)+\gamma\psi_t(s,t),$$

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$$u_\gamma(s,t,\gamma)=w(s,t), \tag{4}$$

where s, t, and γ are the longitudinal, circumferential, and normal (to the midsurface $\gamma=$const) coordinates; u_s, u_t, and u_γ are the corresponding displacements; u, v, and w are the displacements of the midsurface in the directions s, t, and γ; and ψ_s and ψ_t are the angular displacements of a rectilinear element.

According to (4), the strains are expressed as

$$e_s(s,t,\gamma)=\varepsilon_s(s,t)+\gamma\kappa_s(s,t), \qquad e_t(s,t,\gamma)=\varepsilon_t(s,t)+\gamma\kappa_t(s,t),$$

$$e_{st}(s,t,\gamma)=\varepsilon_{st}(s,t)+\gamma 2\kappa_{st}(s,t), \qquad e_{s\gamma}(s,t,\gamma)=\gamma_s(s,t), \qquad e_{t\gamma}(s,t,\gamma)=\gamma_t(s,t). \tag{5}$$

Equations (5) relate the strain components at an arbitrary point of the shell with the strain components of the coordinate surface, which are in turn related to the displacements u, v, w and the angular displacements ψ_s, ψ_t by

$$\varepsilon_s=\frac{\partial u}{\partial s}, \quad \varepsilon_t=\frac{\partial v}{\partial t}+k(t)w, \quad \varepsilon_{st}=\frac{\partial u}{\partial t}+\frac{\partial v}{\partial s}, \quad \kappa_s=\frac{\partial \psi_s}{\partial s}, \quad \kappa_t=\frac{\partial \psi_t}{\partial t}-k(t)\left[\frac{\partial v}{\partial t}+k(t)w\right],$$

$$2\kappa_{st}=\frac{\partial \psi_s}{\partial t}+\frac{\partial \psi_t}{\partial s}-k(t)\frac{\partial u}{\partial t}, \quad \gamma_s=\psi_s-\vartheta_s, \quad \gamma_t=\psi_t-\vartheta_t, \quad \vartheta_s=-\frac{\partial w}{\partial s}, \quad \vartheta_t=-\frac{\partial w}{\partial t}+k(t)v, \tag{6}$$

where $\varepsilon_s, \varepsilon_t, \varepsilon_{st}$ and $\kappa_s, \kappa_t, \kappa_{st}$ are the tangential and flexural strains of the midsurface; $k(t)$ is the curvature of the directrix; ϑ_s and ϑ_t are the angles of rotation of the normal without regard to transverse shear; and γ_s and γ_t are the shear strains, i.e., additional angles of rotation of the normal due to transverse shear.

The system of equilibrium equations is given by

$$\frac{\partial N_t}{\partial t}+\frac{\partial N_{st}}{\partial s}+q_t=0, \qquad \frac{\partial N_s}{\partial s}+\frac{\partial N_{ts}}{\partial t}+k(t)Q_t+q_s=0,$$

$$\frac{\partial Q_s}{\partial s}+\frac{\partial Q_t}{\partial t}-k(t)N_s+q_\gamma=0, \qquad \frac{\partial M_t}{\partial t}+\frac{\partial M_{st}}{\partial s}-Q_t=0,$$

$$\frac{\partial M_s}{\partial s}+\frac{\partial M_{ts}}{\partial t}-Q_t=0, \qquad N_{ts}-N_{st}-k(t)M_{st}=0, \tag{7}$$

where N_s and N_t are the normal forces; N_{st}, N_{ts}, Q_s, and Q_t are the shearing forces; M_s and M_t are the bending moments; M_{st} and M_{ts} are the twisting moments; q_s, q_t, and q_γ are load terms defined in terms of the components $q_s^{\pm}, q_t^{\pm}$, and $q_\gamma^{\pm}$ of the surface load acting of the inside and outside surfaces of the shell,

$$q_s=q_s^++q_s^-, \quad q_t=q_t^++q_t^-, \quad q_\gamma=q_\gamma^++q_\gamma^-, \quad m_s=q_s^+\gamma_n-q_s^-\gamma_0, \quad m_t=q_t^+\gamma_n-q_t^-\gamma_0.$$

The elastic relations for orthotropic shells can be written in the following general form:

$$N_t=C_{11}\varepsilon_t+C_{12}\varepsilon_s, \quad N_s=C_{12}\varepsilon_t+C_{22}\varepsilon_s, \quad N_{ts}=C_{66}\varepsilon_{ts}+k(t)D_{66}2\kappa_{ts},$$

$$N_{st}=C_{66}\varepsilon_{ts}, \quad M_t=D_{11}\kappa_t+D_{12}\kappa_s, \quad M_s=D_{12}\kappa_t+D_{22}\kappa_s,$$

$$M_{ts}=M_{st}=D_{66}2\kappa_{ts}, \quad Q_t=K_1\gamma_t, \quad Q_s=K_2\gamma_s, \tag{8}$$

where C_{ij} and D_{ij} are stiffness characteristics defined in terms of the elastic constants and thicknesses of the shell, and K_1 and K_2 are the shear stiffnesses [2].

Using Eqs. (5)–(8), we arrive at a governing system of partial differential equations of the tenth order, which can be represented in vector form:

$$\frac{\partial \bar{N}}{\partial t}=B_0\bar{N}+B_1\frac{\partial \bar{N}}{\partial s}+B_2\frac{\partial \bar{N}^2}{\partial s^2}+\bar{f}(s,t), \tag{9}$$

$$\bar{N}=\{N_t, N_{ts}, Q_t, M_t, M_{ts}, u, v, w, \psi_t, \psi_s\}^T,$$

$$B_n = ||bij^{(n)}(t)|| \quad (n=0,1,2; \quad i,j=1,2,\ldots,10), \qquad \bar{f}=\{f_1, f_2, \ldots, f_{10}\}^T, \tag{10}$$

where B_n are square matrices whose elements are defined in [3], and $\bar{f}(s,t)$ is the vector of the right-hand side.

Using the Fourier series expansion in Eqs. (9) to separate variables in s, we obtain a system of ordinary differential equations of the tenth order for the nth harmonic (for simplicity, the index n in the notation of unknown functions is omitted):

$$\frac{dN_t}{dt}=\lambda_n N_{ts}-\lambda_n k(t) M_{ts}-q_t,$$

$$\frac{dN_{ts}}{dt}=-\lambda_n d_{61} N_t-\lambda_n k(t)[K_2+d_{63}]w+[k^2(t)K_2+\lambda_n^2 d_{63}]v-k(t)K_2\psi_s-q_s,$$

$$\frac{dM_t}{dt}=Q_t+\lambda_n M_{ts},$$

$$\frac{dM_{ts}}{dt}=-\lambda_n d_{72} M_t-[k(t)K_2+\lambda_n^2 k(t) d_{74})]v+\lambda_n[K_2+k(t)d_{74}]w+\left(K_2+\lambda_n^2 d_{74}\right)\psi_s,$$

$$\frac{\partial Q_t}{\partial t}=k(t)d_{61}N_t-\lambda_n k(t)[K_2+d_{63}]v+[K_1+k^2(t)(d_{63}+\lambda_n^2 k(t)]w+\lambda_n K_2\psi_s-q_\gamma,$$

$$\frac{du}{dt}=d_{11}N_t-\lambda_n d_{13}v+k(t)d_{13}w, \qquad \frac{dv}{dt}=d_{31}N_{ts}+d_{32}M_{ts}-\lambda_n u, \qquad \frac{dw}{dt}=\frac{1}{K_1}Q_t-\psi_t,$$

$$\frac{d\psi_t}{dt}=d_{22}M_t+\lambda_n k(t)d_{24}v+k(t)-k^2(t)d_{24}w-\lambda_n d_{24}\psi_s, \qquad \frac{d\psi_s}{dt}=d_{42}M_{ts}+\lambda_n k(t)u-\lambda_n\psi_t, \tag{11}$$

where $\lambda_n = n\pi / l\,(n=\overline{0,N})$, and

$$d_{11}=\frac{D_{11}}{\Delta}, \quad d_{13}=-\frac{C_{12}D_{11}}{\Delta}, \quad d_{22}=\frac{C_{11}}{\Delta}, \quad d_{24}=-\frac{C_{11}D_{12}}{\Delta}, \quad d_{31}=\frac{D_{66}}{\Delta_1},$$

$$d_{32}=-\frac{k(t)D_{66}}{\Delta_1}, \quad d_{42}=\frac{C_{66}}{\Delta_1}, \quad d_{61}=C_{12}d_{11}, \quad d_{63}=C_{22}+C_{12}d_{13},$$

$$d_{72}=D_{12}d_{22}, \quad d_{74}=D_{22}+D_{12}d_{24}, \quad \Delta=C_{11}D_{11}, \quad \Delta_1=C_{66}D_{66}. \tag{12}$$

The other stress and strain components can be found in terms of the unknown functions (10) from the formulas

$$\varepsilon_t=d_{11}N_t+d_{13}\varepsilon_s, \quad \varepsilon_s=-\lambda_n v+k(t)w, \quad \kappa_t=d_{22}M_t+d_{24}\kappa_s,$$

$$\kappa_s=-\lambda_n\psi_s-k(t)\varepsilon_s, \quad \varepsilon_{ts}=d_{31}N_{ts}+d_{32}M_{ts}, \quad 2\kappa_{ts}=d_{42}M_{ts},$$

$$\gamma_t=\frac{1}{K_1}Q_t, \quad \gamma_s=\psi_s(1+\lambda_n)-k(t)v, \quad Q_s=K_2\gamma_s,$$

$$\sigma_t^{\mp}=B_{11}\varepsilon_t+B_{12}\varepsilon_s+\gamma^{\mp}\left(B_{11}\kappa_t+B_{12}\kappa_s\right), \quad \sigma_s^{\mp}=B_{12}\varepsilon_t+B_{22}\varepsilon_s+\gamma^{\mp}\left(B_{12}\kappa_t+B_{22}\kappa_s\right),$$

$$\tau_{ts}^{\mp}=\left(\varepsilon_{ts}+2\gamma^{\mp}\kappa_{ts}\right)B_{66}, \tag{13}$$

where $B_{11}=B_{22}=\frac{E}{1-\nu^2}$, $B_{12}=\nu B_{11}$, and $B_{66}=\frac{E}{2(1+\nu)}$.

In the isotropic case, C_{ij} and D_{ij} become

$$C_{11} = C_{22} = \frac{Eh}{1-\nu^2}, \quad C_{12} = \nu C_{11}, \quad C_{66} = \frac{Eh}{2(1+\nu)},$$

$$D_{11} = D_{22} = \frac{Eh^3}{12(1-\nu^3)}, \quad D_{12} = \nu D_{11}, \quad D_{66} = \frac{Eh^3}{24(1+\nu)}. \tag{14}$$

Then the governing system of differential equations (11) becomes

$$\frac{dN_t}{dt} = \lambda_n N_{ts} - \lambda_n k(t) M_{ts} - q_t,$$

$$\frac{dN_{ts}}{dt} = -\lambda_n \nu N_t - \lambda_n k(t)[K_2 + Eh]w + [k^2(t)K_2 + \lambda_n^2 Eh)]v - k(t)K_2\psi_s - q_s,$$

$$\frac{dM_t}{dt} = Q_t + \lambda_n M_{ts},$$

$$\frac{dM_{ts}}{dt} = -\lambda_n \nu M_t - \left[k(t)K_2 + \lambda_n^2 k(t)\frac{Eh^3}{12}\right]v + \lambda_n\left[K_2 + k^2(t)\frac{Eh^3}{12}\right]w + \left(K_2 + \lambda_n^2\frac{Eh^3}{12}\right)\psi_s,$$

$$\frac{\partial Q_t}{\partial t} = k(t)\nu N_t - \lambda_n k(t)[K_2 + Eh]v + [K_1 + k^2(t)(Eh + \lambda_n^2 k(t))]w + \lambda_n K_2 \psi_s - q_\gamma,$$

$$\frac{du}{dt} = \frac{1-\nu^2}{Eh}N_t + \lambda_n \nu v - k(t)\nu w, \quad \frac{dv}{dt} = \frac{1}{Gh}[N_{ts} + k(t)M_{ts}] - \lambda_n u, \quad \frac{dw}{dt} = \frac{1}{K_1}Q_t - \psi_t,$$

$$\frac{d\psi_t}{dt} = \frac{12(1-\nu^2)}{Eh^3}M_t - \lambda_n k(t)\nu v + k^2(t)\nu w + \lambda_n \nu \psi_s, \quad \frac{d\psi_s}{dt} = \frac{12}{Gh^3}M_{ts} + \lambda_n k(t)u - \lambda_n \psi_t, \tag{15}$$

where E is Young's modulus; ν is Poisson's ratio; h is the shell thickness; K_1 and K_2 are the shear stiffnesses; and $G_{s\gamma} = G_{t\gamma} = G$ are the shear moduli in the directions s and t.

Since the elliptic cross section of shells under consideration is described in terms of the parameter θ in (1) and the basic variable in Eqs. (15) is the arc length t, these parameters are related by

$$\frac{dF}{dt} = \frac{1}{\omega(\theta)}\frac{dF}{d\theta}, \quad \omega(\theta) = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2}.$$

We have developed an algorithm based on a stable numerical method for solving one-dimensional boundary-value problems [2]. This algorithm has been implemented in a software system to solve static boundary-value problems for noncircular cylindrical shells in a refined formulation that accounts for transverse shear strains.

2. Let us address the three-dimensional formulation of the same problem for a hollow cylinder with elliptic cross section. The midsurface of the cylinder is referred to the coordinate system s, θ, and the cylinder itself to the coordinate system s, θ, γ. We start with the governing equations of the three-dimensional theory of isotropic elasticity [5, 6]. After some transformations, the governing system of partial differential equations for a noncircular cylindrical shell becomes

$$\frac{\partial \bar{\sigma}}{\partial \gamma} = A(s,\theta,\gamma)\,\bar{\sigma} + \bar{f}(s,\theta,\gamma), \quad \bar{\sigma} = \{\sigma_\gamma, \tau_{s\gamma}, \tau_{\theta\gamma}, u_\gamma, u_s, u_\theta\}^T, \tag{16}$$

where σ_γ, $\tau_{s\gamma}$, and $\tau_{\theta\gamma}$ are the stress components; u_γ, u_s, and u_θ are the displacement components; and A is a square matrix with elements defined in [4].

Suppose that the shell is hinged at the ends:

$$\sigma_s = u_\theta = u_\gamma = 0 \quad \text{at} \quad s = 0, l. \tag{17}$$

A surface load is prescribed on the outside surface of the shell. The boundary conditions (17) allow separation of variables in s. To this end, we expand the unknown functions and load components into Fourier series in s.

Substituting these series into the governing system of equations and the boundary conditions on the lateral surfaces and separating variables, we arrive at a two-dimensional boundary-value problem for the amplitudes of the series:

$$\frac{\partial \sigma_{\gamma,n}}{\partial \gamma} = -\frac{1}{1+\gamma/R(\theta)}\frac{1}{R(\theta)}\sigma_{\gamma,n} + \lambda_n \tau_{s\gamma,n} - \frac{1}{\omega(\theta)}\frac{1}{1+\gamma/R(\theta)}\frac{\partial \tau_{\theta\gamma,n}}{\partial \theta} + \frac{1}{1+\gamma/R(\theta)}\frac{1}{R(\theta)}$$

$$\times\left[-\frac{E\nu}{1-\nu^2}\lambda_n u_{s,n} + \frac{\nu}{1-\nu}\sigma_{\gamma,n} + \frac{E}{1-\nu^2}\frac{1}{1+\gamma/R(\theta)}\left(\frac{1}{\omega(\theta)}\frac{\partial u_{\theta,n}}{\partial \theta} + \frac{1}{R(\theta)}u_{\gamma,n}\right)\right],$$

$$\frac{\partial \tau_{s\gamma,n}}{\partial \gamma} = -\frac{1}{1+\gamma/R(\theta)}\frac{1}{R(\theta)}\tau_{s\gamma,n} + \frac{E}{1-\nu^2}\lambda_n^2 u_{s,n} - \frac{E\gamma}{1-\nu^2}\frac{1}{1+\gamma/R(\theta)}\lambda_n\left(\frac{1}{\omega(\theta)}\frac{\partial u_{\theta,n}}{\partial \theta} + \frac{1}{R(\theta)}u_{\gamma,n}\right)$$

$$-\frac{\nu}{1-\nu}\lambda_n\sigma_{\gamma,n} - \frac{1}{\omega(\theta)}\frac{1}{1+\gamma/R(\theta)}\frac{\partial}{\partial\theta}\left[\frac{E}{2(1+\nu)}\left(\frac{1}{\omega(\theta)}\frac{1}{1+\gamma/R(\theta)}\frac{\partial u_{s,n}}{\partial \theta} + \lambda_n u_{\theta,n}\right)\right],$$

$$\frac{\partial \tau_{\theta\gamma,n}}{\partial \gamma} = -\frac{2}{1+\gamma/R(\theta)}\frac{1}{R(\theta)}\tau_{\theta\gamma,n} - \frac{1}{\omega(\theta)}\frac{1}{1+\gamma/R(\theta)}\frac{\partial}{\partial\theta}\left[-\frac{E\nu}{1-\nu^2}\lambda_n u_{s,n} + \frac{E}{1-\nu^2}\frac{1}{1+\gamma/R(\theta)}\right.$$

$$\left.\times\left(\frac{1}{\omega(\theta)}\frac{\partial u_{\theta,n}}{\partial\theta} + \frac{1}{R(\theta)}u_{\gamma,n}\right) + \frac{\nu}{1-\nu}\sigma_{\gamma,n}\right] + \frac{1}{\omega(\theta)}\frac{E}{2(1+\nu)}\lambda_n\frac{1}{1+\gamma/R(\theta)}\frac{\partial u_{s,n}}{\partial\theta} + \frac{E}{2(1+\nu)}\lambda_n^2 u_{\theta,n},$$

$$\frac{\partial u_{\gamma,n}}{\partial\gamma} = \frac{\nu}{1-\nu}\lambda_n u_{s,n} - \frac{\nu}{1-\nu}\frac{1}{1+\gamma/R(\theta)}\left(\frac{1}{\omega(\theta)}\frac{\partial u_{\theta,n}}{\partial\theta} + \frac{1}{R(\theta)}u_{\gamma,n}\right) + \frac{1-\nu-2\nu^2}{E(1-\nu)}\sigma_{\gamma,n},$$

$$\frac{\partial u_{s,n}}{\partial\gamma} = \frac{2(1+\nu)}{E}\tau_{s\gamma,n} - \lambda_n u_{\gamma,n},$$

$$\frac{\partial u_{\theta,n}}{\partial\gamma} = \frac{2(1+\nu)}{E}\tau_{\theta\gamma,n} - \frac{1}{\omega(\theta)}\frac{1}{1+\gamma/R(\theta)}\frac{\partial u_{\gamma,n}}{\partial\theta} + \frac{1}{1+\gamma/R(\theta)}\frac{1}{R(\theta)}u_{\theta,n}$$

$$\left(0 \le \theta \le 2\pi;\quad \gamma_p \le \gamma \le \gamma_q\right) \qquad (n = \overline{0,N}), \tag{18}$$

where $\omega(\theta) = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2} = \sqrt{b^2\sin^2\theta + a^2\cos^2\theta}$, and $R(\theta) = \frac{\omega^3}{ab}$ is the radius of curvature in the cross section of the reference surface.

The boundary conditions on the lateral surfaces are

$$\sigma_{\gamma,n} = 0, \quad \tau_{s\gamma,n} = 0, \quad \tau_{\theta\gamma,n} = 0 \quad \text{for} \quad \gamma = \gamma_p,$$

$$\sigma_{\gamma,n} = q_{\gamma,n}(\theta), \quad \tau_{s\gamma,n} = 0, \quad \tau_{\theta\gamma,n} = 0 \quad \text{for} \quad \gamma = \gamma_q. \tag{19}$$

The radius of curvature $R(\theta)$ appearing in Eqs. (18) does not allow us to separate variables in θ. To overcome this difficulty, we replace the products of unknown functions and the coefficients containing $R(\theta)$ with auxiliary functions:

$$\varphi_1^j = \frac{1}{1+\gamma/R(\theta)}\frac{1}{R(\theta)}\{\sigma_\gamma; \tau_{s\gamma}; u_\gamma; u_s\} \quad (j = \overline{1,4}), \quad \varphi_1^5 = \left(\frac{1}{1+\gamma/R(\theta)}\frac{1}{R(\theta)}\right)^2 u_\gamma,$$

$$\varphi_2^j = \frac{1}{1+\gamma / R(\theta)} \frac{1}{R(\theta)} \left\{ \tau_{\theta\gamma}; u_\theta \right\} \qquad (j=\overline{1,2}),$$

$$\varphi_3^j = \frac{1}{\omega(\theta)} \frac{1}{1+\gamma / R(\theta)} \left\{ \frac{\partial \sigma_\gamma}{\partial \theta}; \frac{\partial u_\gamma}{\partial \theta}; \frac{\partial u_s}{\partial \theta} \right\} \qquad (j=\overline{1,3}),$$

$$\varphi_4^j = \frac{1}{\omega(\theta)} \frac{1}{1+\gamma / R(\theta)} \left\{ \frac{\partial \tau_{\theta\gamma}}{\partial \theta}; \frac{\partial u_\theta}{\partial \theta}; \frac{1}{R(\theta)} \frac{\partial u_\theta}{\partial \theta} \right\} \qquad (j=\overline{1,3}),$$

$$\varphi_5 = \frac{1}{\omega(\theta)} \frac{1}{1+\gamma / R(\theta)} \frac{\partial}{\partial \theta} \varphi_1^3, \qquad \varphi_6 = \frac{1}{\omega(\theta)} \frac{1}{1+\gamma / R(\theta)} \frac{\partial}{\partial \theta} \varphi_3^3,$$

$$\varphi_7 = \frac{1}{\omega(\theta)} \frac{1}{1+\gamma / R(\theta)} \frac{\partial}{\partial \theta} \varphi_4^2. \tag{20}$$

Formally, the resulting governing system of differential equations does not depend on the coordinate θ [8, 9]; however, the auxiliary functions depend on both γ and θ. Next, expending the unknown and auxiliary functions and load components into Fourier series in θ and separating variables, we arrive at a one-dimensional boundary-value problem for the following system of ordinary differential equations (for simplicity, the index n in the notation of unknown and auxiliary functions is omitted):

$$\frac{d\sigma_{\gamma,k}}{d\gamma} = \lambda_n \tau_{s\gamma,k} + \left(\frac{\nu}{1-\nu} - 1 \right) \varphi_{1,k}^1 - \varphi_{4,k}^1 - \frac{E\nu}{1-\nu^2} \lambda_n \varphi_{1,k}^4 + \frac{E}{1-\nu^2} \left(\varphi_{4,k}^3 + \varphi_{1,k}^5 \right),$$

$$\frac{d\tau_{s\gamma,k}}{d\gamma} = -\frac{\nu}{1-\nu} \lambda_n \sigma_{\gamma,k} + \frac{E}{1-\nu^2} \lambda_n^2 u_{s,k} - \varphi_{1,k}^2 - \left(\frac{E\nu}{1-\nu^2} + \frac{E}{2(1+\nu)} \right) \lambda_n \varphi_{4,k}^2 - \frac{E\nu}{1-\nu^2} \lambda_n \varphi_{1,k}^3 - \frac{E}{2(1+\nu)} \varphi_{6,k},$$

$$\frac{d\tau_{\theta\gamma,k}}{d\gamma} = \frac{E}{2(1+\nu)} \lambda_n^2 u_{t,k} - 2\varphi_{2,k}^1 + \left(\frac{E\nu}{1-\nu^2} + \frac{E}{2(1+\nu)} \right) \lambda_n \varphi_{3,k}^3 - \frac{E}{1-\nu^2} \left(\varphi_{7,k} + \varphi_{5,k} \right) - \frac{\nu}{1-\nu} \varphi_{3,k}^1,$$

$$\frac{du_{\gamma,k}}{d\gamma} = \frac{1-\nu-2\nu^2}{(1-\nu)E} \sigma_{\gamma,k} + \frac{\nu}{1-\nu} \left(\lambda_n u_{s,k} - \varphi_{4,k}^2 - \varphi_{1,k}^3 \right),$$

$$\frac{du_{s,k}}{d\gamma} = \frac{2(1+\nu)}{E} \tau_{s\gamma,k} - \lambda_n u_{\gamma,k}, \qquad \frac{du_{\theta,k}}{d\gamma} = \frac{2(1+\nu)}{E} \tau_{\theta\gamma,k} - \varphi_{3,k}^2 + \varphi_{2,k}^2 \tag{21}$$

with the boundary conditions

$$\gamma = \gamma_p: \ \sigma_{\gamma,k} = 0, \quad \tau_{s\gamma,k} = 0, \quad \tau_{\theta\gamma,k} = 0,$$

$$\gamma = \gamma_q: \ \sigma_{\gamma,k} = q_{\gamma,k}, \quad \tau_{s\gamma,k} = 0, \quad \tau_{\theta\gamma,k} = 0 \quad (k = \overline{0,K}). \tag{22}$$

In addition to the amplitudes of the unknown functions, the governing equations (21) include the amplitudes of the auxiliary functions, which are to be calculated separately during the integration of (21) by discrete orthogonalization for all harmonics simultaneously. To this end, at each step, we fix the coordinate γ and calculate the values of the auxiliary functions at some points of the interval $0 \le \theta \le 2\pi$ using the current values of the unknown functions. Next, we expand the functions given on a discrete set of points into Fourier series [7]. The more points there are at which the auxiliary functions are evaluated, the less the discrete Fourier series differs from the exact Fourier series. Thus, it is possible to achieve a very accurate solution. Using, for example, Runge's method, we find the coefficients of these series, substitute them into Eqs. (21), and continue the integration, satisfying the boundary conditions at the ends of the interval $\gamma_p \le \gamma \le \gamma_q$.

We used the refined and spatial models to examine the influence of the cross-sectional ellipticity on the stress state of nonthin elliptic cylindrical shells under load applied to the outside surface for different values of thickness.

TABLE 1

h	Δ	Δ				σ_θ^+ / q_0				σ_θ^- / q_0			
		$\theta = 0$		$\theta = \pi/2$		$\theta = 0$		$\theta = \pi/2$		$\theta = 0$		$\theta = \pi/2$	
		I	II	I	II	I	II	I	II	I	II	I	II
8	0	411.0	392.5	411.0	392.5	8.00	7.42	8.00	7.42	5.9	5.67	5.9	5.67
	0.05	301.6	284.8	531.9	512.5	6.64	6.05	9.27	8.74	6.53	6.17	5.39	5.20
	0.10	204.3	188.9	665.1	644.8	5.28	4.65	10.52	10.00	7.21	6.74	4.85	4.74
	0.15	118.3	104.4	809.9	788.9	3.88	3.19	11.71	11.21	8.05	7.43	4.31	4.25
	0.20	43.3	30.9	965.1	943.8	2.44	1.66	12.82	12.33	9.08	8.26	3.72	3.71
10	0	311.2	294.4	311.2	294.4	6.25	5.69	6.25	5.69	4.33	4.12	4.33	4.12
	0.05	241.4	225.3	387.0	369.7	5.33	4.74	7.12	6.58	4.96	4.65	3.75	3.62
	0.10	178.1	162.9	468.0	450.4	4.36	3.73	7.91	7.40	5.66	5.22	3.19	3.12
	0.15	121.6	107.5	553.2	535.7	3.36	2.67	8.63	8.14	6.46	5.86	2.63	2.62
	0.20	72.0	59.1	641.3	624.2	2.32	1.54	9.27	8.79	7.41	6.59	2.06	2.10
12	0	244.3	228.9	244.3	228.9	5.07	4.51	5.07	4.51	3.28	3.11	3.28	3.11
	0.05	197.9	182.8	293.5	278.0	4.42	3.84	5.66	5.13	3.87	3.60	2.73	2.64
	0.10	155.1	140.5	344.9	329.6	3.73	3.10	6.18	5.67	4.52	4.12	2.20	2.18
	0.15	116.3	102.3	397.4	382.7	3.00	2.31	6.63	6.14	5.23	4.69	1.70	1.73
	0.20	81.8	68.8	450.1	436.2	2.24	1.47	7.00	6.53	6.09	5.30	1.19	1.28
14	0	196.6	182.6	196.6	182.6	4.21	3.66	4.21	3.66	2.55	2.43	2.55	2.43
	0.05	164.8	150.8	229.7	216.0	3.76	3.17	4.62	4.09	3.09	2.86	2.06	2.01
	0.10	134.9	121.0	263.5	250.3	3.25	2.62	4.97	4.46	3.67	3.32	1.59	1.60
	0.15	107.3	93.9	297.5	284.8	2.71	2.03	5.25	4.77	4.32	3.80	1.14	1.21
	0.20	82.3	69.5	330.3	318.9	2.15	1.39	5.48	5.01	5.07	4.32	0.72	0.82
16	0	161.2	148.8	161.2	148.8	3.57	3.02	3.57	3.02	2.03	1.94	2.03	1.94
	0.05	139.1	126.1	184.5	172.2	3.24	2.65	3.86	3.33	2.50	2.32	1.59	1.58
	0.10	117.5	104.5	207.4	195.9	2.87	2.41	4.09	3.59	3.02	2.72	1.18	1.23
	0.15	97.2	84.4	229.9	219.3	2.47	1.79	4.28	3.80	3.61	3.13	0.81	0.89
	0.20	78.6	66.2	251.5	242.1	2.04	1.29	4.41	3.95	4.27	3.57	0.45	0.56
18	0	134.9	123.4	134.9	123.4	3.08	2.52	3.08	2.52	1.65	1.60	1.65	1.60
	0.05	118.6	106.7	151.3	140.4	2.84	2.25	3.29	2.76	2.06	1.92	1.26	1.27
	0.10	102.7	90.6	167.4	157.3	2.56	1.93	3.45	2.94	2.53	2.26	0.91	0.97
	0.15	87.5	75.4	182.9	173.8	2.26	1.58	3.57	3.08	3.05	2.62	0.59	0.68
	0.20	73.2	61.3	197.5	189.6	1.93	1.18	3.65	3.18	3.64	2.99	0.28	0.41
20	0	114.4	103.4	114.4	103.4	2.69	2.14	2.69	2.14	1.36	1.33	1.36	1.33
	0.05	102.3	91.3	126.4	116.6	2.52	1.92	2.85	2.31	1.73	1.61	1.03	1.05
	0.10	90.3	78.9	137.9	129.3	2.31	1.68	2.96	2.45	2.14	1.92	0.72	0.79
	0.15	78.5	67.1	148.9	141.3	2.08	1.39	3.04	2.55	2.61	2.22	0.44	0.55
	0.20	67.4	56.1	159.2	152.4	1.82	1.08	3.07	2.62	3.14	2.54	0.18	0.31

Table 1 presents solutions obtained with the following parameter values: $l = 60$, $R = 60$, $\nu = 0.3$, $\Delta = 0$, 0.05, 0.10, 0.15, 0.20, $h = 8$, 10, 12, 14, 16, 18, 20, and $s = l/2$.

Table 1 summarizes the values of the deflection $w\,(u_\gamma)$ in the midsection of the shell for two values of θ and series of values of ellipticity Δ and thickness h obtained in the spatial (I) and refined (II) formulations.

It can be seen that as the thickness of the circular shell ($\Delta = 0$) increases from 8 to 20, the difference between the refined and exact solutions changes from 5 to 9%. For the elliptic shell with Δ increasing from 0.05 to 0.20, this difference calculated at the major vertex ($\theta = 0$) changes as follows: from 5 to 29% for $h = 8$; from 7 to 18% for $h = 10$; from 7 to 16% for $h = 12$; from 8 to 14% for $h = 14$; from 9 to 16% for $h = 16$; from 10 to 16% for $h = 18$; and from 10 to 17% for $h = 20$. Hence, with nonzero ellipticity Δ, the difference between the two theories for smaller thicknesses ($h = 8$) is greater than for larger thicknesses ($h = 20$). The following pattern is observed at the vertex of the softer region ($\theta = \pi / 2$): the greater the ellipticity, the less the difference between the deflections by two theories.

Thus, Table 1 suggests that the magnitude of the deflection at the major vertex of the elliptic cross section of nonthin cylindrical shells, unlike circular cylindrical shells, is determined by the degree of increase in the curvature of the shell.

If $R = 60\,(\Delta = 0)$, then while the ellipticity changes as $\Delta = 0.05$; 0.10; 0.15; 0.20 at $\theta = 0$, the radius of curvature changes as 51.5, 44.1, 37.5, 31.1. The associated ratios of thickness to radius of curvature h / R_θ are: 0.13, 0.16, 0.18, 0.21, 0.25 for $h = 8$; 0.23, 0.27, 0.31, 0.37, 0.44 for $h = 14$; and 0.33, 0.39, 0.45, 0.55, 0.63 for $h = 20$. While $h / R_\theta \leq 0.05$ is assumed for thin shells, the ratio h / R_θ increases with the ellipticity and thickness, which is the cause of the difference between the deflections predicted by the refined and spatial models.

Thus, for $R = 60$ and $\theta = 0$, the deflections by the two theories differ by no greater than 10% for $\Delta = 0.05$; 0.10 when the thickness h changes from 8 to 12 and only for $\Delta = 0.05$ when the thickness h changes from 14 to 20. For $\theta = \pi / 2$, these deflections differ by no greater than 8%.

Noteworthy is the following feature of the refined theory of shells based on the straight-line hypothesis: the difference between the deflections predicted by the refined and spatial theories is greater for thinner shells ($h = 8$) than for thicker shells ($10 \leq h \leq 20$) (see Table 1). This feature can apparently be attributed to the fact that the system of differential equations of the refined theory becomes stiffer, i.e., ill-conditioned, with decreasing thickness. These equations do not go over, by passing to the limit, into the classical equations of shells and their numerical solution is an unstable process.

Table 1 also collects the values of the stresses σ_θ^+ and σ_θ^- (on the outside and inside surfaces) for $\theta = 0$, $\theta = \pi /2$; $\Delta = 0$, 0.05, 0.10, 0.15, 0.20; and $h = 8$, 10, 12, 14, 16, 18, 20. It can be seen that the stress σ_θ^+ in the stiffer region ($\theta = 0$) decreases considerably with increase in Δ for fixed h. The stress σ_θ^+ also somewhat decreases with increase in h. The difference between the values of σ_θ^+ predicted by the refined and spatial models increases significantly with the ellipticity Δ and insignificantly with the thickness h. When $\theta = \pi / 2$, the stress σ_θ^+ increases a little with increase in Δ and decreases with increase in h. The difference between the two theories is less than 10% for $h < 18$.

Table 1 shows that when $\theta = 0$, σ_θ^- increases almost twofold with increase in Δ and decreases a little with increase in h. The difference between the stresses predicted by the two models increases with the ellipticity Δ and reaches 19% for $h = 20$. For $\theta = \pi / 2$, the difference between the two values of σ_θ^- is slightly less than that between the values of σ_θ^+.

Since the stresses $\sigma_s^\pm$ are less than the stresses $\sigma_\theta^\pm$, the difference between the values of the former predicted by the two theories is greater than the difference between the values of the latter.

We have analyzed the stress state of nonthin cylindrical shells with elliptic cross section and may conclude that the stress–strain analysis of nonthin shells with a coordinate surface of varying curvature should account for the increase in the thickness and the value of the ratio h / R_θ in different regions of the coordinate surface and should involve assessment of the resulting error.

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