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To the Beginning of the Third Millennium

STRESS SOLUTIONS TO THE THREE-DIMENSIONAL PROBLEM OF ELASTICITY

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New representations of the stress tensor in the linear theory of elasticity and thermoelasticity are proposed. These representations satisfy the equilibrium equations and the strain compatibility equation. The stress tensor is expressed in terms of a harmonic tensor or a harmonic vector. The second boundary-value problem for an elastic half-space and an elastic layer is solved as an example

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It is preferred to solve the three-dimensional problem of elasticity in terms of stresses rather than displacements, especially if stresses are prescribed on the surface (second boundary-value problem). For a review of general solutions to the three-dimensional problem of elasticity, see [1, 17, 21, 23, 30, 32].

In this paper, we briefly summarize stress solutions to the three-dimensional problem of elasticity obtained in [5–12, 36, 37].

1. Classical Formulation of the Stress Problem in Elasticity. Consider an isotropic, homogeneous elastic material occupying a volume V in three-dimensional Euclidean space R^3. Points of the space R^3 are denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$. Body forces are absent.

Stress boundary-value solutions in the linear theory of elasticity require that the stress tensor $\hat{T}$ satisfy, in V, the equation of statics

$$\operatorname{div}\hat{T} = 0 \tag{1.1}$$

and the Beltrami compatibility equation

$$\Delta\hat{T} + \frac{1}{1+\nu}\nabla\nabla\sigma = 0, \tag{1.2}$$

where ν is Poisson's ratio; ∇ is the inverted delta in R^3; $\Delta = \nabla\cdot\nabla$ is the Laplacian in R^3; and $\sigma = I_1(\hat{T})$ is the first invariant of the stress tensor.

The well-known representations of the stress tensor in terms of the Maxwell and Morera functions satisfying Eq. (1.1) are not invariant. An invariant representation for $\hat{T}$ was independently obtained by Finci, Krutkov, and Blokh [3]:

$$\hat{T} = \operatorname{rot}*(\operatorname{rot}\hat{\Phi})^* = \operatorname{Ink}\hat{\Phi}, \tag{1.3}$$

where $\hat{\Phi}$ is a symmetric tensor of the second rank.

Representation (1.3) satisfies Eq. (1.1). Substituting (1.3) into (1.2), Krutkov has derived a rather complicated differential equation for the tensor $\hat{\Phi}$ [18].

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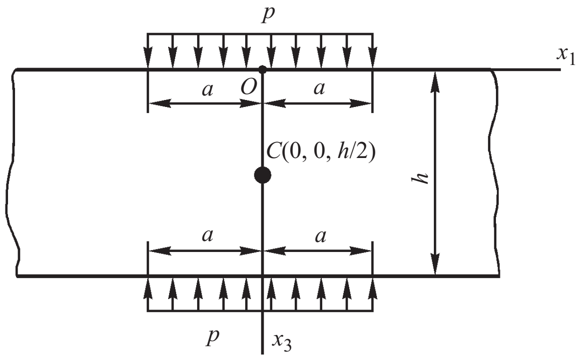

Fig. 1

Two methods may be used to find general stress solutions. One method, followed by Krutkov, is to find a representation for the stress tensor that would satisfy Eq. (1.1) and to substitute it into (1.2). One may proceed in reverse: find a representation for $\hat{T}$ that would satisfy Eq. (1.2) and then substitute it into (1.1). The other method seeks to represent the stress tensor in terms of simpler functions (harmonic, for one), with a representation for $\hat{T}$ being not postulated beforehand but rather found in the process of solution. We will follow the latter method to derive a new general stress solution.

Obviously,

$$2\nabla\varphi = \Delta(\mathbf{R}\varphi) - \mathbf{R}\Delta\varphi\,, \tag{1.4}$$

where φ is a scalar and $\mathbf{R} = \boldsymbol{i}_s x_s$ is the position vector.

Applying (1.4) to σ yields

$$2\nabla\sigma = \Delta(\mathbf{R}\sigma), \tag{1.5}$$

where σ is assumed to be a harmonic function ($\Delta\sigma = 0$) in the absence of body forces.

Substituting (1.5) into (1.2), we obtain

$$\Delta\left[\hat{T} + \frac{1}{2(1+\nu)}\nabla(\mathbf{R}\sigma)\right] = 0. \tag{1.6}$$

Expression (1.6) is a new form of the Beltrami compatibility equation.

It can be shown that

$$\nabla(\mathbf{R}\sigma) = (\nabla\sigma)\mathbf{R} + \sigma\hat{E}, \tag{1.7}$$

where $\hat{E}$ is a unit tensor of the second rank.

With (1.7), Eq. (1.6) becomes

$$\Delta\left[\hat{T} + \frac{(\nabla\sigma)\mathbf{R} + \sigma\hat{E}}{2(1+\nu)}\right] = 0. \tag{1.8}$$

If

$$\hat{G} = \hat{T} + \frac{(\nabla\sigma)\mathbf{R} + \sigma\hat{E}}{2(1+\nu)}\,, \tag{1.9}$$

then Eq. (1.8) yields

$$\Delta\hat{G} = 0, \tag{1.10}$$

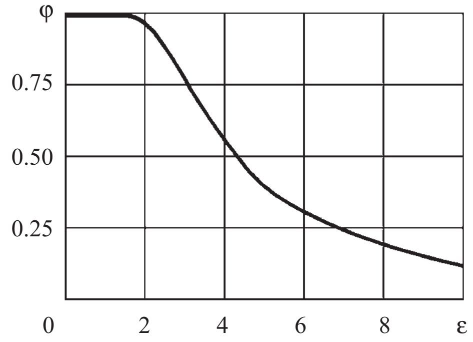

Fig. 2

i.e., $\hat{G}$ is a harmonic tensor. The properties of harmonic tensors are detailed in [3]. Formula (1.9) expresses the harmonic tensor $\hat{G}$ in terms of the stress tensor $\hat{T}$ and its first invariant σ.

Formula (1.9) yields

$$\hat{T} = \hat{G} - \frac{(\nabla\sigma)\mathbf{R} + \sigma\hat{E}}{2(1+\nu)}. \tag{1.11}$$

Representation (1.11) satisfies the strain compatibility equation (1.2). Let the components of the stress tensor also satisfy the equilibrium equation

$$\nabla \cdot \hat{T} = 0. \tag{1.12}$$

Taking the divergence of (1.11), we obtain

$$\nabla \cdot \hat{T} = \nabla \cdot \hat{G} - \frac{1}{2(1+\nu)} \Delta (\mathbf{R}\sigma).$$

Substituting Eqs. (1.5) and (1.12) into this relation, we get

$$\nabla \cdot \hat{G} = \frac{1}{1+\nu} \nabla\sigma, \tag{1.13}$$

where σ is assumed known.

Thus, if the components of the tensor $\hat{G}$ satisfy Eq. (1.13), then the components of the stress tensor $\hat{T}$ satisfy the equilibrium equation (1.12).

Expressed in terms of Cartesian coordinates, Eq. (1.13) takes the form

$$\frac{\partial b_{11}}{\partial x_1} + \frac{\partial b_{21}}{\partial x_2} + \frac{\partial b_{31}}{\partial x_3} = 2\alpha \frac{\partial \sigma}{\partial x_1}, \qquad \frac{\partial b_{12}}{\partial x_1} + \frac{\partial b_{22}}{\partial x_2} + \frac{\partial b_{32}}{\partial x_3} = 2\alpha \frac{\partial \sigma}{\partial x_2},$$

$$\frac{\partial b_{13}}{\partial x_1} + \frac{\partial b_{23}}{\partial x_2} + \frac{\partial b_{33}}{\partial x_3} = 2\alpha \frac{\partial \sigma}{\partial x_3}, \tag{1.14}$$

where b_{ij} are the components of the harmonic tensor $\hat{G}$; $\alpha = 1/2(1+\nu)$.

Thus, the components b_{ij} are related by (1.14).

By the reciprocity of tangential stresses ($\sigma_{ij} = \sigma_{ji}$, $i \neq j$), Eq. (1.11) leads to

$$b_{21} = b_{12} + \alpha\left(x_1 \frac{\partial\sigma}{\partial x_2} - x_2 \frac{\partial\sigma}{\partial x_1}\right), \qquad b_{31} = b_{13} + \alpha\left(x_1 \frac{\partial\sigma}{\partial x_3} - x_3 \frac{\partial\sigma}{\partial x_1}\right),$$

$$b_{32} = b_{23} + \alpha\left(x_2 \frac{\partial\sigma}{\partial x_3} - x_3 \frac{\partial\sigma}{\partial x_2}\right). \tag{1.15}$$

The nine components b_{ij} of the asymmetric harmonic tensor can be determined by satisfying three boundary conditions and the six relations (1.14), (1.15).

Relation (1.11) yields

$$\sigma_{ii} = b_{ii} - \alpha\left(\sigma + x_i \frac{\partial\sigma}{\partial x_i}\right), \quad \sigma_{ij} = b_{ji} - \alpha x_i \frac{\partial\sigma}{\partial x_j} \quad (i \neq j). \tag{1.16}$$

Let us determine σ appearing in (1.14)–(1.16).

If there are no body forces, then

$$\sigma = \frac{2(1+v)}{1-2v}\mu\theta,$$

where μ is the shear modulus, and θ is dilatation.

Betti and Cerruti [22, 30] proposed a method to find θ when either surface displacements or surface loads are known. However, finding θ is as difficult as solving the original problem. We will demonstrate below that there is no need for the preliminary determination of σ—it can be found while solving the boundary-value problem to determine σ_{ij}.

The idea of the method of determining σ is as follows. If, for example, we are solving the second (static) boundary-value problem with the plane $x_3 = 0$ as a boundary, then the following stresses are prescribed on it: σ_{31}, σ_{32}, and σ_{33}. Formulas (1.16) yield

$$\sigma_{31} = b_{13} - \alpha x_3 \frac{\partial\sigma}{\partial x_1}, \quad \sigma_{32} = b_{23} - \alpha x_3 \frac{\partial\sigma}{\partial x_2}, \quad \sigma_{33} = (b_{33} - \alpha\sigma) - \alpha x_3 \frac{\partial\sigma}{\partial x_3}. \tag{1.17}$$

The theory of harmonic functions and formulas (1.17) allow us to determine the components b_{13}, b_{23}, and b_{33} from the known boundary stresses. Substituting then these components into the third relation in (1.14), we find σ. Let us exemplify the rest of the procedure.

1.1. Given an elastic isotropic half-space $x_3 \geq 0$, solve the second (static) boundary-value problem. The following stresses act on the boundary $x_3 = 0$ of the half-space:

$$\sigma_{3i} = \begin{cases} -f_i(x_1, x_2) & \text{if } (x_1, x_2) \in \Omega_t, \\ 0 & \text{if } (x_1, x_2) \notin \Omega_t, \end{cases} \tag{1.18}$$

where $i = 1, 2, 3$, and Ω_i are the loaded regions in the plane $x_3 = 0$.

Introduce the functions

$$N_i(x_1, x_2, x_3) = \frac{1}{2\pi}\iint_{\Omega_i} f_i(y_1, y_2)\ln(x_3 + r)dy_1 dy_2 \quad (i = 1, 2, 3), \tag{1.19}$$

$$r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2, \quad \frac{\partial}{\partial x_3}\ln(x_3 + r) = \frac{1}{r}. \tag{1.20}$$

The functions N_i are harmonic in the half-space $x_3 > 0$, and

$$\lim_{x_3 \to +0} \frac{\partial^2 N_i}{\partial x_3^2} = \frac{1}{2\pi} \lim_{x_3 \to +0} \frac{\partial}{\partial x_3} \iint_{\Omega_i} f_i(y_1, y_2) r^{-1} \partial y_1 \partial y_2 = \begin{cases} -f_i(x_1, x_2), & (x_1, x_2) \in \Omega_i, \\ 0, & (x_1, x_2) \notin \Omega_i \end{cases} \tag{1.21}$$

on the plane $x_3 = 0$.

With (1.17)–(1.21), the harmonic functions b_{13}, b_{23}, and b_{33} can be expressed as

$$b_{13}=\frac{\partial^2 N_1}{\partial x_3^2}, \quad b_{23}=\frac{\partial^2 N_2}{\partial x_3^2}, \quad b_{33}=\frac{\partial^2 N_3}{\partial x_3^2}+\alpha\sigma. \tag{1.22}$$

Thus expressed, the components b_{i3} (i = 1, 2, 3) satisfy the boundary conditions (1.18).

Substituting (1.22) into the last relation in (1.14), we get

$$\sigma=2(1+\nu)\frac{\partial\varphi}{\partial x_3}, \quad \varphi=\frac{\partial N_1}{\partial x_1}+\frac{\partial N_2}{\partial x_2}+\frac{\partial N_3}{\partial x_3}. \tag{1.23}$$

Formula (1.23) yields σ, which is the first invariant of the stress tensor.

Thus, we have found σ during the problem solving process and, hence, there is no need to use the Betti–Cerruti method to determine it preliminarily.

Substituting (1.22) into (1.17) yields

$$\sigma_{31}=\frac{\partial^2 N_1}{\partial x_3^2}-\alpha x_3\frac{\partial\sigma}{\partial x_1}, \quad \sigma_{32}=\frac{\partial^2 N_2}{\partial x_3^2}-\alpha x_3\frac{\partial\sigma}{\partial x_2}, \quad \sigma_{33}=\frac{\partial^2 N_3}{\partial x_3^2}-\alpha x_3\frac{\partial\sigma}{\partial x_3}, \tag{1.24}$$

where N_i (i = 1, 2, 3) and σ are defined by (1.19) and (1.23), respectively. Expressions (1.24) allow us to determine the stresses on the area elements perpendicular to the x_3-axis.

To determine the other stress components σ_{11}, σ_{22}, and σ_{12}, it is first necessary to find the harmonic functions b_{11}, b_{22}, b_{12}, b_{21}, b_{31}, and b_{32}. To this end, we use Eqs. (1.14) and (1.15).

Doing this gives

$$b_{12}=\frac{\partial^2 N_1}{\partial x_2\partial x_3}+\frac{\partial^2 N_2}{\partial x_1\partial x_3}-\frac{\partial^2 N_3}{\partial x_1\partial x_2}+2\nu\frac{\partial^2\Phi}{\partial x_1\partial x_2}+\frac{\partial}{\partial x_1}\left(x_2\frac{\partial\varphi}{\partial x_3}-x_3\frac{\partial\varphi}{\partial x_2}\right),$$

$$b_{11}=\frac{\partial^2 N_1}{\partial x_1\partial x_3}-\frac{\partial^2 N_2}{\partial x_2\partial x_3}+\frac{\partial^2 N_3}{\partial x_2^2}+\frac{\partial\varphi}{\partial x_3}-2\nu\frac{\partial^2\Phi}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(x_1\frac{\partial\varphi}{\partial x_3}-x_3\frac{\partial\varphi}{\partial x_1}\right),$$

$$b_{22}=-\frac{\partial^2 N_1}{\partial x_1\partial x_3}+\frac{\partial^2 N_2}{\partial x_2\partial x_3}+\frac{\partial^2 N_3}{\partial x_1^2}+\frac{\partial\varphi}{\partial x_3}-2\nu\frac{\partial^2\Phi}{\partial x_1^2}+\frac{\partial}{\partial x_2}\left(x_2\frac{\partial\varphi}{\partial x_3}-x_3\frac{\partial\varphi}{\partial x_2}\right). \tag{1.25}$$

Thus, (1.15), (1.22), and (1.25) define all the nine components b_{ij} of the harmonic asymmetric tensor. Formulas (1.25) include a function $\Phi(x_1, x_2, x_3)$ related to $\varphi(x_1, x_2, x_3)$ by $\varphi=\partial\Phi/\partial x_3$.

After determining all the harmonic functions, we substitute them into (1.16) to obtain formulas for the stress components:

$$\sigma_{11}=\frac{\partial^2 N_1}{\partial x_1\partial x_3}-\frac{\partial^2 N_2}{\partial x_2\partial x_3}+\frac{\partial^2 N_3}{\partial x_2^2}+\frac{\partial\varphi}{\partial x_3}-x_3\frac{\partial^2\varphi}{\partial x_1^2}-2\nu\frac{\partial^2\Phi}{\partial x_2^2},$$

$$\sigma_{22}=-\frac{\partial^2 N_1}{\partial x_1\partial x_3}+\frac{\partial^2 N_2}{\partial x_2\partial x_3}+\frac{\partial^2 N_3}{\partial x_1^2}+\frac{\partial\varphi}{\partial x_3}-x_3\frac{\partial^2\varphi}{\partial x_2^2}-2\nu\frac{\partial^2\Phi}{\partial x_1^2},$$

$$\sigma_{33}=\frac{\partial^2 N_3}{\partial x_3^2}-x_3\frac{\partial^2\varphi}{\partial x_3^2}, \quad \sigma_{12}=\frac{\partial^2 N_1}{\partial x_2\partial x_3}+\frac{\partial^2 N_2}{\partial x_1\partial x_3}-\frac{\partial^2 N_3}{\partial x_1\partial x_2}-x_3\frac{\partial^2\varphi}{\partial x_1\partial x_2}+2\nu\frac{\partial^2\Phi}{\partial x_1\partial x_2},$$

$$\sigma_{23}=\frac{\partial^2 N_2}{\partial x_3^2}-x_3\frac{\partial^2\varphi}{\partial x_2\partial x_3}, \quad \sigma_{31}=\frac{\partial^2 N_1}{\partial x_3^2}-x_3\frac{\partial^2\varphi}{\partial x_1\partial x_3}. \tag{1.26}$$

Thus, the method being discussed helped us to find a general solution to the second boundary-value problem for an elastic half-space. Special solutions to this problem may be found in monographs on elasticity theory (see, e.g., [21, 27, 34]).

To test the formulas, let us examine special cases of distributed load and concentrated normal force. The results obtained in [27, 34] for these cases follow from (1.26).

1.2. Consider an isotropic elastic layer ($0 \le x_3 \le h$). Three-dimensional problems for an elastic layer were addressed in [14, 21, 33, 35, 39, 43, 44]. We will use a rectangular coordinate frame x_1, x_2, x_3, with the x_3-axis being perpendicular to the boundary surfaces of the layer. Impose the following boundary conditions for stresses:

$$\sigma_{3j} = \begin{cases} f_j^0(x_1, x_2) & \text{on} \quad x_3 = 0, \\ f_j^h(x_1, x_2) & \text{on} \quad x_3 = h, \end{cases} \qquad j = 1, 2, 3, \tag{1.27}$$

where σ_{33} is the normal stress, and σ_{31} and σ_{32} are the tangential stresses. Let the functions f_j^0 and f_j^h be such that all the six equations of statics hold.

The stress tensor $\hat{T}$ must satisfy the equilibrium equation (1.1) and the Beltrami compatibility equation (1.2).

For simplicity, we assume that body forces are absent.

It was shown in [8] that Eq. (1.2) holds if

$$\hat{T} = \hat{G} - \frac{(\nabla\sigma)\mathbf{R} + \sigma\hat{E}}{2(1+\nu)}, \tag{1.28}$$

where $\hat{G}$ is a harmonic tensor, i.e., $\Delta\hat{G} = 0$; $\hat{E}$ is a unit tensor of the second rank; and $\mathbf{R} = \boldsymbol{i}_j x_j$ is the position vector.

The harmonic tensor $\hat{G}$ is asymmetric; therefore, it has nine components g_{mn}.

The stress tensor $\hat{T}$ defined by (1.28) will satisfy the equilibrium equation (1.1) if

$$\nabla \cdot \hat{G} = \frac{1}{1+\nu} \nabla\sigma. \tag{1.29}$$

By the reciprocity of tangential stresses ($\sigma_{mn} = \sigma_{nm}$, $m \ne n$), formula (1.28) leads to

$$g_{nm} = g_{mn} + \frac{1}{2(1+\nu)} \left(x_m \frac{\partial\sigma}{\partial x_n} - x_n \frac{\partial\sigma}{\partial x_m} \right) \qquad (m \ne n). \tag{1.30}$$

Relations (1.29) and (1.30) are equivalent to the six equations for the components g_{mn}. The other three components of the harmonic tensor can be found by satisfying the boundary conditions (1.27).

Each component g_{mn} of the harmonic tensor $\hat{G}$ satisfies the Laplace equation

$$\Delta g_{mn} = 0, \qquad m, n = 1, 2, 3. \tag{1.31}$$

Let us use the Fourier transformation. The two-dimensional Fourier transform of some function $f(x_1, x_2, x_3)$ is given by the following formula [29, 38]:

$$\bar{f}(\xi_1, \xi_2, x_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_3) e^{i(\xi_1 x_1 + \xi_2 x_2)} dx_1 dx_2.$$

Taking the two-dimensional Fourier transform of Eq. (1.31) and solving the resulting ordinary differential equation, we get

$$\bar{g}_{mn}(\xi_1, \xi_2, x_3) = A_{mn}(\xi_1, \xi_2) \sinh(kx_3) + B_{mn}(\xi_1, \xi_2) \cosh(kx_3), \tag{1.32}$$

where $k^2 = \xi_1^2 + \xi_2^2$ and $m, n = 1, 2, 3$.

The same is true of the function $\sigma(x_1, x_2, x_3)$:

$$\bar{\sigma}(\xi_1, \xi_2, x_3) = A_0(\xi_1, \xi_2) \sinh(kx_3) + B_0(\xi_1, \xi_2) \cosh(kx_3). \tag{1.33}$$

The stresses σ_{31}, σ_{32}, and σ_{33} are prescribed on the layer boundaries $x_3 = 0$ and $x_3 = h$. Equations (1.28) and (1.30) yield

$$\sigma_{31} = g_{13} - \alpha x_3 \frac{\partial \sigma}{\partial x_1}, \quad \sigma_{32} = g_{23} - \alpha x_3 \frac{\partial \sigma}{\partial x_2}, \quad \sigma_{33} = (g_{33} - \alpha\sigma) - \alpha x_3 \frac{\partial \sigma}{\partial x_3}, \quad \alpha = \frac{1}{2(1+\nu)}. \tag{1.34}$$

Taking the two-dimensional Fourier transform of formulas (1.34) and considering that the stresses decay at infinity, we obtain

$$\bar{\sigma}_{31} = \bar{g}_{13} + i\xi_1 \alpha x_3 \bar{\sigma}, \quad \bar{\sigma}_{32} = \bar{g}_{23} + i\xi_2 \alpha x_3 \bar{\sigma}, \quad \bar{\sigma}_{33} = (\bar{g}_{33} - \alpha\bar{\sigma}) - \alpha x_3 \frac{\partial \bar{\sigma}}{\partial x_3}. \tag{1.35}$$

Taking the Fourier transform of the boundary conditions (1.27), we obtain

$$\bar{\sigma}_{3m} = \begin{cases} \bar{f}_m^0(\xi_1, \xi_2) & \text{on} \quad x_3 = 0, \\ \bar{f}_m^h(\xi_1, \xi_2) & \text{on} \quad x_3 = h, \end{cases} \qquad m = 1, 2, 3. \tag{1.36}$$

Satisfying conditions (1.36) and using formulas (1.32), (1.33), and (1.35), we obtain the system of equations

$$B_{13} = \bar{f}_1^0, \quad B_{23} = \bar{f}_2^0, \quad B_{33} - \alpha B_0 = \bar{f}_3^0,$$

$$A_{13}\sinh(kh) + B_{13}\cosh(kh) + i\xi_1 \alpha h \left[A_0 \sinh(kh) + B_0 \cosh(kh)\right] = \bar{f}_1^h,$$

$$A_{23}\sinh(kh) + B_{23}\cosh(kh) + i\xi_2 \alpha h \left[A_0 \sinh(kh) + B_0 \cosh(kh)\right] = \bar{f}_2^h,$$

$$A_{33}\sinh(kh) + B_{33}\cosh(kh) - \alpha A_0 \left[\sinh(kh) + kh\cosh(kh)\right] - \alpha B_0 \left[\cosh(kh) + kh\sinh(kh)\right] = \bar{f}_3^h,$$

$$i\xi_1 A_{13} + i\xi_2 A_{23} - \kappa B_{33} + 2\alpha k B_0 = 0, \qquad i\xi_1 B_{13} + i\xi_2 B_{23} - \kappa A_{33} + 2\alpha k A_0 = 0, \tag{1.37}$$

where $\bar{f}_m^0$ and $\bar{f}_m^h$ (m = 1, 2, 3) are assumed known. The last two equations in (1.37) have been derived using the Fourier-transform of relation (1.29). Solving Eqs. (1.37) yields A_{13}, B_{13}, A_{23}, B_{23}, A_{33}, B_{33}, A_0, and B_0. Then, we can use formulas (1.32) and (1.33) to find $\bar{g}_{13}, \bar{g}_{23}, \bar{g}_{33}$, and $\bar{\sigma}$.

The system of equations (1.37) can be solved symbolically using Cramer's rule. For example, the determinant D of (1.37) is given by

$$D = k^2 \sinh(kh)\left[\sinh^2(kh) - (kh)^2\right]. \tag{1.38}$$

The general-form expressions for A_{13}, B_{13}, A_{23}, B_{23}, A_{33}, B_{33}, A_0, and B_0 are omitted as awkward. Let us consider an example to detail the procedure of determining these coefficients.

Solving the system of equations (1.37), we find $\bar{g}_{13}, \bar{g}_{23}, \bar{g}_{33}$, and $\bar{\sigma}$. Now we can find the other Fourier-transformed harmonic functions $\bar{g}_{11}, \bar{g}_{22}, \bar{g}_{12}, \bar{g}_{21}, \bar{g}_{31}$, and $\bar{g}_{32}$:

$$\bar{g}_{21} = \bar{g}_{12} + \alpha\left(-\xi_2 \frac{\partial \bar{\sigma}}{\partial \xi_1} + \xi_1 \frac{\partial \bar{\sigma}}{\partial \xi_2}\right), \quad \bar{g}_{31} = \bar{g}_{13} + \alpha\left(-i\frac{\partial^2 \bar{\sigma}}{\partial \xi_1 \partial x_3} + i\xi_1 x_3 \bar{\sigma}\right), \tag{1.39}$$

$$\bar{g}_{32} = \bar{g}_{23} + \alpha\left(-i\frac{\partial^2 \bar{\sigma}}{\partial \xi_2 \partial x_3} + i\xi_2 x_3 \bar{\sigma}\right), \quad k^2 \bar{g}_{12} = -i\xi_2 \frac{\partial \bar{g}_{13}}{\partial x_3} - i\xi_1 \frac{\partial \bar{g}_{32}}{\partial x_3} + \xi_1 \xi_2 \bar{g}_{33} - 2\nu\alpha\xi_1\xi_2\bar{\sigma},$$

$$k^2 \bar{g}_{11} = -i\xi_1 \frac{\partial \bar{g}_{31}}{\partial x_3} + i\xi_2 \frac{\partial \bar{g}_{23}}{\partial x_3} - \xi_2^2 \bar{g}_{33} + 2\alpha\left(k^2 + \nu\xi_2^2\right)\bar{\sigma},$$

$$k^2 \bar{g}_{22} = i\xi_1 \frac{\partial \bar{g}_{13}}{\partial x_3} - i\xi_2 \frac{\partial \bar{g}_{32}}{\partial x_3} - \xi_1^2 \bar{g}_{33} + 2\alpha\left(k^2 + \nu\xi_1^2\right)\bar{\sigma}. \tag{1.40}$$

Thus, $\bar{g}_{13}, \bar{g}_{23}, \bar{g}_{33}$, and $\bar{\sigma}$ follow from the system of equations (1.37), and the other harmonic functions from (1.39) and (1.40).

After finding $\bar{g}_{mn}$, we recover the harmonic functions g_{mn} using the inverse Fourier transform:

$$g_{mn}(x_1, x_2, x_3) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{g}_{mn}(\xi_1, \xi_2, x_3)\, e^{-i(x_1\xi_1 + x_2\xi_2)} d\xi_1 d\xi_2 .$$

The components of the stress tensor are determined from formula (1.28).

Example. Consider an elastic layer (of thickness h) compressed by normal forces uniformly distributed over squares with side length $2a$ located in the planes $x_3 = 0$ and $x_3 = h$ (Fig. 1).

Put

$$f_1^0 = f_2^0 = f_1^h = f_2^h = 0, \qquad f_3^0 = f_3^h = \begin{cases} -p, & |x_1| \le a,\ |x_2| \le a, \\ 0, & |x_1| \ge a, |x_2| \ge a \end{cases}$$

in the boundary conditions (1.27).

Hence,

$$\bar{f}_1^0 = \bar{f}_2^0 = \bar{f}_1^h = \bar{f}_2^h = 0,$$

$$\bar{f}_3^0 = \bar{f}_3^h = -\frac{p}{2\pi}\int_{-a}^{a}\int_{-a}^{a} e^{i(\xi_1 x_1 + \xi_2 x_2)} dx_1 dx_2 = -\frac{p}{2\pi}\frac{\sin a\xi_1 \sin a\xi_2}{\xi_1 \xi_2} \tag{1.41}$$

in (1.37).

We have solved the system of equations (1.37) with (1.41) symbolically using Cramer's rule and obtained the following:

$$A_{13} = \bar{f}_3 \frac{i\xi_1 k\beta}{D}\sinh\beta\,(\beta - \sinh\beta), \quad B_{13} = 0, \quad A_{23} = \bar{f}_3 \frac{i\xi_2 k\beta}{D}\sinh\beta(\beta - \sinh\beta), \quad B_{23} = 0,$$

$$A_{33} = \bar{f}_3 \frac{2k^2}{D}\sinh\beta(\cosh\beta - 1)(\beta - \sinh\beta), \quad B_{33} = \bar{f}_3 \frac{k^2}{D}\sinh\beta(2\sinh^2\beta - \beta\sinh\beta - \beta^2),$$

$$\alpha A_0 = \bar{f}_3 \frac{k^2}{D}\sinh\beta(\cosh\beta - 1)(\beta - \sinh\beta), \quad \alpha B_0 = \bar{f}_3 \frac{k^2}{D}\sinh^2\beta(\sinh\beta - \beta), \tag{1.42}$$

where $\beta = kh$, $\bar{f}_3 = \bar{f}_3^0 = \bar{f}_3^h$, and D is defined by (1.38).

Next, we have

$$\bar{g}_{13} = A_{13}\sinh(kx_3), \quad \bar{g}_{23} = A_{23}\sinh(kx_3), \quad \bar{g}_{33} = A_{33}\sinh(kx_3) + B_{33}\cosh(kx_3),$$

$$\bar{\sigma} = A_0\sinh(kx_3) + B_0\cosh(kx_3). \tag{1.43}$$

The other Fourier-transformed harmonic functions can be found from (1.39) and (1.40).

Now we will determine stresses on the area elements perpendicular to the x_3-axis. Substituting (1.42) and (1.43) into (1.35), we obtain

$$\bar{\sigma}_{31}(\xi_1, \xi_2, x_3) = -\frac{i\xi_1 \bar{f}_3}{\beta + \sinh\beta}\left[(h - x_3)\sinh(kx_3) + x_3\sinh(\beta - kx_3)\right],$$

$$\bar{\sigma}_{32}(\xi_1, \xi_2, x_3) = -\frac{i\xi_2 \bar{f}_3}{\beta + \sinh\beta}\left[(h - x_3)\sinh(kx_3) + x_3\sinh(\beta - kx_3)\right],$$

$$\bar{\sigma}_{33}(\xi_1,\xi_2,x_3)=\frac{\bar{f}_3}{\beta+\sinh\beta}\left[\sinh(kx_3)+(\beta-kx_3)\cosh(kx_3)+\sinh(\beta-kx_3)+kx_3\cosh(\beta-kx_3)\right], \tag{1.44}$$

where $\bar{f}_3$ is defined by (1.41). Taking the inverse Fourier transform of (1.44) yields the stresses $\sigma_{31}(x_1, x_2, x_3)$, $\sigma_{32}(x_1, x_2, x_3)$, and $\sigma_{33}(x_1, x_2, x_3)$.

Let us demonstrate the rest of the procedure by determining the normal stress σ_{33}. We have

$$\sigma_{33}(x_1,x_2,x_3)=-\frac{p}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(x_1\xi_1+x_2\xi_2)}\frac{\sin(a\xi_1)\sin(a\xi_2)}{\xi_1\xi_2\left[(kh)+\sinh(kh)\right]}$$

$$\times\left[\sinh(kx_3)+\sinh(kh-kx_3)+k(h-x_3)\cosh(kx_3)+kx_3\cosh(kh-kx_3)\right]d\xi_1 d\xi_2, \tag{1.45}$$

which can be rearranged as

$$\sigma_{33}(y_1,y_2,y_3)=-\frac{4p}{\pi^2}\int_0^{\infty}\int_0^{\infty}\frac{\sin\gamma_1\sin\gamma_2\cos(y_1\gamma_1)\cos(y_2\gamma_2)}{\gamma_1\gamma_2(\varepsilon\gamma+\sinh\varepsilon\gamma)}F(y_3,\gamma)d\gamma_1 d\gamma_2, \tag{1.46}$$

where $\varepsilon=h/a$, $\gamma_1=a\xi_1$, $\gamma_2=a\xi_2$, $\gamma=(\gamma_1^2+\gamma_2^2)^{1/2}$, $y_j=x_j/a$, j = 1, 2, 3, $-\infty < y_1, y_2 < \infty$, $0 \le y_3 \le \varepsilon$, and $F(y_3,\gamma)=\sinh(y_3\gamma)+\sinh(\varepsilon\gamma-y_3\gamma)+\gamma(\varepsilon-y_3)\cosh(y_3\gamma)+y_3\gamma\cosh(\varepsilon\gamma-y_3\gamma)$.

We evaluated the double integral in (1.46) numerically using quadrature formulas and replacing the upper limit ∞ with 50. Increasing the upper limit to 100 does not affect the result up to the fifth decimal place.

Next we used formula (1.46) to determine the stress σ_{33} at the point C with coordinates 0, 0, $h/2$ (Fig. 1) for different values of ε. Doing this gives $\sigma_{33}(C) = -p\varphi(\varepsilon)$. The function $\varphi(\varepsilon)$ is plotted in Fig. 2.

1.3. Let us discuss another general solution to Eqs. (1.1) and (1.2).

Consider an isotropic homogeneous elastic material occupying a volume V in three-dimensional space R^3. Body forces are absent.

The solution of Eq. (1.2) is sought in the form

$$\hat{T}=\hat{G}+\frac{1}{4\pi}\nabla\nabla\int_V\frac{\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y}), \tag{1.47}$$

where $\hat{G}$ is a harmonic symmetric tensor of the second rank; and $\varphi(\mathbf{x})$ is a scalar.

Representation (1.47) gives

$$\sigma=I_1(\hat{G})+\Delta f,\qquad f(x)=\frac{1}{4\pi}\int_V\frac{\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y}), \tag{1.48}$$

where $I_1(\hat{G})=g_{11}+g_{22}+g_{33}$ is the first invariant of the harmonic tensor $\hat{G}$.

Taking the Laplacian of Eq. (1.47) yields

$$\Delta\hat{T}=\nabla\nabla(\Delta f), \tag{1.49}$$

where $\Delta\hat{G}=0$.

Next, we use a well-known formula from Newtonian potential theory (potential for distributed masses)

$$\frac{1}{4\pi}\Delta\int_V\frac{\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}dV(\mathbf{y})=-\varphi(\mathbf{x}),\qquad \mathbf{x}\in V. \tag{1.50}$$

Formula (1.50) is valid if $\varphi(x)$ is piecewise-continuously differentiable. The functions $\varphi(\mathbf{x})$ to be used below are precisely such.

Formulas (1.48) and (1.50) yield

$$\Delta f(\mathbf{x}) = -\varphi(\mathbf{x}). \tag{1.51}$$

With (1.51), Eqs. (1.48) and (1.49) become

$$\sigma = I_1(\hat{G}) - \varphi, \qquad \Delta\hat{T} = -\nabla\nabla\varphi. \tag{1.52}$$

Substituting (1.52) into Eq. (1.2), we get

$$\varphi = \frac{I_1(\hat{G})}{2+\nu}. \tag{1.53}$$

With (1.53), formula (1.47) becomes

$$\hat{T} = \hat{G} + \frac{1}{4\pi(2+\nu)} \nabla\nabla \int_V \frac{I_1(\hat{G})}{|\mathbf{x}-\mathbf{y}|} dV(\mathbf{y}). \tag{1.54}$$

Since the tensors $\hat{T}$ and $\nabla\nabla f$ are symmetric, the tensor $\hat{G}$ must be such. Hence, the harmonic tensor $\hat{G}$ has six independent components g_{ij} $(i, j = 1, 2, 3)$.

Thus, formula (1.54) represents the stress tensor $\hat{T}$ in terms of the symmetric harmonic tensor $\hat{G}$ and its first invariant $I_1(\hat{G})$. In the Cartesian frame $Ox_1x_2x_3$, the components g_{ij} of the tensor $\hat{G}$ satisfy the Laplace equation

$$\Delta g_{ij} = 0 \qquad i, j = 1, 2, 3.$$

It is easy to verify that representation (1.54) satisfies the Beltrami compatibility equation (1.2). Using (1.52) and (1.53), we obtain

$$\Delta\hat{T} = -\frac{1}{2+\nu} \nabla\nabla I_1(\hat{G}), \qquad \sigma = \frac{1+\nu}{2+\nu} I_1(\hat{G}). \tag{1.55}$$

These expressions satisfy Eq. (1.2), which can be checked by direct substitution.

The displacement solution to the problem of elasticity is frequently based on the so-called Naghdi–Hsu transformation [45]

$$\mathbf{u}(\mathbf{x}) = \mathbf{B}(\mathbf{x}) + \frac{1}{8\pi(1-\nu)} \nabla \int_V \frac{\nabla \cdot \mathbf{B}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} dV(\mathbf{y}),$$

$$\mathbf{B}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - \frac{1}{4\pi(1-2\nu)} \nabla \int_V \frac{\nabla \cdot \mathbf{u}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} dV(\mathbf{y}), \tag{1.56}$$

where $\mathbf{u}(\mathbf{x})$ is the displacement vector, and $\mathbf{B}(\mathbf{x})$ is a harmonic vector. The first formula in (1.56) defines $\mathbf{u}$ in terms of $\mathbf{B}$, and the second formula in (1.56), vice versa, expresses the harmonic vector in terms of the displacement vector.

Let us show that the stress solution to the three-dimensional problem of elasticity is also based on a transformation similar to (1.56). The second relation (1.55) yields

$$I_1(\hat{G}) = \frac{2+\nu}{1+\nu}\sigma. \tag{1.57}$$

Substituting (1.57) into (1.54), we obtain

$$\hat{G} = \hat{T} - \frac{1}{4\pi(1+\nu)} \nabla\nabla \int_V \frac{\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} dV(\mathbf{y}), \tag{1.58}$$

where the harmonic symmetric tensor $\hat{G}$ is expressed in terms of the stress tensor $\hat{T}$ and its first invariant $\sigma = I_1(\hat{T})$.

Expressions (1.54) and (1.58) constitute a transformation for the three-dimensional stress problem of elasticity.

Representation (1.54) of the stress tensor $\hat{T}$ in terms of the symmetric harmonic tensor $\hat{G}$, i.e., six harmonic functions g_{ij}, satisfies the Beltrami equation (1.2).

Let (1.54) also satisfy the equilibrium equation (1.1). Taking the divergence of (1.54) yields

$$\nabla \cdot \hat{T} = \nabla \cdot \hat{G} + \frac{1}{4\pi(2+\nu)} \nabla \cdot \nabla \nabla \int_V \frac{I_1(\hat{G})}{|\mathbf{x}-\mathbf{y}|} dV(\mathbf{y}).$$

Considering the fact that $\nabla \cdot \nabla = \Delta$ and formula (1.50), we get

$$\nabla \cdot \hat{T} = \nabla \cdot \hat{G} - \frac{1}{(2+\nu)} \nabla I_1(\hat{G}). \tag{1.59}$$

In order that $\nabla \cdot \hat{T} = 0$, it is necessary to substitute the following equality into (1.59):

$$\nabla \cdot \hat{G} = \frac{1}{(2+\nu)} \nabla I_1(\hat{G}). \tag{1.60}$$

Thus, for representation (1.54) to satisfy the equilibrium equation (1.1) and the compatibility equation (1.2), it is necessary that the harmonic tensor $\hat{G}$ appearing in (1.54) be symmetric and satisfy Eq. (1.60).

In Cartesian coordinates, (1.60) consists of three equations. Hence, the six components of the tensor $\hat{G}$ are related by three additional constraints that follow from (1.60), i.e., the tensor $\hat{G}$ has only three independent components g_{ij}.

2. Second Formulation of the Stress Problem in Elasticity. The stress tensor $\hat{T}$ must satisfy the equation of statics

$$\operatorname{div} \hat{T} = 0 \tag{2.1}$$

and the strain compatibility equation

$$\operatorname{Ink} \hat{\varepsilon} = \frac{1}{2\mu} \operatorname{Ink} \left(\hat{T} - \frac{\nu}{1+\nu} \sigma \hat{E} \right) = 0, \tag{2.2}$$

where $\operatorname{Ink} \hat{Q} = \operatorname{rot}(\operatorname{rot} \hat{Q})^*$; $\hat{E}$ is a unit tensor; ν is Poisson's ratio; μ is the shear modulus; $\sigma = I_1(\hat{T})$ is the first invariant of the stress tensor; and $\hat{\varepsilon}$ is the linear strain tensor. Assume that body forces are absent.

Let us now address the compatibility equation (2.2). If the Ink operation on a symmetric tensor gives zero, then this tensor is deformation of some vector. Hence,

$$\hat{T} - \frac{\nu}{1+\nu} \sigma \hat{E} = \operatorname{def} \mathbf{c}, \tag{2.3}$$

where $\mathbf{c}$ is a vector,

$$\operatorname{def} \mathbf{c} = [(\nabla \mathbf{c})^* + \nabla \mathbf{c}]/2.$$

Equating the tensor traces on the left- and right-hand sides of (2.3), we find

$$\frac{1-2\nu}{1+\nu} \sigma = \operatorname{div} \mathbf{c}, \tag{2.4}$$

where $I_1(\operatorname{def} \mathbf{c}) = \operatorname{div} \mathbf{c}$ and $I_1(\hat{E}) = 3$.

With (2.4), expression (2.3) becomes

$$\hat{T} = \frac{\nu}{1-2\nu} \hat{E} \operatorname{div} \mathbf{c} + \operatorname{def} \mathbf{c}. \tag{2.5}$$

This stress tensor $\hat{T}$ identically satisfies the compatibility condition (2.2). Denote $\mathbf{c} = 2\mu \mathbf{c}_0$. Then (2.5) becomes

$$\hat{T}=2\mu\left(\frac{\nu}{1-2\nu}\hat{E}\operatorname{div}\mathbf{c}_0+\operatorname{def}\mathbf{c}_0\right). \tag{2.6}$$

Let us now express the vector $\mathbf{c}_0$ in terms of a harmonic vector $\mathbf{B}$ $(\nabla^2\mathbf{B}=0)$ and some scalar φ:

$$\mathbf{c}_0=\mathbf{B}-x_3\nabla\varphi, \tag{2.7}$$

where x_1, x_2, and x_3 are rectangular coordinates. The scalar function φ will be related below to the harmonic vector $\mathbf{B}$.

Substituting (2.7) into (2.6) yields

$$\hat{T}=2\mu\left\{\frac{\nu}{1-2\nu}\hat{E}\operatorname{div}\left(\mathbf{B}-x_3\nabla\varphi\right)+\operatorname{def}\left(\mathbf{B}-x_3\nabla\varphi\right)\right\}. \tag{2.8}$$

Representation (2.8) satisfies the compatibility condition (2.2). For (2.8) to satisfy also the differential equilibrium equation (2.1), it is necessary that

$$\nabla^2\varphi=0,\qquad \frac{\partial\varphi}{\partial x_3}=\frac{1}{3-4\nu}\nabla\cdot\mathbf{B}. \tag{2.9}$$

Thus, (2.8) represents the stress tensor $\hat{T}$ in terms of the harmonic vector $\mathbf{B}$ and the harmonic scalar φ related by (2.9). If $\nabla^2\mathbf{B}=0$ in addition to (2.9), then (2.8) satisfies Eqs. (2.1) and (2.2).

Expression (2.8) can be simplified. Since

$$\operatorname{div}\left(x_3\nabla\varphi\right)=\frac{\partial\varphi}{\partial x_3}+x_3\nabla^2\varphi,$$

we have

$$\operatorname{div}\left(\mathbf{B}-x_3\nabla\varphi\right)=2(1-2\nu)\frac{\partial\varphi}{\partial x_3},$$

where (2.9) has been taken into account.

Hence, representation (2.8) takes the final form

$$\hat{T}=2\mu\left[2\nu\hat{E}\frac{\partial\varphi}{\partial x_3}+\operatorname{def}\left(\mathbf{B}-x_3\nabla\varphi\right)\right]. \tag{2.10}$$

Expression (2.10) can be written component-wise:

$$\sigma_{st}=\mu\left[4\nu\delta_{st}\frac{\partial\varphi}{\partial x_3}+\left(\frac{\partial B_s}{\partial x_t}+\frac{\partial B_t}{\partial x_s}\right)-\frac{\partial}{\partial x_t}\left(x_3\frac{\partial\varphi}{\partial x_s}\right)-\frac{\partial}{\partial x_s}\left(x_3\frac{\partial\varphi}{\partial x_t}\right)\right],\qquad s,t=1,2,3, \tag{2.11}$$

where δ_{st} is the Kronecker delta.

Representation (2.10) satisfies the equation of statics (2.1) and the compatibility equations (2.2). Expression (2.11) allows us to determine the components B_1, B_2, and B_3 of the harmonic vector $\mathbf{B}$ from known boundary stresses, and the harmonic scalar φ from (2.9).

This is how we can derive the final expression for the stress tensor $\hat{T}$ in a specific boundary-value problem.

2.1. Let us test expression (2.10) against the second boundary-value problem for an elastic half-space in the static case. The following stresses are prescribed on the boundary $x_3 = 0$ of the half-space:

$$\sigma_{3t}=\begin{cases}-f_t(x_1,x_2) & \text{if}\quad (x_1,x_2)\in\Omega_t,\\ 0 & \text{if}\quad (x_1,x_2)\notin\Omega_t,\end{cases} \tag{2.12}$$

where Ω_t are the loaded regions in the plane $x_3 = 0$ (t = 1, 2, 3).

Formula (2.11) yields expressions for the stresses σ_{31}, σ_{32}, and σ_{33} on $x_3 = 0$. Then

$$\sigma_{31} = \mu\left(\frac{\partial B_3}{\partial x_1} + \frac{\partial B_1}{\partial x_3} - \frac{\partial \varphi}{\partial x_1}\right), \quad \sigma_{32} = \mu\left(\frac{\partial B_3}{\partial x_2} + \frac{\partial B_2}{\partial x_3} - \frac{\partial \varphi}{\partial x_2}\right), \quad \sigma_{33} = 2\mu\left(\frac{\partial B_3}{\partial x_3} - (1-2\nu)\frac{\partial \varphi}{\partial x_3}\right). \tag{2.13}$$

Introduce functions N_t harmonic in the half-space $x_3 > 0$:

$$N_t(x_1, x_2, x_3) = \frac{1}{2\pi}\iint_{\Omega_t} f_t(y_1, y_2)\ln(x_3 + r)dy_1 dy_2, \qquad t = 1, 2, 3. \tag{2.14}$$

Then

$$\lim_{x_3 \to +c} \frac{\partial^2 N_t}{\partial x_3^2} = \begin{cases} -f_t(x_1, x_2) & \text{if } \ (x_1, x_2) \in \Omega_t, \\ 0 & \text{if } \ (x_1, x_2) \notin \Omega_t. \end{cases} \tag{2.15}$$

Using (2.12)–(2.15), we arrive at a system of equations whose solution is

$$B_1 = \frac{1}{2\mu}\left(2\frac{\partial N_1}{\partial x_3} - \frac{\partial N_3}{\partial x_1}\right) + 2\nu\frac{\partial \Psi}{\partial x_1}, \qquad B_2 = \frac{1}{2\mu}\left(2\frac{\partial N_2}{\partial x_3} - \frac{\partial N_3}{\partial x_2}\right) + 2\nu\frac{\partial \Psi}{\partial x_2},$$

$$B_3 = \frac{1}{2\mu}\frac{\partial N_3}{\partial x_3} + (1-2\nu)\varphi, \qquad \varphi = \frac{\partial \Psi}{\partial x_3}. \tag{2.16}$$

It follows from (2.16) that

$$\nabla \cdot \mathbf{B} = \frac{1}{\mu}\frac{\partial}{\partial x_3}(\nabla \cdot \mathbf{N}) + (1-4\nu)\frac{\partial \varphi}{\partial x_3}, \qquad \mathbf{N} = (N_1, N_2, N_3). \tag{2.17}$$

Relations (2.9) and (2.17) yield

$$\varphi = \frac{1}{2\mu}(\nabla \cdot \mathbf{N}). \tag{2.18}$$

Thus, the harmonic vector $\mathbf{B} = (B_1, B_2, B_3)$ and the harmonic scalar φ appearing in (2.10) are defined by (2.16) and (2.18) in the second boundary-value problem for an elastic half-space. Substituting these expressions into (2.11), we obtain the final formulas for the stresses σ_{st}.

2.2. Let us show how formula (2.10) can be used to determine stresses in an elastic layer. Consider an isotropic elastic layer ($0 \le x_3 \le h$) with the following boundary conditions for the stresses σ_{31}, σ_{32}, and σ_{33}:

$$\sigma_{3t} = \begin{cases} f_t^0(x_1, x_2) & \text{on } \ x_3 = 0, \\ f_t^h(x_1, x_2) & \text{on } \ x_3 = h, \end{cases} \qquad (t = 1, 2, 3). \tag{2.19}$$

Formula (2.11) yields

$$\sigma_{31} = \mu\left(\frac{\partial B_3}{\partial x_1} + \frac{\partial B_1}{\partial x_3} - \frac{\partial \varphi}{\partial x_1} - 2x_3\frac{\partial^2 \varphi}{\partial x_1 \partial x_3}\right),$$

$$\sigma_{32} = \mu\left(\frac{\partial B_3}{\partial x_2} + \frac{\partial B_2}{\partial x_3} - \frac{\partial \varphi}{\partial x_2} - 2x_3\frac{\partial^2 \varphi}{\partial x_2 \partial x_3}\right),$$

$$\sigma_{33} = 2\mu\left[\frac{\partial B_3}{\partial x_3} - (1-2\nu)\frac{\partial \varphi}{\partial x_3} - x_3\frac{\partial^2 \varphi}{\partial x_3^2}\right]. \tag{2.20}$$

Each component B_m of the harmonic vector **B** satisfies the Laplace equation

$$\nabla^2 B_m = 0, \qquad m = 1, 2, 3. \tag{2.21}$$

The two-dimensional Fourier transform of some function $f(x_1, x_2, x_3)$ is given by the following formula [38]:

$$\bar{f}(\xi_1, \xi_2, x_3) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, e^{i(\xi_1 x_1 + \xi_2 x_2)}\, dx_1 dx_2. \tag{2.22}$$

Taking the Fourier transform (2.22) of Eq. (2.21) and solving the resulting ordinary differential equation, we obtain

$$\overline{B}_m(\xi_1, \xi_2, x_3) = A_m(\xi_1, \xi_2)\sinh(\kappa x_3) + C_m(\xi_1, \xi_2)\cosh(\kappa x_3), \tag{2.23}$$

where $\kappa = \xi_1^2 + \xi_2^2$ and $m = 1, 2, 3$.

The same is true for the function φ appearing in the equation $\nabla^2\varphi = 0$:

$$\overline{\varphi}(\xi_1, \xi_2, x_3) = A_0(\xi_1, \xi_2)\sinh(\kappa x_3) + C_0(\xi_1, \xi_2)\cosh(\kappa x_3). \tag{2.24}$$

Taking the Fourier transform (2.22) of formulas (2.20) and considering that the stresses decay at infinity, we obtain

$$\overline{\sigma}_{31} = \mu\left(-i\xi_1\overline{B}_3 + \frac{\partial \overline{B}_1}{\partial x_3} + i\xi_1\overline{\varphi} + 2i\xi_1 x_3\frac{\partial\overline{\varphi}}{\partial x_3}\right),$$

$$\overline{\sigma}_{32} = \mu\left(-i\xi_2\overline{B}_3 + \frac{\partial \overline{B}_2}{\partial x_3} + i\xi_2\overline{\varphi} + 2i\xi_2 x_3\frac{\partial\overline{\varphi}}{\partial x_3}\right),$$

$$\overline{\sigma}_{33} = 2\mu\left(\frac{\partial \overline{B}_3}{\partial x_3} - (1-2\nu)\frac{\partial\overline{\varphi}}{\partial x_3} - x_3\frac{\partial^2\overline{\varphi}}{\partial x_3^2}\right). \tag{2.25}$$

Subjected to the Fourier transform, the boundary conditions (2.19) become

$$\overline{\sigma}_{3t} = \begin{cases} \bar{f}_t^0(\xi_1, \xi_2) & \text{on} \quad x_3 = 0, \\ \bar{f}_t^h(\xi_1, \xi_2) & \text{on} \quad x_3 = h \end{cases} \qquad (t = 1, 2, 3). \tag{2.26}$$

Satisfying conditions (2.26) and using formulas (2.23)–(2.25), we obtain the system of equations

$$-i\xi_1 C_3 + \kappa A_1 + \xi_1 C_0 = \bar{f}_1^0/\mu, \quad -i\xi_2 C_3 + \kappa A_2 + \xi_2 C_0 = \bar{f}_2^0/\mu, \quad \kappa A_3 - (1-2\nu)\kappa A_0 = \bar{f}_3^0/2\mu,$$

$$-i\xi_1\left[A_3\sinh(\kappa h) + C_3\cosh(\kappa h)\right] + \kappa\left[A_1\cosh(\kappa h) + C_1\sinh(\kappa h)\right]$$

$$+ i\xi_1\left[A_0\sinh(\kappa h) + C_0\cosh(\kappa h)\right] + 2i\xi_1\kappa h\left[A_0\cosh(\kappa h) + C_0\sinh(\kappa h)\right] = \bar{f}_1^h/\mu,$$

$$-i\xi_2\left[A_3\sinh(\kappa h) + C_3\cosh(\kappa h)\right] + \kappa\left[A_2\cosh(\kappa h) + C_2\sinh(\kappa h)\right]$$

$$+ i\xi_2\left[A_0\sinh(\kappa h) + C_0\cosh(\kappa h)\right] + 2i\xi_2\kappa h\left[A_0\cosh(\kappa h) + C_0\sinh(\kappa h)\right] = \bar{f}_2^h/\mu,$$

$$\kappa\left[A_3\cosh(\kappa h) + C_3\sinh(\kappa h)\right] - (1-2\nu)\kappa\left[A_0\cosh(\kappa h) + C_0\sinh(\kappa h)\right]$$

$$-\kappa^2 h\left[A_0\sinh(\kappa h) + C_0\cosh(\kappa h)\right] = \bar{f}_3^h/2\mu,$$

$$(3-4\nu)\kappa A_0 + i\xi_1 C_1 + \xi_2 C_2 - \kappa A_3 = 0, \quad (3-4\nu)\kappa C_0 + i\xi_1 A_1 + \xi_2 A_2 - \kappa C_3 = 0. \tag{2.27}$$

The last two equations in (2.27) have been derived using the Fourier transform of expression (2.9).

The system of equations (2.27) can be solved symbolically using Cramer's rule to find the coefficients A_0, C_0, A_1, C_1, A_2, C_2, A_3, and C_3. Next, $\overline{B}_1, \overline{B}_2, \overline{B}_3$, and $\overline{\varphi}$ can be found from (2.23) and (2.24) and then $\overline{\sigma}_{31}, \overline{\sigma}_{32}$, and $\overline{\sigma}_{33}$ from formulas (2.25). Finally, applying the inverse two-dimensional Fourier transform, we recover the stresses σ_{31}, σ_{32}, and σ_{33} as functions of the coordinates x_1, x_2, and x_3.

The other three stresses σ_{11}, σ_{22}, and σ_{12} can be determined from (2.11). Next, we find $\overline{\sigma}_{11}, \overline{\sigma}_{22}$, and $\overline{\sigma}_{12}$ (i.e., formulas similar to (2.25)). These formulas again include $\overline{B}_1, \overline{B}_2, \overline{B}_3$, and $\overline{\varphi}$. The original functions can easily be recovered numerically using the inverse Fourier transform.

3. Third Formulation of the Stress Problem in Elasticity. We have proved, using the Fourier transformation, that the six strain compatibility equations split into two groups, three equations in each. The equations of the first group can be derived from the equations of the second group and vice versa. Only three compatibility equations are independent. A similar result has been obtained for the stress compatibility equations. Therefore, it is advisable to use three equilibrium equations and three stress compatibility equations to solve the three-dimensional stress problem of elasticity.

The classical three-dimensional stress problem of linear elasticity reduces to a system of nine differential equations (three equilibrium equations and six Beltrami–Michell equations) with appropriate boundary conditions. Washizu [46] showed that the components of the strain incompatibility tensor are related by three Bianchi identities. For this reason, Kozak [42] proposed to determine the six independent stress components from three equilibrium equations and three Beltrami–Michell equations chosen appropriately. The other three Beltrami–Michell equations must only hold on the boundary. Belov et al. [2] derived three strain compatibility equations, one is algebraic and the other two are of the third order. This made it possible to reduce the stress problem to three equilibrium equations for three tangential stresses.

3.1. Consider the following compatibility equations [27]:

$$\varepsilon_{ij,km} + \varepsilon_{km,ij} - \varepsilon_{ik,jm} - \varepsilon_{jm,ik} = 0, \tag{3.1}$$

where ε_{ij} are the components of the strain tensor; $i, j, k, m = 1, 2, 3$.

There are total 81 equations (3.1), of which six are strain compatibility equations, some are satisfied identically, and the remaining repeat [34]. Contracting the tensors in (3.1) with respect to the indices k and m, we obtain somewhat different compatibility equations:

$$\nabla^2 \varepsilon_{ij} + e_{,ij} - \varepsilon_{ik,jk} - \varepsilon_{jk,ik} = 0, \quad e = \varepsilon_{kk}. \tag{3.2}$$

The compatibility equations (3.1) are six partial differential equations of the second order for six components of the strain tensor ε_{ij}. Some monographs on elasticity theory [27] consider these six equations independent.

Equations (3.1) can be written in a compact form: $\mathrm{Ink}\,\hat{\varepsilon} = \mathrm{rot}\,\mathrm{rot}\,\hat{\varepsilon} = 0$, where $\hat{\varepsilon}$ is the strain tensor.

Back in 1892, Beltrami concluded that only three compatibility equations are independent. He wrote them as rather awkward partial differential equations of the third order.

Thus, the six compatibility equations (3.1) are not independent. However, there is no need to use the Beltrami equations. We will show below that Eqs. (3.1) decompose into two dependent groups, three equations in each.

In Cartesian coordinates, these two groups are represented as

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}, \quad \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{xx}}{\partial z \partial x} \tag{3.3}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) = 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z}, \quad \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x},$$

$$\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)=2\frac{\partial^2\varepsilon_z}{\partial x\partial y}. \tag{3.4}$$

The three-dimensional Fourier transform of some function $f(x, y, z)$ is represented by the following formula [38]:

$$\bar{f}(\alpha,\beta,\gamma)=\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y,z)\,e^{i(\alpha x+\beta y+\gamma z)}\,dxdydz.$$

Taking the three-dimensional Fourier transform of Eqs. (3.3) and (3.4) and considering that the strains decay at infinity, we obtain the following two groups of equations:

$$\beta^2\bar{\varepsilon}_x+\alpha^2\bar{\varepsilon}_y=\alpha\beta\bar{\gamma}_{xy},\qquad \gamma^2\bar{\varepsilon}_y+\beta^2\bar{\varepsilon}_z=\beta\gamma\bar{\gamma}_{yz},\qquad \alpha^2\bar{\varepsilon}_z+\gamma^2\bar{\varepsilon}_x=\gamma\alpha\bar{\gamma}_{zx} \tag{3.5}$$

and

$$\alpha\left(\beta\bar{\gamma}_{zx}+\gamma\bar{\gamma}_{xy}-\alpha\bar{\gamma}_{yz}\right)=2\beta\gamma\bar{\varepsilon}_x,\quad \beta\left(\gamma\bar{\gamma}_{xy}+\alpha\bar{\gamma}_{yz}-\beta\bar{\gamma}_{zx}\right)=2\gamma\alpha\bar{\varepsilon}_y,\quad \gamma\left(\alpha\bar{\gamma}_{yz}+\beta\bar{\gamma}_{zx}-\gamma\bar{\gamma}_{xy}\right)=2\alpha\beta\bar{\varepsilon}_z. \tag{3.6}$$

For simplicity, we will consider an unbounded elastic medium. This is possible because we deal only with differential equations that describe the behavior of the medium, disregarding the boundary conditions.

Equations (3.5) and (3.6) are algebraic. Resolving the system of linear equations (3.5) for $\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$, and $\bar{\varepsilon}_z$, we arrive at Eqs. (3.6); and, conversely, resolving the system of equations (3.6) for $\bar{\gamma}_{xy}$, $\bar{\gamma}_{yz}$, and $\bar{\gamma}_{zx}$, we arrive at Eqs. (3.5).

Thus, only three of the six strain compatibility equations are independent. Hence, either Eqs. (3.3) or Eqs. (3.4) should be used in practical applications and there is no need to set up new three compatibility equations by combing Eqs. (3.3) and (3.4).

3.2. Let us now address the three-dimensional stress problem of elasticity. In the classical formulation, this problem reduces to the differential equilibrium equations

$$\sigma_{ij,j}+f_i=0 \tag{3.7}$$

and the Beltrami–Michell equations

$$\sigma_{ij,kk}+\frac{1}{1+\nu}s_{,ij}=-\frac{1}{1-\nu}\delta_{ij}f_{k,k}-(f_{i,j}+f_{j,i}), \tag{3.8}$$

where σ_{ij} are the components of the stress tensor; $s=\sigma_{ii}$; ν is Poisson's ratio; δ_{ij} is the Kronecker delta; and f_i are body force components.

Thus, we have a system of nine differential equations for the six stress components σ_{ij}.

There were many attempts to find a general solution to the system of equations (3.7), (3.8). Using an invariant representation of the stress tensor, Krutkov reduced this system of equations to one rather awkward differential equation for the stress function tensor.

Numerical methods are widely used to solve problems in elasticity. However, such methods face serious difficulties in solving the three-dimensional stress problem of elasticity based on Eqs. (3.7) and (3.8) because there are nine equations for six stress components.

To solve this problem, it is sufficient to discard three equations in system (3.7), (3.8). However, the question remains: Exactly which equations should be rejected? There seems to be little sense in discarding the equilibrium equations (3.7); hence, such three equations are among Eqs. (3.8). To find them, we should split the six equations (3.8) into two dependent groups. However, such a splitting appears impossible.

To resolve this problem, we have to abandon the Beltrami–Michell equations (3.8) and use the stress compatibility equations instead. As shown below, the stress compatibility equations can be split into two dependent groups. The system of equations (3.8) has been derived by combining the stress compatibility equations and the equilibrium equations (3.7). This is probably the reason why Eqs. (3.8) cannot be divided into two dependent groups.

Substituting Hooke's law

$$\varepsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\delta_{ij}s\right),$$

where μ is the shear modulus, into (3.1), we obtain the system of six stress compatibility equations:

$$\sigma_{ij,km} + \sigma_{km,ij} - \sigma_{im,jk} - \sigma_{jk,im} = \frac{\nu}{1+\nu}\left(\delta_{ij}s_{,km} + \delta_{km}s_{,ij} - \delta_{im}s_{,jk} - \delta_{jk}s_{,im}\right). \tag{3.9}$$

Let us show that (3.9) can be split into two dependent groups. In Cartesian coordinates, the system of equations (3.9) can be written as

$$\frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial x^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2}\right) = 2\frac{\partial^2\tau_{xy}}{\partial x\partial y},$$

$$\frac{\partial^2\sigma_y}{\partial z^2} + \frac{\partial^2\sigma_z}{\partial y^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}\right) = 2\frac{\partial^2\tau_{yz}}{\partial y\partial z},$$

$$\frac{\partial^2\sigma_z}{\partial x^2} + \frac{\partial^2\sigma_x}{\partial z^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2 s}{\partial z^2} + \frac{\partial^2 s}{\partial x^2}\right) = 2\frac{\partial^2\tau_{zx}}{\partial z\partial x}, \tag{3.10}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\tau_{zx}}{\partial y} + \frac{\partial\tau_{xy}}{\partial z} - \frac{\partial\tau_{yz}}{\partial x}\right) = \frac{\partial^2}{\partial y\partial z}\left(\sigma_x - \frac{\nu}{1+\nu}s\right),$$

$$\frac{\partial}{\partial y}\left(\frac{\partial\tau_{xy}}{\partial z} + \frac{\partial\tau_{yz}}{\partial x} - \frac{\partial\tau_{zx}}{\partial y}\right) = \frac{\partial^2}{\partial z\partial x}\left(\sigma_y - \frac{\nu}{1+\nu}s\right),$$

$$\frac{\partial}{\partial z}\left(\frac{\partial\tau_{yz}}{\partial x} + \frac{\partial\tau_{zx}}{\partial y} - \frac{\partial\tau_{xy}}{\partial z}\right) = \frac{\partial^2}{\partial x\partial y}\left(\sigma_z - \frac{\nu}{1+\nu}s\right). \tag{3.11}$$

Let us prove that the first group (3.10) depends on the second group (3.11). To this end, we take the three-dimensional Fourier transform of Eqs. (3.10) and (3.11) and take into account the conditions at infinity. Doing so gives:

$$\beta^2\bar{\sigma}_x + \alpha^2\bar{\sigma}_y - \frac{\nu}{1+\nu}(\alpha^2 + \beta^2)\,\bar{s} = 2\alpha\beta\bar{\tau}_{xy},$$

$$\gamma^2\bar{\sigma}_y + \beta^2\bar{\sigma}_z - \frac{\nu}{1+\nu}(\beta^2 + \gamma^2)\,\bar{s} = 2\beta\gamma\bar{\tau}_{yz},$$

$$\alpha^2\bar{\sigma}_z + \gamma^2\bar{\sigma}_x - \frac{\nu}{1+\nu}(\gamma^2 + \alpha^2)\,\bar{s} = 2\gamma\alpha\beta\bar{\tau}_{zx}, \tag{3.12}$$

$$\alpha\left(\beta\bar{\tau}_{zx} + \gamma\bar{\tau}_{xy} - \alpha\bar{\tau}_{yz}\right) = \beta\gamma\left(\bar{\sigma}_x - \frac{\nu}{1+\nu}\bar{s}\right),$$

$$\beta\left(\gamma\bar{\tau}_{xy} + \alpha\bar{\tau}_{yz} - \beta\bar{\tau}_{zx}\right) = \gamma\alpha\left(\bar{\sigma}_y - \frac{\nu}{1+\nu}\bar{s}\right),$$

$$\gamma\left(\alpha\bar{\tau}_{yz} + \beta\bar{\tau}_{zx} - \gamma\bar{\tau}_{xy}\right) = \alpha\beta\left(\bar{\sigma}_z - \frac{\nu}{1+\nu}\bar{s}\right). \tag{3.13}$$

Equations (3.12) and (3.13) are linear algebraic equations. Resolving the system of equations (3.13) for $\bar{\tau}_{xy}$, $\bar{\tau}_{yz}$, and $\bar{\tau}_{zx}$, we arrive at Eqs. (3.12); and, conversely, resolving the system of equations (3.12) for

$$\left(\bar{\sigma}_x - \frac{\nu}{1+\nu}\bar{s}\right), \quad \left(\bar{\sigma}_y - \frac{\nu}{1+\nu}\bar{s}\right), \quad \left(\bar{\sigma}_z - \frac{\nu}{1+\nu}\bar{s}\right),$$

we arrive at Eqs. (3.13).

Hence, only three of the six stress compatibility equations (3.9) are independent. Therefore, either Eqs. (3.10) or Eqs. (3.11) should be used when a Cartesian coordinate frame is used.

Thus, in solving the three-dimensional stress problem of elasticity, it is expedient to use three equilibrium equations and three stress compatibility equations.

3.3. Let us show that the third problem formulation based on the three compatibility equations (3.10) or (3.11) and three equilibrium equations allows solving stress boundary-value problems in elasticity. Let us consider, as an example, an elastic half-space with three stress components prescribed on the boundary.

We will use the coordinates x_1, x_2, x_3 instead of x, y, z.

In the new notation, the compatibility equations (3.10) and the equilibrium equations become

$$\frac{\partial^2\sigma_{11}}{\partial x_2^2} + \frac{\partial^2\sigma_{22}}{\partial x_1^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2\sigma}{\partial x_1^2} + \frac{\partial^2\sigma}{\partial x_2^2}\right) = 2\frac{\partial^2\sigma_{12}}{\partial x_1 \partial x_2},$$

$$\frac{\partial^2\sigma_{22}}{\partial x_3^2} + \frac{\partial^2\sigma_{33}}{\partial x_2^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2\sigma}{\partial x_2^2} + \frac{\partial^2\sigma}{\partial x_3^2}\right) = 2\frac{\partial^2\sigma_{23}}{\partial x_2 \partial x_3}, \tag{3.14}$$

$$\frac{\partial^2\sigma_{33}}{\partial x_1^2} + \frac{\partial^2\sigma_{11}}{\partial x_3^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2\sigma}{\partial x_3^2} + \frac{\partial^2\sigma}{\partial x_1^2}\right) = 2\frac{\partial^2\sigma_{31}}{\partial x_3 \partial x_1},$$

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} = 0, \quad \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} = 0, \quad \frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} = 0. \tag{3.15}$$

Let body forces be absent.

The two-dimensional Fourier transform of some function $f(x_1, x_2, x_3)$ is given by the formula

$$\bar{f}(\alpha_1, \alpha_2, x_3) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3) e^{i(\alpha_1 x_1 + \alpha_2 x_2)} dx_1 dx_2.$$

Taking the two-dimensional Fourier transform of Eqs. (3.14) and (3.15), we obtain a system of ordinary differential equations:

$$-\alpha_2^2\bar{\sigma}_{11} - \alpha_1^2\bar{\sigma}_{22} + \gamma\left(\alpha_1^2\bar{\sigma} + \alpha_2^2\bar{\sigma}\right) = -2\alpha_1\alpha_2\bar{\sigma}_{12},$$

$$\frac{d^2\bar{\sigma}_{22}}{dx_3^2} - \alpha_2^2\bar{\sigma}_{33} + \gamma\left(\alpha_2^2\bar{\sigma} - \frac{d^2\bar{\sigma}}{dx_3^2}\right) = -2i\alpha_2\frac{d\bar{\sigma}_{23}}{dx_3}, \tag{3.16}$$

$$-\alpha_1^2\bar{\sigma}_{33} + \frac{d^2\bar{\sigma}_{11}}{dx_3^2} + \gamma\left(\alpha_1^2\bar{\sigma} - \frac{d^2\bar{\sigma}}{dx_3^2}\right) = -2i\alpha_2\frac{d\bar{\sigma}_{31}}{dx_3},$$

$$-i\alpha_1\bar{\sigma}_{11} - i\alpha_2\bar{\sigma}_{12} + \frac{d\bar{\sigma}_{13}}{dx_3} = 0, \quad -i\alpha_1\bar{\sigma}_{21} - i\alpha_2\bar{\sigma}_{22} + \frac{d\bar{\sigma}_{23}}{dx_3} = 0,$$

$$-i\alpha_1\bar{\sigma}_{31} - i\alpha_2\bar{\sigma}_{32} + \frac{d\bar{\sigma}_{33}}{dx_3} = 0 \qquad (\gamma = \nu/(1+\nu)). \tag{3.17}$$

Resolving Eqs. (3.16) for $\bar{\sigma}_{12}$, $d\bar{\sigma}_{23}/dx_3$, and $d\bar{\sigma}_{13}/dx_3$ and substituting them into Eqs. (3.17), we get

$$\frac{d^2\bar{\sigma}_{11}}{dx_3^2} - \kappa^2\bar{\sigma}_{11} - \gamma\frac{d^2\bar{\sigma}}{dx_3^2} + \left[\gamma\left(\kappa^2 + \alpha_1^2\right) - \alpha_1^2\right]\bar{\sigma} = 0,$$

$$\frac{d^2\bar{\sigma}_{22}}{dx_3^2} - \kappa^2\bar{\sigma}_{22} - \gamma\frac{d^2\bar{\sigma}}{dx_3^2} + \left[\gamma\left(\kappa^2 + \alpha_2^2\right) - \alpha_2^2\right]\bar{\sigma} = 0,$$

$$\frac{d^2\bar{\sigma}_{33}}{dx_3^2} - \kappa^2\bar{\sigma}_{33} + (1-2\gamma)\frac{d^2\bar{\sigma}}{dx_3^2} + \gamma\kappa^2\bar{\sigma} = 0 \qquad (\kappa^2 = \alpha_1^2 + \alpha_2^2). \tag{3.18}$$

Summing Eqs. (3.18) yields a differential equation for the function $\bar{\sigma}(\alpha_1, \alpha_2, x_3)$:

$$\frac{d^2\bar{\sigma}}{dx_3^2} - \kappa^2\bar{\sigma} = 0. \tag{3.19}$$

Solving the system of differential equations (3.18), (3.19), we find the two-dimensional Fourier transforms of the normal stresses σ_{11}, σ_{22}, σ_{33} and the function σ. Then, we use Eqs. (3.16) to find the Fourier-transformed tangential stresses. The arbitrary constants are determined from the boundary conditions.

To find the solution of Eqs. (3.18), (3.19), we need to define the domain occupied by the elastic body. Let it be the half-space $x_3 \ge 0$.

3.4. To find the solution for the elastic half-space $x_3 \ge 0$, we will first solve Eq. (3.19). Considering that stresses decay at infinity, we obtain

$$\bar{\sigma} = C_0 e^{-\kappa x_3} \qquad (C_0 = C_0(\alpha_1, \alpha_2)). \tag{3.20}$$

Substituting (3.20) into (3.18) yields

$$\frac{d^2\bar{\sigma}_{11}}{dx_3^2} - \kappa^2\bar{\sigma}_{11} = (1-\gamma)\alpha_1^2 C_0 e^{-\kappa x_3}, \qquad \frac{d^2\bar{\sigma}_{22}}{dx_3^2} - \kappa^2\bar{\sigma}_{22} = (1-\gamma)\alpha_2^2 C_0 e^{-\kappa x_3},$$

$$\frac{d^2\bar{\sigma}_{33}}{dx_3^2} - \kappa^2\bar{\sigma}_{33} = (1-\gamma)\kappa^2 C_0 e^{-\kappa x_3}. \tag{3.21}$$

In finding partial solutions of the inhomogeneous equations (3.21), it should be remembered that κ is a simple root of characteristic equations [24]. Therefore, the general solutions of Eqs. (3.21) have the form

$$\bar{\sigma}_{11} = \left[C_1 - (1-\gamma)C_0\frac{\alpha_1^2}{2\kappa}x_3\right]e^{-\kappa x_3}, \qquad \bar{\sigma}_{22} = \left[C_2 - (1-\gamma)C_0\frac{\alpha_2^2}{2\kappa}x_3\right]e^{-\kappa x_3},$$

$$\bar{\sigma}_{33} = \left[C_3 + (1-\gamma)C_0\frac{\kappa}{2}x_3\right]e^{-\kappa x_3}, \tag{3.22}$$

where $C_m = C_m(\alpha_1, \alpha_2)$, $m = 1, 2, 3$. Summing formulas (3.22), we get

$$C_0 = C_1 + C_2 + C_3. \tag{3.23}$$

Next, substituting formulas (3.22) into Eqs. (3.16), we find the two-dimensional Fourier transforms of the tangential stresses:

$$\bar{\sigma}_{12}=\frac{1}{2\alpha_1\alpha_2}\left[\alpha_2^2C_1+\alpha_1^2C_2-\gamma\kappa^2C_0-(1-\gamma)\frac{\alpha_1^2\alpha_2^2}{\kappa}C_0x_3\right]e^{-\kappa x_3},$$

$$\bar{\sigma}_{23}=-\frac{i}{2\alpha_2\kappa}\left[\alpha_2^2C_1+\left(\alpha_2^2+\kappa^2\right)C_2-\gamma\kappa^2C_0-(1-\gamma)\,\alpha_2^2C_0\left(1+\kappa x_3\right)\right]e^{-\kappa x_3},$$

$$\bar{\sigma}_{31}=-\frac{i}{2\alpha_1\kappa}\left[\left(\alpha_1^2+\kappa^2\right)C_1+\alpha_1^2C_2-\gamma\kappa^2C_0-(1-\gamma)\,\alpha_1^2C_0\left(1+\kappa x_3\right)\right]e^{-\kappa x_3}. \tag{3.24}$$

The coefficients C_1, C_2, and C_3 appearing in (3.22) and (3.24) can be determined from the boundary conditions, and C_0 from (3.23).

On the boundary $x_3 = 0$ of the elastic half-space, we have

$$\sigma_{3j}=f_j\,(x_1,x_2)\qquad (j=1,2,3), \tag{3.25}$$

where f_j are known functions. Taking the two-dimensional Fourier transform of (3.25), we obtain (on $x_3 = 0$)

$$\bar{\sigma}_{3j}=\bar{f}_j(\alpha_1,\alpha_2)\qquad (j=1,2,3). \tag{3.26}$$

Formulas (3.22) and (3.24) yield (on $x_3 = 0$)

$$\bar{\sigma}_{31}=-\frac{i}{2\alpha_1\kappa}\left[\left(\alpha_1^2+\kappa^2\right)C_1+\alpha_1^2C_2-\left(\alpha_1^2+\gamma\alpha_2^2\right)C_0\right],$$

$$\bar{\sigma}_{32}=-\frac{i}{2\alpha_2\kappa}\left[\alpha_2^2C_1+\left(\alpha_2^2+\kappa^2\right)C_2-\left(\alpha_2^2+\gamma\alpha_1^2\right)C_0\right],\qquad \bar{\sigma}_{33}=C_3. \tag{3.27}$$

Using (3.23), (3.26), and (3.27), we obtain a system of algebraic equations for C_m ($m = 0, 1, 2, 3$):

$$\left(\alpha_1^2+\kappa^2\right)C_1+\alpha_1^2C_2-\left(\alpha_1^2+\gamma\alpha_2^2\right)C_0=2i\alpha_1\kappa\bar{f}_1,$$

$$\alpha_2^2C_1+\left(\alpha_2^2+\kappa^2\right)C_2-\left(\alpha_2^2+\gamma\alpha_1^2\right)C_0=2i\alpha_2\kappa\bar{f}_2,$$

$$C_3=\bar{f}_3,\qquad C_1+C_2+C_3-C_0=0.$$

Solving it yields

$$C_1=\frac{1}{(1-\gamma)\kappa^4}\left\{2i\alpha_1\kappa\left(\kappa^2-\gamma\alpha_1^2\right)\bar{f}_1+2i\gamma\alpha_2^3\kappa\bar{f}_2+\left[(1-\gamma)\,\alpha_1^4+(1+\gamma)\,\alpha_1^2\alpha_2^2+2\gamma\alpha_2^4\right]\bar{f}_3\right\},$$

$$C_2=\frac{1}{(1-\gamma)\kappa^4}\left\{2i\gamma\alpha_1^3\kappa\bar{f}_1+2i\alpha_2\kappa\left(\kappa^2-\gamma\alpha_2^2\right)\bar{f}_2+\left[(1-\gamma)\,\alpha_2^4+(1+\gamma)\,\alpha_1^2\alpha_2^2+2\gamma\alpha_1^4\right]\bar{f}_3\right\},$$

$$C_3=\bar{f}_3,\qquad C_0=\frac{2}{(1-\gamma)\kappa}\left(\alpha_1\bar{f}_1+\alpha_2\bar{f}_2+\kappa\bar{f}_3\right). \tag{3.28}$$

Substituting (3.28) into (3.22) and (3.24), we arrive at the final formulas for the Fourier-transformed components of the stress tensor. The original stress components $\sigma_{j\kappa}$ (j, $\kappa = 1, 2, 3$) can be recovered using the inverse Fourier transform.

Now, by determining the stress σ_{33}, we will demonstrate the rest of the procedure. Substituting the expressions for C_3 and C_0 from (3.28) into the third formula in (3.22), we obtain

$$\bar{\sigma}_{33}(\alpha_1,\alpha_2,x_3)=\left[\bar{f}_3+\left(i\alpha_1\bar{f}_1+i\alpha_2\bar{f}_2+\kappa\bar{f}_3\right)x_3\right]e^{-\kappa x_3}. \tag{3.29}$$

Applying the inversion formula and the convolution theorem [29] to (3.29), we find

$$\sigma_{33}(x_1,x_2,x_3)=-\frac{\partial Q_3}{\partial x_3}+x_3\frac{\partial}{\partial x_3}\left(\frac{\partial Q_1}{\partial x_1}+\frac{\partial Q_2}{\partial x_2}+\frac{\partial Q_3}{\partial x_3}\right), \tag{3.30}$$

where $Q_j(x_1,x_2,x_3)=\frac{1}{2\pi}\iint_{\Omega_j} f_j(y_1,y_2)\frac{dy_1dy_2}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+x_3^2}}$ $(j=1,2,3)$; Ω_j are the domains in the plane $x_3 = 0$ in which $f_j(x_1, x_2) \neq 0$.

Formula (3.30) is in agreement with the results of [10].

The formulas for the other stress components can be derived in a similar way.

4. Stress Problem in Thermoelasticity. Two new solutions of the stress problem in thermoelasticity were proposed in [12, 37]. Aspects of problem solving and specific problems in thermoelasticity are addressed in [16, 17, 21, 23, 25, 26].

Consider a homogeneous isotropic material occupying a half-space or a layer of finite thickness. Assume that there are neither body forces nor thermal sources and that temperature is constant.

The three-dimensional stress problem in thermoelasticity includes the equilibrium equation

$$\nabla\cdot\hat{T}=0, \tag{4.1}$$

the strain compatibility equations

$$\nabla\hat{T}+\frac{1}{1+\nu}\nabla\nabla\sigma=-2\mu\alpha\left(\nabla\nabla\theta+\frac{1+\nu}{1-\nu}\hat{E}\nabla^2\theta\right), \tag{4.2}$$

and the heat-conduction equations

$$\nabla^2\theta=0. \tag{4.3}$$

We will examine the static case. Given a distribution of surface forces **F**, the boundary condition on the surface O is defined by

$$\mathbf{n}\cdot\hat{T}\Big|_o=F. \tag{4.4}$$

The thermal boundary condition is given by

$$\theta|_o=\theta_0, \tag{4.5}$$

where $\hat{T}$ is the stress tensor; $\sigma=I_I(\hat{T})$ is the first variant of the stress tensor; ∇^2 is the Laplacian; ∇ is the inverted delta; ν is Poisson's ratio; μ is the shear modulus; α is the coefficient of linear expansion; **n** is the unit outward normal vector to the surface; θ is the temperature measured starting from the temperature of the natural state; and $\hat{E}$ is a unit tensor of the second rank. The heat-conduction problem is treated independently of the thermoelastic problem. In this case,

$$\nabla^2\sigma=-4\mu\frac{1+\nu}{1-\nu}\alpha\nabla^2\theta.$$

Taking Eq. (4.3) into account, we obtain

$$\nabla^2\sigma=0. \tag{4.6}$$

Then

$$2\nabla\varphi=\nabla^2(\mathbf{R}\varphi)-\mathbf{R}\nabla^2\varphi, \tag{4.7}$$

where φ is a scalar, and $\mathbf{R} = \mathbf{i}_s x_s$ is the position vector.

Applying (4.7) to σ and taking (4.6) into account, we find $2\nabla\sigma = \nabla^2(\mathbf{R}\sigma)$. It can be shown that $\nabla(\mathbf{R}\sigma) = (\nabla\sigma)\mathbf{R} + \sigma\hat{E}$. Hence

$$\nabla\nabla\sigma = \nabla\frac{1}{2}\nabla^2(\mathbf{R}\sigma) = \frac{1}{2}\nabla^2\left[(\nabla\sigma)\mathbf{R} + \sigma\hat{E}\right]. \tag{4.8}$$

Considering (4.3), we similarly arrive at

$$\nabla\nabla\theta = \frac{1}{2}\nabla^2\left[(\nabla\theta)\mathbf{R} + \theta\hat{E}\right]. \tag{4.9}$$

With (4.8) and (4.9), Eq. (4.2) becomes

$$\nabla^2\left\{\hat{T} + \frac{(\nabla\sigma)\mathbf{R} + \sigma\hat{E}}{2(1+v)} + \mu\alpha\left[(\nabla\theta)\mathbf{R} + \theta\hat{E}\right] + 2\mu\alpha\frac{1+\nu}{1-\nu}\theta\hat{E}\right\} = 0.$$

Considering (4.3) and (4.6), we finally obtain

$$\nabla^2\left\{\hat{T} + \frac{1}{2(1+v)}(\nabla\sigma)\mathbf{R} + \mu\alpha(\nabla\theta)\mathbf{R}\right\} = 0. \tag{4.10}$$

Denote

$$\hat{G} = \hat{T} + \frac{1}{2(1+v)}(\nabla\sigma)\mathbf{R} + \mu\alpha(\nabla\theta)\mathbf{R}. \tag{4.11}$$

Then Eq. (4.10) takes the form

$$\nabla^2\hat{G} = 0. \tag{4.12}$$

Hence, $\hat{G}$ is a harmonic asymmetric tensor of the second rank.

From (4.11), it follows that

$$\hat{T} = \hat{G} - \frac{1}{2(1+v)}(\nabla\sigma)\mathbf{R} - \mu\alpha(\nabla\theta)\mathbf{R}. \tag{4.13}$$

Representation (4.13) satisfies the strain compatibility equation (4.2). Let the tensor $\hat{T}$ also satisfy the equilibrium equation (4.1). We have

$$\nabla\cdot[(\nabla\sigma)\mathbf{R}] = \mathbf{R}\nabla\cdot\nabla\sigma + \nabla\sigma\cdot\nabla\mathbf{R} = \mathbf{R}\nabla^2\sigma + \nabla\sigma\cdot\hat{E} = \nabla\sigma$$

because $\nabla^2\sigma = 0$ and $\nabla\mathbf{R} = \hat{E}$. Then, we arrive at

$$\nabla\cdot T = \nabla\cdot\hat{G} - \frac{\nabla\sigma}{2(1+v)} - \mu\alpha\nabla\theta.$$

Considering (4.1), we obtain

$$\nabla\cdot\hat{G} = \frac{\nabla\sigma}{2(1+v)} + \mu\alpha\nabla\theta. \tag{4.14}$$

When written component-wise, relation (4.14) is equivalent to three equations. Thus, if the components of the tensor $\hat{G}$ satisfy relation (4.14), then the stress tensor $\hat{T}$ will also satisfy the equilibrium equation (4.1).

Representation (4.13) has the following component-wise form:

$$\sigma_{ij} = g_{ij} - \frac{1}{2(1+v)} x_j \frac{\partial \sigma}{\partial x_i} - \mu \alpha x_j \frac{\partial \theta}{\partial x_i}, \tag{4.15}$$

where σ_{ij} and g_{ij} are the components of the tensors $\hat{T}$ and $\hat{G}$, respectively.

By the reciprocity of the tangential stresses ($\sigma_{ij} = \sigma_{ji}$), it follows from (4.15) that

$$g_{ij} = g_{ji} + \frac{1}{2(1+v)} \left(x_j \frac{\partial \sigma}{\partial x_i} - x_i \frac{\partial \sigma}{\partial x_j} \right) + \mu \alpha \left(x_j \frac{\partial \theta}{\partial x_i} - x_i \frac{\partial \theta}{\partial x_j} \right). \tag{4.16}$$

Substituting (4.16) into (4.15), we obtain

$$\sigma_{ij} = g_{ji} - \frac{1}{2(1+v)} x_i \frac{\partial \sigma}{\partial x_j} - \mu \alpha x_i \frac{\partial \theta}{\partial x_j} \qquad (i, j = 1, 2, 3). \tag{4.17}$$

Formula (4.17) is more convenient for further analysis than (4.15). In particular, if $i = 3$, then

$$\sigma_{3j} = g_{j3} - \frac{1}{2(1+v)} x_3 \frac{\partial \sigma}{\partial x_j} - \mu \alpha x_3 \frac{\partial \theta}{\partial x_j} \qquad (j = 1, 2, 3). \tag{4.18}$$

Formula (4.18) describes the stresses on the area elements parallel to the plane x_1Ox_2.

The rest of the procedure is similar to that for the problem without thermal terms.

4.1. Another formulation of the stress problem in thermoelasticity. Consider an elastic isotropic body occupying a half-space or a layer of finite thickness.

The strain compatibility condition reads

$$\operatorname{Ink} \hat{\varepsilon} = 0, \tag{4.19}$$

where

$$\operatorname{Ink} \hat{\varepsilon} = \operatorname{rot} \left(\operatorname{rot} \hat{\varepsilon} \right)^*,$$

$\hat{\varepsilon}$ is the (symmetric) linear strain tensor.

Because of the thermal terms, the relationship between the strain tensor $\hat{\varepsilon}$ and the stress tensor $\hat{T}$ takes the form

$$\hat{\varepsilon} = \frac{1}{2\mu} \left(\hat{T} - \frac{\nu}{1+\nu} \sigma \hat{E} \right) + \alpha \theta \hat{E}. \tag{4.20}$$

Substituting (4.20) into (4.19), we obtain the stress compatibility equations with thermal terms:

$$\operatorname{Ink} \left[\frac{1}{2\mu} \left(\hat{T} - \frac{\nu}{1+\nu} \sigma \hat{E} \right) + \alpha \theta \hat{E} \right] = 0. \tag{4.21}$$

In the absence of body forces, the differential equilibrium equation is as follows:

$$\operatorname{div} \hat{T} = 0. \tag{4.22}$$

Assume that the temperature is constant and there are no thermal sources. Then, we arrive at the heat-conduction equation

$$\nabla^2 \theta = 0. \tag{4.23}$$

Thus, the stress problem in thermoelasticity reduces to the equations of statics (4.22), the strain compatibility equations (4.21), and the Laplace equations (4.23).

Let the distributed forces (4.4) be prescribed on the surface O of the elastic body, and the thermal boundary condition have the form (4.5).

Let us first solve the compatibility equation (4.21). According to [21], a symmetric tensor with zero incompatibility (Ink) is a deformation of some vector. Applying this rule to (4.21), we get

$$\frac{1}{2\mu}\left(\hat{T}-\frac{\nu}{1+\nu}\sigma\hat{E}\right)+\alpha\theta\hat{E}=\operatorname{def}\mathbf{b}, \qquad \operatorname{def}\mathbf{b}=\frac{1}{2}\left[(\nabla\mathbf{b})^{*}+\nabla\mathbf{b}\right], \tag{4.24}$$

where **b** is some vector.

From (4.24) it follows that

$$\hat{T}=2\mu\operatorname{def}\mathbf{b}+\left(\frac{\nu}{1+\nu}\sigma-2\mu\alpha\theta\right)\hat{E}. \tag{4.25}$$

Since $I_1(\operatorname{def}\mathbf{b}) = \operatorname{div}\mathbf{b}$, $I_1(\hat{E})=3$, and $I_1(\hat{T})=\sigma$, formula (4.25) becomes

$$\frac{\nu}{1+\nu}\sigma=\frac{2\nu\mu}{1-2\nu}\left(\operatorname{div}\mathbf{b}-3\alpha\theta\right).$$

Substituting this expression into (4.25), we obtain

$$\hat{T}=2\mu\left[\operatorname{def}\mathbf{b}+\frac{\nu\hat{E}}{1-2\nu}\operatorname{div}\mathbf{b}-\frac{1+\nu}{1-2\nu}\alpha\theta\hat{E}\right]. \tag{4.26}$$

This representation of the stress tensor $\hat{T}$ identically satisfies the compatibility equation (4.21).

Let us express the vector **b** in terms of a harmonic vector **B** ($\nabla^2\mathbf{B} = 0$) and a scalar φ:

$$\mathbf{b} = \mathbf{B} - x_3\nabla\varphi. \tag{4.27}$$

Substituting (4.27) into (4.26) yields

$$\hat{T}=2\mu\left[\operatorname{def}\left(\mathbf{B}-x_3\nabla\varphi\right)+\frac{\nu\hat{E}}{1-2\nu}\operatorname{div}\left(\mathbf{B}-x_3\nabla\varphi\right)-\frac{1+\nu}{1-2\nu}\alpha\theta\hat{E}\right]. \tag{4.28}$$

Representation (4.28) satisfies the compatibility equation (4.21). The stress tensor $\hat{T}$ must also satisfy the equation of statics (4.22). Therefore, substituting (4.28) into (4.22), we get

$$\operatorname{div}\left[\operatorname{def}\left(\mathbf{B}-x_3\nabla\varphi\right)+\frac{\nu\hat{E}}{1-2\nu}\operatorname{div}\left(\mathbf{B}-x_3\nabla\varphi\right)-\frac{1+\nu}{1-2\nu}\alpha\theta\hat{E}\right]=0. \tag{4.29}$$

Since $\operatorname{div}(\hat{E}\operatorname{div}\mathbf{b})$ = grad div **b** and $\operatorname{div}\operatorname{def}\mathbf{b}=(\nabla^2\mathbf{b}+\nabla\operatorname{div}\mathbf{b})/2$, Eq. (4.29) becomes

$$\frac{1}{2(1-2\nu)}\nabla\operatorname{div}\left(\mathbf{B}-x_3\nabla\varphi\right)-\frac{1}{2}\nabla^2\left(x_3\nabla\varphi\right)-\frac{1+\nu}{1-2\nu}\alpha\nabla\theta=0, \tag{4.30}$$

where $\nabla^2\mathbf{B} = 0$ has been taken into account.

Considering the equalities

$$\operatorname{div}\left(x_3\nabla\varphi\right)=x_3\nabla^2\varphi+\frac{\partial\varphi}{\partial x_3}, \qquad \nabla^2\left(x_3\nabla\varphi\right)=x_3\nabla\left(\nabla^2\varphi\right)+2\nabla\frac{\partial\varphi}{\partial x_3},$$

we rearrange Eq. (4.30) to the form

$$\nabla\left[\operatorname{div}\mathbf{B}-(3-4\nu)\frac{\partial\varphi}{\partial x_3}-2(1+\nu)\alpha\theta\right]-\nabla\left(x_3\nabla^2\varphi\right)-(1-2\nu)x_3\nabla\left(\nabla^2\varphi\right)=0. \tag{4.31}$$

Let

$$\nabla^2\varphi = 0, \qquad \frac{\partial\varphi}{\partial x_3} = \frac{1}{3-4\nu}\left[\operatorname{div}\mathbf{B} - 2(1+\nu)\alpha\theta\right], \tag{4.32}$$

then Eq. (4.31) becomes an identity.

Thus, representation (4.28) satisfies the compatibility condition (4.21) and the equation of statics (4.22) if the function φ satisfies conditions (4.32) and the function **B** is a harmonic vector.

Expression (4.28) can be simplified somewhat. Considering (4.32), we get

$$\operatorname{div}\left(x_3\nabla\varphi\right) = \frac{\partial\varphi}{\partial x_3}, \qquad \operatorname{div}\mathbf{B} = (3-4\nu)\frac{\partial\varphi}{\partial x_3} + 2(1+\nu)\alpha\theta.$$

Hence,

$$\operatorname{div}\left(\mathbf{B} - x_3\nabla\varphi\right) = 2(1-2\nu)\frac{\partial\varphi}{\partial x_3} + 2(1+\nu)\alpha\theta. \tag{4.33}$$

Substituting (4.33) into (4.28), we finally obtain

$$\hat{T} = 2\mu\left\{\left[2\nu\frac{\partial\varphi}{\partial x_3} - (1+\nu)\alpha\theta\right]\hat{E} + \operatorname{def}\left(\mathbf{B} - x_3\nabla\varphi\right)\right\}, \tag{4.34}$$

or in component-wise form:

$$\sigma_{st} = \mu\left\{\left[4\nu\frac{\partial\varphi}{\partial x_3} - 2(1+\nu)\alpha\theta\right]\delta_{st} + \left(\frac{\partial B_s}{\partial x_t} + \frac{\partial B_t}{\partial x_s}\right) - \left[\frac{\partial}{\partial x_t}\left(x_3\frac{\partial\varphi}{\partial x_s}\right) + \frac{\partial}{\partial x_s}\left(x_3\frac{\partial\varphi}{\partial x_t}\right)\right]\right\} \quad (s, t = 1, 2, 3), \tag{4.35}$$

where σ_{st} are the components of the stress tensor, and δ_{st} is the Kronecker delta.

The stress tensor (4.34) satisfies the strain compatibility equation (4.21) and the equilibrium equation (4.22). Specific expressions for the harmonic vector **B** and the harmonic scalar φ can be derived from the boundary conditions (4.4), (4.5) and heat-conduction equation (4.23).

Representation (4.34) can be used to solve the three-dimensional stress problem of thermoelasticity. For example, expression (4.34) or (4.35) can be applied to the second boundary-value problem for an elastic half-space or an elastic layer of finite thickness.

Elastic Half-Space. Consider an elastic half-space with the following stresses and temperature prescribed on its boundary ($x_3 = 0$):

$$\sigma_{3j} = \begin{cases} -f_j(x_1, x_2), & (x_1, x_2) \in \Omega_j, \\ 0, & (x_1, x_2) \notin \Omega_j, \end{cases} \quad (j = 1, 2, 3) \tag{4.36}$$

$$\theta = \begin{cases} \theta_0(x_1, x_2), & (x_1, x_2) \in \Omega, \\ 0, & (x_1, x_2) \notin \Omega, \end{cases} \tag{4.37}$$

where Ω_j are the loaded regions on the plane $x_3 = 0$, and Ω is the domain within which a nonzero temperature is maintained.

Each component B_m of the harmonic vector **B** satisfies the Laplace equation

$$\nabla^2\mathbf{B}_m = 0, \qquad m = 1, 2, 3. \tag{4.38}$$

Moreover,

$$\nabla^2\varphi = 0, \qquad \nabla^2\theta = 0. \tag{4.39}$$

Taking the two-dimensional Fourier transform of Eqs. (4.38) and (4.39), we obtain

$$\overline{B}_m(\xi_1,\xi_2,x_3)=A_m e^{-kx_3}, \qquad \overline{\varphi}(\xi_1,\xi_2,x_3)=A_0 e^{-kx_3},$$

$$\overline{\theta}(\xi_1,\xi_2,x_3)=C_0 e^{-kx_3}, \qquad k^2=\xi_1^2+\xi_2^2, \tag{4.40}$$

and taking the two-dimensional Fourier transform of Eqs. (4.35), we get

$$\overline{\sigma}_{31}=\mu\left(-i\xi_1\overline{B}_3+\frac{\partial\overline{B}_1}{\partial x_3}+i\xi_1\overline{\varphi}\right), \qquad \overline{\sigma}_{32}=\mu\left(-i\xi_2\overline{B}_3+\frac{\partial\overline{B}_2}{\partial x_3}+i\xi_2\overline{\varphi}\right),$$

$$\overline{\sigma}_{33}=2\mu\left\{\frac{\partial\overline{B}_3}{\partial x_3}-\left[(1-2\nu)\frac{\partial\overline{\varphi}}{\partial x_3}+(1+\nu)\alpha\overline{\theta}\right]\right\}. \tag{4.41}$$

On the boundary (for $x_3 = 0$) of the half-space, we have

$$\overline{\sigma}_{3j}=\overline{\sigma}_{3j}(\xi_1,\xi_2). \tag{4.42}$$

Using (4.32) and (4.40)–(4.42), we arrive at the system of algebraic equations

$$-i\xi_1 A_3-kA_1+i\xi_1 A_0=\mu^{-1}\overline{\sigma}_{31}, \qquad -i\xi_2 A_3-kA_2+i\xi_2 A_0=\mu^{-1}\overline{\sigma}_{32},$$

$$-kA_3+(1-2\nu)kA_0=0.5\mu^{-1}\overline{\sigma}_{33}+(1+\nu)\alpha C_0,$$

$$-(3-4\nu)kA_0+i\xi_1A_1+i\xi_2A_2+kA_3=-2(1+\nu)\alpha C_0. \tag{4.43}$$

The system of equations (4.43) can be solved symbolically, using Cramer's formulas, to determine A_0, A_1, A_2, and A_3. Then, we use formulas (4.40) to find $\overline{B}_m$ $(m = 1, 2, 3)$ and $\overline{\varphi}$. The original functions $B_m(x_1, x_2, x_3)$ and $\varphi(x_1, x_2, x_3)$ can be recovered using the inverse Fourier transform. The components of the stress tensor are found from (4.35).

To demonstrate the rest of the procedure, let us consider the special case where $\sigma_{31} = \sigma_{32} = \sigma_{33} = 0$ on $x_3 = 0$.

If the temperature distribution (4.37) is prescribed on the boundary of the half-space, then solving Eqs. (4.43) yields

$$A_0=0, \qquad A_1=(1+\nu)\alpha C_0\frac{i\xi_1}{k^2}, \qquad A_2=(1+\nu)\alpha C_0\frac{i\xi_2}{k^2}, \qquad A_3=-(1+\nu)\alpha C_0\frac{1}{k}.$$

Hence,

$$\overline{B}_1=(1+\nu)\alpha C_0\frac{i\xi_1}{k^2}e^{-kx_3}, \qquad \overline{B}_2=(1+\nu)\alpha C_0\frac{i\xi_2}{k^2}e^{-kx_3},$$

$$\overline{B}_3=-(1+\nu)\alpha C_0\frac{1}{k}e^{-kx_3}, \qquad \overline{\varphi}=0, \tag{4.44}$$

according to formulas (4.40).

In this case, $\sigma_{31} = \sigma_{32} = \sigma_{33} = 0$ on the area elements parallel to the boundary of the half-space. Let us determine the other stress components (σ_{11}, σ_{22}, and σ_{12}). Formulas (4.35) and (4.44) yield

$$\overline{\sigma}_{11}=-2\mu(1+\nu)\alpha C_0\frac{\xi_2^2}{k^2}e^{-kx_3}, \qquad \overline{\sigma}_{22}=-2\mu(1+\nu)\alpha C_0\frac{\xi_1^2}{k^2}e^{-kx_3},$$

$$\overline{\sigma}_{12}=2\mu(1+\nu)\alpha C_0\frac{\xi_1\xi_2}{k^2}e^{-kx_3}. \tag{4.45}$$

Let us now determine C_0. According to (4.37), we have $\overline{\theta}(\xi_1,\xi_2,0)=\overline{\theta}_0(\xi_1,\xi_2)$.

From (4.40) it follows that $\bar{\theta}(\xi_1,\xi_2,0)=C_0$. Hence, $C_0=\bar{\theta}_0(\xi_1,\xi_2)$, i.e.,

$$C_0=\frac{1}{2\pi}\iint_{\Omega}\theta_0(x_1,x_2)\,e^{i(\xi_1x_1+\xi_2x_2)}dx_1dx_2. \tag{4.46}$$

Taking the inverse Fourier transform of (4.45), we obtain

$$\sigma_{11}=-\frac{(1+\nu)\mu\alpha}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_0(\xi_1,\xi_2)\frac{\xi_2^2}{k^2}e^{-kx_3}e^{i(\xi_1x_1+\xi_2x_2)}d\xi_1d\xi_2,$$

$$\sigma_{22}=-\frac{(1+\nu)\mu\alpha}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_0(\xi_1,\xi_2)\frac{\xi_1^2}{k^2}e^{-kx_3}e^{i(\xi_1x_1+\xi_2x_2)}d\xi_1d\xi_2,$$

$$\sigma_{12}=\frac{(1+\nu)\mu\alpha}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_0(\xi_1,\xi_2)\frac{\xi_1\xi_2}{k^2}e^{-kx_3}e^{i(\xi_1x_1+\xi_2x_2)}d\xi_1d\xi_2, \tag{4.47}$$

where C_0 is defined by (4.46).

To achieve specific results, it is necessary to specify the function $\theta_0(x_1, x_2)$ and the domain Ω.

Let the domain Ω be a square with side length $2a$ in the plane $x_3 = 0$, and let the function $\theta_0(x_1, x_2)$ be described by the formula

$$\theta_0(x_1,x_2)=\theta^0\left(1-\beta\frac{|x_1|}{a}\right)\left(1-\beta\frac{|x_2|}{a}\right),\qquad (x_1,x_2)\in\Omega, \tag{4.48}$$

β changing from 0 to 1.

Substituting (4.48) into (4.46), we obtain

$$C_0=\frac{2}{\pi}\int_0^a\int_0^a\theta^0\left(1-\beta\frac{|x_1|}{a}\right)\left(1-\beta\frac{|x_2|}{a}\right)\cos\xi_1x_1\cos\xi_2x_2\,dx_1dx_2.$$

Evaluating the integrals yields

$$C_0(\xi_1,\xi_2)=\frac{2\theta^0}{\pi}\left[\frac{1-\beta}{\xi_1}\sin a\xi_1+\frac{\beta}{a\xi_1^2}\left(1-\cos a\xi_1\right)\right]\left[\frac{1-\beta}{\xi_2}\sin a\xi_2+\frac{\beta}{a\xi_2^2}\left(1-\cos a\xi_2\right)\right]. \tag{4.49}$$

We will restrict ourselves to calculating the stress σ_{11} in (4.47) because the stresses σ_{22} and σ_{12} can be determined in much the same way.

Substituting (4.49) into the first formula in (4.47) results in

$$\sigma_{11}=-\frac{8(1+\nu)\mu a\theta^0}{\pi^2}\int_0^{\infty}\int_0^{\infty}\frac{\xi_2^2}{k^2}\left[\frac{1-\beta}{\xi_1}\sin a\xi_1+\frac{\beta}{a\xi_1^2}\left(1-\cos a\xi_1\right)\right]$$

$$\times\left[\frac{1-\beta}{\xi_2}\sin a\xi_2+\frac{\beta}{a\xi_2^2}\left(1-\cos a\xi_2\right)\right]e^{-kx_3}\cos x_1\xi_1\cos x_2\xi_2\,d\xi_1d\xi_2. \tag{4.50}$$

If $\gamma_1=a\xi_1,\gamma_2=a\xi_2,\gamma=(\gamma_1^2+\gamma_2^2)^{1/2}$, $y_i=x_i/a$ (i = 1, 2, 3), then formula (4.50) becomes

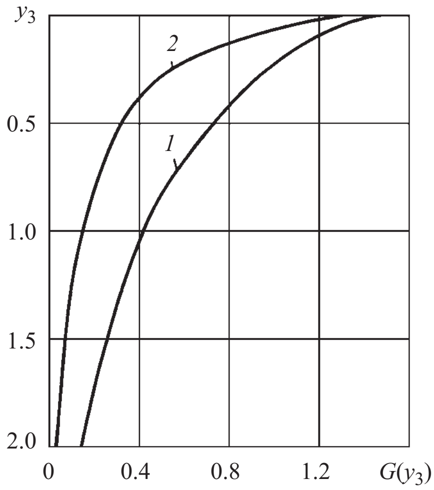

Fig. 3

$$\sigma_{11}(y_1, y_2, y_3) = -\frac{8(1+\nu)\mu a\theta^0}{\pi^2}\int_0^{\infty}\int_0^{\infty} F(\gamma_1,\gamma_2)\frac{\gamma_2^2}{\gamma^2} e^{-\gamma y_3} \cos y_1\gamma_1 \cos y_2\gamma_2 \, d\gamma_1 d\gamma_2,$$

$$F(\gamma_1,\gamma_2) = \left[\frac{1-\beta}{\gamma_1}\sin\gamma_1 + \frac{\beta}{\gamma_1^2}\left(1-\cos\gamma_1\right)\right]\left[\frac{1-\beta}{\gamma_2}\sin\gamma_2 + \frac{\beta}{\gamma_2^2}\left(1-\cos\gamma_2\right)\right]. \tag{4.51}$$

Since $y_1 = y_2 = 0$ on the y_3-axis, then

$$\sigma_{11}(0,0,y_3) = -\frac{8(1+\nu)\mu a\theta^0}{\pi^2}\int_0^{\infty}\int_0^{\infty} F(\gamma_1,\gamma_2)\frac{\gamma_2^2}{\gamma^2} e^{-\gamma y_3} d\gamma_1 d\gamma_2 .$$

If

$$G(y_3) = \int_0^{\infty}\int_0^{\infty} F(\gamma_1,\gamma_2)\frac{\gamma_2^2}{\gamma^2} e^{-\gamma y_3} d\gamma_1 d\gamma_2, \tag{4.52}$$

then

$$\sigma_{11}(0,0,y_3) = -\frac{8(1+\nu)\mu a\theta^0}{\pi^2} G(y_3).$$

The double integral in (4.52) can be evaluated numerically using quadrature formulas. The results are presented in Fig. 3, where curves *1* and *2* correspond to $\beta = 0, 1$.

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