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IMPACT OF A LONG THIN BODY ON A CYLINDRICAL CAVITY IN LIQUID: A PLANE PROBLEM

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An approach is developed to the investigation of the shock interaction between a long thin cylindrical body and a cylindrical cavity in an infinite compressible perfect liquid. This process accompanies the supercavitation of the body. Three typical cases of cross-sectional dimensions of the body and the cavity are examined. For each case, a mixed nonstationary boundary-value problem with an unknown moving boundary is formulated. The unknown quantities are expanded into Fourier series. An auxiliary problem is solved using the Laplace transform to establish the relationship between the pressure and the velocity on the cavity surface. As a result, the problem is reduced to an infinite system of Volterra equations of the second kind solved simultaneously with the equation of transverse motion and the equation of the contact boundary. An asymptotic solution valid at the initial stage of interaction is obtained for all the three cases, and a numerical solution is found for the most typical case

Keywords: shock interaction, supercavitation, mixed problem

1. Introduction. A promising line of development of water (or, more specifically, underwater) transport is the creation of so-called supercavitating vehicles. Supercavitation is observed when a large vapor or gas bubble occurs behind a body passed by a rapidly streaming water. If such a bubble completely envelopes the moving body, the drag on it reduces significantly. In this case, the body, in effect, flies in a vapor/gas cavity and contacts with the liquid only at its nose equipped with a cavitator. The principles of high-speed hydrodynamics were developed and outlined in, e.g., [9]. Over the last decades, studies of supercavitation have been conducted in Russia, the USA, and Ukraine. Velocities close to the speed of sound in water were reached under laboratory conditions [11]. The prospects of implementing the supercaviation technology seem appealing. This explains the interest to supercavitation as a problem of high-speed hydromechanics.

A specific feature of the process in question is the almost inevitable transverse motion of the body within the supercavity because of the instability of the primary (longitudinal) motion. The motion manifests itself over a wide range of velocities and consists in alternating impacts of the body on the lower and upper walls of the supercavity. This undesirable phenomenon causes the primary motion to loss energy or even to cease. Studying the laws governing the existence and development of transverse shock motion would help to create means for the stabilization of longitudinal motion. Previous studies (see, e.g., [12, 18]) were mainly based on the assumption that the liquid is incompressible and the transverse velocity is given. The incompressible liquid model does not make reliable predictions at the early stage of shock interaction (this model predicts infinite hydrodynamic pressure at the initial moment of interaction, which defies common sense). Moreover, it appears expedient to study this process for an arbitrary relationship between the typical cross-sectional dimensions of the cavity and the body.

The present paper sets out to formulate a shock-interaction problem and to develop approaches to study the shock interaction between a long rigid circular cylinder and a circular cylindrical cavity in a perfect compressible liquid. Section 2 gives a statement of a nonstationary initial–boundary-value problem with an unknown time-dependent boundary. The statement includes a coupled system of equations (hydrodynamic equations describing the body–liquid interaction and the equations of

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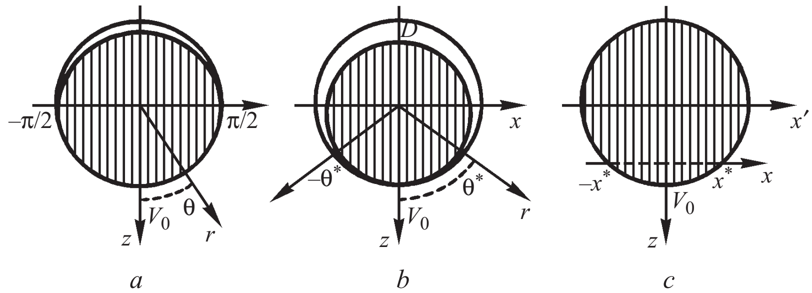

Fig. 1

motion of the body) and boundary conditions separately prescribed on the body–liquid contact area and on the free surface of the cavity. Methods of solving this (generally nonlinear) problem are developed for three typical relationships between typical cross-sectional dimensions of the body and the cavity: (i) slightly different radii (Sec. 3); (ii) equal radii (Sec. 4); and (iii) significantly different radii (Sec. 5).

We will expand the unknown quantities into Fourier series. To solve the auxiliary nonmixed problem, we will use the Laplace transform. For all cases, we will obtain an infinite system of Volterra equations of the second kind for the Fourier coefficients of the series-expanded pressure (velocity). Since the transverse velocity of the body is only specified at the initial instant of interaction, such a system should be solved simultaneously with the equation of motion of the body and the equation of the boundary of the contact area. We will also present governing equations of simpler form that require much less effort to be solved. Section 6 presents an asymptotic solution for the early stage of interaction in all the three cases, and a numerical solution for case (iii) as most practical. The results obtained are analyzed in Sec. 7.

It should be pointed out that the liquid dealt with here is at rest. A separate study is planned to correct the solution for the axial motion of the liquid and to asses the contributions of this motion and the transverse shock motion of the body in the cavity.

2. General Problem Formulation. As already mentioned in the introduction, it is experimentally established fact [11] that a bubble forms around a supercavitating body. This supercavity has the form of a strongly elongated ellipsoid (aspect ratio of 70 to 200) that wholly encloses the moving body and contacts it only at the nose. Thus, drag is applied to the head of the body, i.e., ahead of its center of mass, which means violation of the classical stability condition for motion through a continuous medium. Because of this, initial perturbations of the angle of attack and angular velocity of motion at velocities of 300 to 1000 m/sec result in transverse motions of the body and impacts of its tail against the interior surface of the cavity [12, 16]. The body may undergo steady or decaying periodic transverse motions accompanied by alternating impacts on the upper and lower walls of the cavity. Being unstable in the small, such a motion can remain stable in the large unless the nose is partially wetted. To reduce losses of energy of the primary motion and to exclude possible instability, one should be able to predict the transverse motion of the body within the supercavity. It is the problem of transverse motion of a supercavitating body and some numerical data illustrating different approaches to its solution that are addressed in the present paper.

Since the longitudinal dimensions of the body and the cavity are much greater than their transverse dimensions, which, as a rule, differ insignificantly, it would be appropriate to describe the shock interaction between the supercavitating body and the supercavity in the following manner (this mathematical description being valid far from the ends of the body):

Consider an infinitely long circular cylindrical cavity of radius R_c in an infinite liquid. The cavity encloses a rigid circular cylinder of radius R_b. Denote by D the air gap between the surface of the body and the surface of the cavity, $R_c - R_b = D$ (Fig. 1). In the general case, R_c is not equal to R_b, and $1 \le R_b/R_c < \infty$. Let us analyze the process in some cross section of the cavity (plane problem). Suppose that at time $t = 0$ the body reaches the interior surface of the cavity and starts penetrating it with initial velocity $V_0(0)$. The task is to determine the hydrodynamic loads on and the law of motion of the body.

The interaction between the body and the liquid, on the one hand, causes acoustic waves, which carry a portion of the impact energy away and, on the other hand, changes the velocity of the body, even to the point of sign reversal.

For adequate modeling of the process in question, it is necessary to solve a coupled system of equations that includes the hydrodynamic equations describing the shock motion of the liquid and the equations of motion of the body. This system should be supplemented with the appropriate boundary and initial conditions. It is obvious that boundary conditions should be prescribed within the body–liquid contact area and on the free surface of the cavity. Natural boundary conditions would be no flow through the body surface (equality of the normal (to the interface) velocities of the body and the liquid) within the contact area and constancy of the pressure (may be set at zero for simplicity) on the free surface of the cavity.

Note that when the body penetrates the liquid, the position of the contact boundary on the cavity surface is generally unknown. This boundary moves over the surface with unknown velocity, and its position at each instant is determined from the solution of the general coupled problem.

Initial conditions are also obvious: initial position and initial velocity $V_0 = V_0(t)|_{t=0}$ of the body ($V_0(t)$ is the unknown velocity of penetrating the liquid), the liquid being at rest at the initial time.

Suppose that, in contrast to the longitudinal velocity, the transverse velocity V_0 of the body in the supercavity is much less than the speed of sound in the liquid (we will not, however, neglect the compressibility of the liquid to allow for the impact energy carried away by acoustic waves), and viscosity is negligible. Consequently, the liquid can be considered perfect and compressible. It is also natural to assume that the depth of penetration of the body into the liquid is small. Then, we can use the linear wave equation to describe the motion of the liquid (so-called acoustic approximation) and, moreover, to specify boundary conditions on the undeformed surface of the cavity.

Thus, we can now formulate a mixed initial–boundary-value problem with an unknown time- and space-dependent boundary.

The geometry of the problem is shown in Fig. 1*b*. Let us introduce dimensionless variables needed for further analysis:

$$\bar{x} = \frac{x}{R_\text{b}}, \quad \bar{z} = \frac{z}{R_\text{b}}, \quad \bar{r} = \frac{r}{R_\text{b}}, \quad \bar{t} = \frac{ct}{R_\text{b}},$$

$$\bar{p} = \frac{p}{\gamma c^2}, \quad \bar{V} = \frac{V}{c}, \quad \bar{M} = \frac{M}{\gamma \pi R_\text{b}^2}, \quad \bar{Q} = \frac{Q}{\gamma c^2 R_\text{b}}, \tag{2.1}$$

where x and z are Cartesian coordinates; r and θ are polar coordinates (the z-axis coincides with the direction of transverse motion of the body), with the pole being at the center of the cavity; t is time; p is the hydrodynamic pressure; V is the velocity; c is the speed of sound; γ is the density of the liquid; R_b is the typical linear dimension (cylinder radius); $\overline{M}$ is the mass per unit length of the body (according to (2.1), it is divided by the mass of the liquid displaced by a body of unit length); and Q is the hydrodynamic drag. Since only the dimensionless quantities (2.1) are to be used below, the overbar will be omitted.

The motion of a perfect compressible liquid is described by a wave potential that satisfies the equation

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} - \frac{\partial^2 \Phi}{\partial t^2} = 0. \tag{2.2}$$

The pressure and velocity of the liquid are given by the formulas

$$p = -\frac{\partial \Phi}{\partial t}, \quad V = \nabla \Phi.$$

Let us specify boundary conditions on the cavity surface $r = 1$. The points with coordinates θ^* and $-\theta^*$ (Fig. 1*b*) separate the contact area and the free surface. Hence, the boundary condition within the contact area (equality of normal velocities) is given by

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=1} = V_0(t)\cos\theta, \quad |\theta| < \theta^*. \tag{2.3}$$

A second condition imposed on the solution within the contact area is no negative pressure:

$$p\big|_{r=1} \geq 0, \quad |\theta| < \theta^*. \tag{2.4}$$

The boundary condition on the free surface of the supercavity is pressure:

$$\left.\frac{\partial \Phi}{\partial t}\right|_{r=1} = 0, \quad |\theta| > \theta^*. \tag{2.5}$$

Expressions (2.3)–(2.5) allow for the symmetry about the z-axis. One more condition to be imposed is that wave disturbances decay at infinity:

$$\Phi \to 0 \quad \text{as} \quad r \to \infty. \tag{2.6}$$

Initial conditions for the potential of the liquid at rest are

$$\Phi|_{t=0} = \left.\frac{\partial \Phi}{\partial t}\right|_{t=0} = 0. \tag{2.7}$$

The motion of the body is described by the equation

$$M\frac{dV}{dt} = \frac{1}{\pi}Q, \quad V|_{t=0} = V_0, \tag{2.8}$$

where Q is the transverse drag on the body,

$$Q = -2\int_0^{\theta^*} p(t,\theta)\cos\theta d\theta, \tag{2.9}$$

where $p(t,\theta)$ is the liquid pressure within the contact area, and θ^* is the polar angle of the intersection point of the circular cross-sectional boundaries of the supercavity and the body; it is determined from a geometrical problem depending on the relationship between the typical dimensions of the cavity and the body and on the unknown depth of penetration $w(t)$,

$$\theta^* = \arccos\frac{(1+w)D + w^2/2}{D+w}. \tag{2.10}$$

Note that the gap D is determined as $D = R_c - R_b$.

Relations (2.2)–(2.10) constitute a mathematical formulation of the problem. The problem is generally nonlinear because of condition (2.4), and because, in particular, the right-hand side of Eq. (2.8) includes a complex function of the penetration depth w (see formula (2.10)), which, in turn, depends on the unknown velocity $V(t)$ of the body.

The nature of transverse motions of the body within the cavity, as well as the nature of the primary motion, essentially depends on the size of the air gap between the surfaces of the body and the cavity [10]. Providing motion with a minimum gap seems to be less energy-intensive. However, a very small air gap may result in partial wetting of the body's head, even at small deviations of the body axis from the course, and, consequently, in instability. A nonsmall gap would require much energy, yet contribute to stability. The merits and demerits of both alternatives make it expedient to study the transverse motion for each of them. Specific features of shock interaction at small and nonsmall gaps require different approaches to the solution of the problem. The difference between the approaches would only be due to different natural assumptions. For example, if the gap is small, then the contact area will have a curvilinear boundary commensurable with the typical transverse dimension of the cavity and will vary in size from a wetted point up to fully wetted surface of the body. In the limiting case where the gap is so small that it can be neglected, the contact area is constant and equal to half the cross-sectional area of the cavity. If the cross-sectional radius of the body differs from that of the cavity by a finite amount, the contact area will be small compared with the typical dimension and its curvature will be so small that it can be considered plane. All the three cases are schematized in Fig. 1.

We will discuss below three different approaches to the solution of the problem depending on the size of the air gap (small, zero, and nonsmall).

3. Small Air Gap. If the gap D between the cavity and the body in the undisturbed state (Fig. 1*b*) is small, yet nonzero, then the contact area resulting from impacts of the body on the interior surface of the cavity will not be small. Moreover, this contact area will obviously be a time-varying curved surface. Its area and evolution with time will be determined in solving the problem.

Let us first solve an auxiliary problem for the wave equation (2.2) to relate the pressure and velocity on the surface of the cavity. Assume that the pressure on this surface is described by some function $P(t,\theta)$,

$$-\frac{\partial}{\partial t}\Phi\bigg|_{r=1}=P(t,\theta), \tag{3.1}$$

that can be expanded into Fourier series

$$P(t,\theta)=\sum_{n=0}^{\infty}P_n(t)\cos n\theta. \tag{3.2}$$

The initial conditions are zero. To solve this nonmixed problem, we take the Laplace transform [2] of Eq. (2.2), marking Laplace-transformed quantities with the superscript L,

$$f^L(s)=\int_0^{\infty}e^{-st}f(t)dt.$$

Then, Eq. (2.2) with (2.7) becomes

$$\frac{\partial^2\Phi^L}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi^L}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi^L}{\partial\theta^2}-s^2\Phi^L=0. \tag{3.3}$$

Its general solution decaying at infinity (condition (2.6)) is given by [4]

$$\Phi^L=\sum_{n=0}^{\infty}A_n(s)K_n(sr)\cos n\theta, \tag{3.4}$$

where $K_n(sr)$ is the modified Bessel function of the second kind [3], and A_n are undetermined coefficients.

The pressure and velocity of the liquid are also expanded into Fourier series:

$$p=\sum_{n=0}^{\infty}p_n(t)\cos n\theta, \qquad V=\sum_{n=0}^{\infty}V_n(t)\cos n\theta. \tag{3.5}$$

Then the boundary condition (3.1) yields

$$A_n=-\frac{P_n^L(s)}{sK_n(s)},$$

and the Fourier term for the Laplace-transformed velocity on the liquid surface becomes

$$V_n^L(s)=-P_n^L(s)\frac{K_n'(s)}{K_n(s)}, \tag{3.6}$$

where $K_n'(s)=\left[\frac{\partial}{\partial r}K_n(sr)\right]_{r=1}$.

This expression can be modified somewhat:

$$V_n^L(s)=-P_n(s)\frac{K_n(s)-K_n(s)+K_n'(s)}{K_n(s)}$$

or

$$V_n^L(s)=P_n^L(s)-P_n^L(s)R_n^L(s),$$

where

$$R_n^L(s)=\frac{K_n(s)+K_n'(s)}{K_n(s)}. \tag{3.7}$$

The expression can be transformed back to the time domain using the convolution theorem of operational calculus [2]:

$$V_n(t)=P_n(t)-\int_0^t P_n(\tau)R_n(t-\tau)d\tau, \tag{3.8}$$

where $R_n(t)$ is the time-domain function corresponding to $R_n^L(s)$ (3.7). Methods for finding it will be outlined below. Relation (3.8) allows us to determine the velocity of the cavity if pressure is prescribed on its surface,

$$V(t)=P(t)-\sum_{n=0}^{\infty}\int_0^t P_n(\tau)R_n(t-\tau)d\tau\cos n\theta. \tag{3.9}$$

The solution of the auxiliary problem is used to solve the original mixed problem. Using representations (3.5) and (3.8) and the boundary conditions (2.3) and (2.5), we obtain

$$\sum_{n=0}^{\infty} p_n(t)\cos n\theta-\sum_{n=0}^{\infty}\int_0^t p_n(\tau)R_n(t-\tau)d\tau\cos n\theta=V_0(t)\cos\theta, \quad \theta<\theta^*,$$

$$\sum_{n=0}^{\infty} p_n(t)\cos n\theta=0, \quad \theta>\theta^*,$$

or in compact form

$$\sum_{n=0}^{\infty} p_n(t)\cos n\theta=H(\theta^*-\theta)\left[V_0(t)\cos\theta+\sum_{n=0}^{\infty}\int_0^t p_n(\tau)R_n(t-\tau)d\tau\cos n\theta\right], \tag{3.10}$$

where $H(\theta)$ is the Heaviside function,

$$H(x)=\begin{cases}1, & x>0,\\ 0, & x<0.\end{cases}$$

The right-hand side of Eq. (3.10) as a function of the angular variable θ can be expanded into a Fourier series. Then, considering the orthogonality and uniqueness of the Fourier series, we equate the coefficients of like harmonics on the left- and right-hand sides. As a result, we obtain an infinite system of Volterra equations of the second kind for the unknown Fourier coefficients

$$p_n(t)-\sum_{m=0}^{\infty}\beta_{mn}(\theta^*)\int_0^t p_m(\tau)R_m(t-\tau)d\tau=V_{0n}(t), \quad n=0,1,\ldots,\infty, \tag{3.11}$$

where

$$V_{0n}(t)=\xi_n V_0(t), \quad \xi_0=\frac{\sin\theta^*}{\pi}, \quad \xi_1=\frac{\theta^*}{\pi}+\frac{1}{2\pi}\sin 2\theta^*, \quad \xi_n=\frac{\sin(n-1)\theta^*}{\pi(n-1)}+\frac{\sin(n+1)\theta^*}{\pi(n+1)},$$

$$\beta_{00}=\pi^{-1}\theta^*, \quad \beta_{mm}=\frac{\theta^*}{\pi}+\frac{\sin 2m\theta^*}{2m\pi}, \quad m\neq 0, \quad \beta_{mn}=\frac{1}{\pi}\left(\frac{\sin(m-n)\theta^*}{m-n}+\frac{\sin(m+n)\theta^*}{m+n}\right). \tag{3.12}$$

Thus, solving the system of equations (3.11) yields the pressure on the surface of the cavity. Its velocity is calculated from formulas (3.8) or (3.9). Note that system of equations (3.11) should be solved simultaneously with (2.10) and the equation of motion

$$M\frac{dV}{dt}=Q,\quad V|_{t=0}=V_0,\quad Q=-\sum_{n=0}^{\infty}p_n(t)\left[\frac{\sin(n+1)\theta^*}{n+1}+\frac{\sin(n-1)\theta^*}{n-1}\right]. \tag{3.13}$$

In setting up the governing system of equations (3.11), we did not touch on the nontrivial problem of evaluating the kernel of the integral operator, i.e., the function $R_n(t)$, $n=0,1,\ldots$, whose Laplace transform is given by formula (3.7). To demonstrate how to do this, we will use the asymptotic representation of modified Bessel functions of the second kind with a large argument [3]:

$$K_n(s)\approx\sqrt{\frac{\pi}{2s}}e^{-s}\left(1+\frac{4n^2-1}{8s}+\ldots\right),\quad K_n'(s)\approx-\sqrt{\frac{\pi}{2s}}e^{-s}\left(1+\frac{4n^2+3}{8s}+\ldots\right), \tag{3.14}$$

which yields the first terms of the asymptotic expansion of the function $R_n^L(s)$:

$$R_n^L(s)\approx\frac{-\frac{1}{4s}e^{-s}\sqrt{2\frac{\pi}{s}}}{\sqrt{\frac{\pi}{2s}}e^{-s}\left(1+\frac{4n^2+3}{8s}+\ldots\right)}=-\frac{1}{2s}\left(1-\frac{4n^2+3}{8s}+\ldots\right)=-\frac{1}{2s}+\frac{(4n^2+3)}{16s^2}-\ldots\,.$$

Separating out the principal term of the asymptotic expansion in (3.7) and using the recurrence relations for cylinder functions

$$-2nK_n(z)=zK_{n-1}(z)-zK_{n+1}(z),\quad \frac{d}{dz}K_n(z)=-K_{n-1}(z)-\frac{n}{z}K_n(z),$$

$$\frac{d}{dz}K_n(z)=-K_{n+1}(z)+\frac{n}{z}K_n(z), \tag{3.15}$$

we obtain

$$R_n^L(s)=-\frac{1}{8s}+\frac{\left(n+\frac{1}{2}\right)\frac{1}{s}K_n(s)+K_n(s)-K_{n+1}(s)}{K_n(s)}=-\frac{1}{2s}+R_{n1}^L(s),$$

$$R_{n1}^L(s)=\frac{\left(n+\frac{1}{2}\right)\frac{1}{s}K_n(s)+K_n(s)-K_{n+1}(s)}{K_n(s)}\,. \tag{3.16}$$

Taking the inverse Laplace transform yields

$$R_n(t)=-\frac{1}{2}H(t)+R_{n1}(t), \tag{3.17}$$

where $R_{n1}(t)$ is a sufficiently smooth function, since the step part of $R_n(t)$ has been separated out in (3.17). It remains to evaluate the function $R_{n1}(t)$. To this end, we rearrange formula (3.16) as

$$e^sK_n(s)R_{n1}^L(s)=e^s\left[\left(n+\frac{1}{2}\right)\frac{1}{s}K_n(s)+K_n(s)-K_{n+1}(s)\right]. \tag{3.18}$$

To transform expression (3.18) back to the time domain, we will use the convolution theorem of operational calculus [2]. The modified Bessel function of the second kind is transformed using the following relations [2, 4]:

$$e^s K_n(s) \leftrightarrow \frac{z_1^n + z_2^n}{Z}, \qquad \frac{e^s}{s} K_0(s) \leftrightarrow \ln z_1, \qquad \frac{e^s}{s} K_n(s) \leftrightarrow \frac{z_1^n - z_2^n}{2n},$$

$$z_{1,2} = t+1 \pm Z, \qquad Z = \sqrt{(t+1)^2 - 1}. \tag{3.19}$$

We obtain the following Volterra equation of the first kind for the functions $R_{n1}(t)$:

$$\int_0^t R_{n1}(\tau) k_n(t-\tau) d\tau = r_n(t), \qquad n = 0, 1, \ldots, \infty, \tag{3.20}$$

its kernel and right-hand side are

$$k_n(t) = \frac{z_1^n + z_2^n}{Z} = \frac{\left(t+1+\sqrt{(t+1)^2-1}\right)^n + \left(t+1-\sqrt{(t+1)^2-1}\right)^n}{\sqrt{(t+1)^2-1}},$$

$$r_0(t) = \frac{1}{2} \ln\left(t+1+\sqrt{(t+1)^2-1}\right) - \frac{2t}{\sqrt{(t+1)^2-1}},$$

$$r_n(t) = \left(n + \frac{1}{2}\right) \frac{\left(t+1+\sqrt{(t+1)^2-1}\right)^n - \left(t+1-\sqrt{(t+1)^2-1}\right)^n}{2n}$$

$$+ \frac{\left(t+1+\sqrt{(t+1)^2-1}\right)^n + \left(t+1-\sqrt{(t+1)^2-1}\right)^n}{\sqrt{(t+1)^2-1}}$$

$$- \frac{\left(t+1+\sqrt{(t+1)^2-1}\right)^{n+1} + \left(t+1-\sqrt{(t+1)^2-1}\right)^{n+1}}{\sqrt{(t+1)^2-1}}, \qquad n \neq 0.$$

The integral equation (3.20) is an equation with a weak singularity and can be solved numerically using, for example, the method proposed in [5].

Thus, the task is to solve simultaneously the equation of motion (2.8) and the infinite system of Volterra equations of the second kind (3.11). The kernel of the integral operator in (3.11) is determined, according to formula (3.17), by solving the system of Volterra equations of the first kind (3.20). Note that due to (3.5) the right-hand side of (2.8) is defined by

$$Q = -2 \sum_{n=0}^{\infty} p_n(t) \sin \theta^*(t). \tag{3.21}$$

A more convenient approximate algorithm may be used to determine the kernel in the system of Volterra equations of the second kind (3.11). To this end, we will use the well-known asymptotic properties of cylinder functions of high order [1]:

$$K_\nu(\nu z) \approx \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu\eta}}{(1+z^2)^{1/4}} \{1 + \ldots\}, \qquad K'_\nu(\nu z) \approx -\sqrt{\frac{\pi}{2\nu}} \frac{(1+z^2)^{1/4}}{z} e^{-\nu\eta} \{1 + \ldots\},$$

$$\eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}.$$

Thus, we obtain the following asymptotics with nonsmall n for the kernel $R_n(t)$:

$$R_n^L(s) = \frac{K'_n(s) + K_n(s)}{K_n(s)} \approx 1 - \sqrt{1 + \left(\frac{n}{s}\right)^2} = -\left(\frac{n}{s}\right)^2 \frac{s}{s + \sqrt{s^2 + n^2}}.$$

This expression can be transformed back to the time domain according to [2]:

$$R_n(t) = -n\int_0^t \frac{J_1(n\tau)}{\tau}\,d\tau, \tag{3.22}$$

where $J_1(x)$ is a Bessel function [3].

Expression (3.22) can be used for sufficiently large n. When n is small, one should use either relations (3.17) and (3.20) or represent the modified Bessel functions of the second kind by rational approximations [1], which, for small n, ensure good accuracy and produce expressions easily subjected to the inverse Laplace transform with the help of either the above-cited integral equations or partial-fraction expansion and residue theory.

In conclusion, it should be pointed out that the infinite system of equations (3.11) can be set up by solving another auxiliary problem where not pressure, as in (3.1), but velocity is specified on the cavity surface:

$$-\frac{\partial}{\partial r}\Phi\bigg|_{r=1} = V(t,\theta), \quad V = \sum_{n=0}^{\infty} V_n(t)\cos n\theta. \tag{3.23}$$

Solving the boundary-value problem (3.23) yields the Fourier coefficients for pressure expressed in terms of the coefficients $V_n(t)$:

$$P_n(t) = V_n(t) + \int_0^t V_n(\tau)\,\overline{R}_n(t-\tau)\,d\tau,$$

where the kernel $\overline{R}_n(t)$ is determined by taking the inverse Laplace transform of the expression

$$\overline{R}_n^L(s) = \frac{K_n(s)}{K_n(s) + K_n'(s)}.$$

Accordingly, the pressure is expressed in terms of the velocity $V(t, \theta)$ as follows:

$$P(t) = V(t) + \sum_{n=0}^{\infty} \int_0^t V_n(\tau)\,\overline{R}_n(t-\tau)\,d\tau \cos n\theta. \tag{3.24}$$

Expression (3.24) used together with (3.5) to satisfy the mixed boundary conditions (2.3) and (2.5) leads to an infinite system of Volterra equations of the second kind for the coefficients $V_n(t)$:

$$V_n(t) + \sum_{m=0}^{\infty} \overline{\beta}_{mn}(\theta^*) \int_0^t V_m(\tau)\overline{R}_m(t-\tau)\,d\tau = V_{0n}(t), \quad n = 0, 1, \ldots, \infty.$$

Since the kernel $\overline{R}_n(t)$ and the coefficients $\overline{\beta}_n(t)$ are evaluated in a similar way as those for the infinite system of equations (3.11), we omit the details here.

4. Negligible Gap ($D = 0$). Suppose that the radius of the cavity is negligibly different from the radius of the body, i.e., $D = 0$. In this case, the polar angle θ^* defining the boundary of the free surface is obviously constant and is equal to $\pi/2$ (it should be born in mind that the penetration depth is small) (Fig. 1*a*). The formulation and rigorous solution of the problem for $D = 0$ follows from the relations of Sec. 3 in which $\theta^* = \pi/2$. The governing infinite system of Volterra equations of the second kind becomes

$$p_n(t) - \sum_{m=0}^{\infty} \beta_{mn} \int_0^t p_m(\tau) R_m(t-\tau)\,d\tau = V_{0n}(t), \quad n = 0, 1, \ldots, \tag{4.1}$$

where

$$V_{0n}(t)=\xi_n V_0(t), \quad \xi_0=\pi^{-1}, \quad \xi_1=\frac{1}{2}+\frac{1}{2\pi}, \quad \xi_n=\begin{cases}0, & n=3,5,7,\ldots \\ (-1)^{n-1}\dfrac{2n}{\pi(n^2-1)}, & n=2,4,6,\ldots,\end{cases}$$

$$\beta_{00}=\frac{1}{2}, \quad \beta_{mm}=\begin{cases}\dfrac{1}{2}, & m=2,4,\ldots \\ \dfrac{1}{2}+(-1)^{\frac{m-1}{2}}\dfrac{1}{2m\pi}, & m=1,3,\ldots\end{cases} \quad \beta_{mn}=\begin{cases}0, & m\pm n \text{ is even} \\ (-1)^{m-n}\dfrac{2m}{m^2-n^2}, & m\pm n \text{ is odd.}\end{cases}$$

The system of equations (4.1) is solved simultaneously with the equation of motion

$$M\frac{dV}{dt}=Q, \quad V|_{t=0}=V_0(0), \quad Q=-\sum_{n=0}^{\infty} p_n(t)\left(\frac{1}{n+1}\sin\frac{(n+1)\pi}{2}+\frac{1}{n-1}\sin\frac{(n-1)\pi}{2}\right). \tag{4.2}$$

The functions $R_n(t)$, $n=0,1,\ldots$, are defined by the formulas derived in Sec. 3.

To find a simpler approximate solution, we make two assumptions, which are believed not to distort significantly the problem formulation and expected results. One assumption has already been made above: $\theta^*=\pi/2$. The other assumption is that condition (2.5) can be replaced by the nondeformability condition for the cavity

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=1}=0, \quad |\theta|>\theta^*. \tag{4.3}$$

The error introduced by this assumption can be estimated after finding the rigorous solution of problem (2.2)–(2.10).

Under these assumptions, the Laplace transform reduces the problem to Eq. (3.3) with the following boundary condition:

$$\left.\frac{\partial\Phi^L}{\partial r}\right|_{r=1}=\begin{cases}V_0^L(s)\cos\theta, & |\theta|<\dfrac{\pi}{2}, \\ 0, & |\theta|>\dfrac{\pi}{2}.\end{cases} \tag{4.4}$$

As follows from (4.4), this boundary-value problem is no longer mixed. The right-hand side of condition (4.4) can be expanded into a Fourier series:

$$V^L=H\left(\frac{\pi}{2}-\theta\right)V_0^L(s)\cos\theta=\sum_{n=0}^{\infty}\varepsilon_n V_{0n}^L(s)\cos n\theta,$$

$$V_{0n}^L(s)=\xi_n V_0(s), \quad \varepsilon_n=\begin{cases}1/2, & n=0, \\ 1, & n>0,\end{cases} \tag{4.5}$$

where $H(t)$ is the Heaviside function, and the coefficients ξ_n have been defined earlier.

Substituting solution (3.5) into condition (4.4), we obtain

$$A_0=\frac{V_{0n}^L(s)}{\pi s K_0'(s)}, \quad A_1=\frac{1}{2}\frac{V_0^L(s)}{sK_1'(s)}, \quad A_n=\xi_n\frac{V_0}{sK_n'(s)}, \quad n=2,\ldots,\infty, \quad K_n'(x)=\frac{\partial}{\partial x}\{K_n(x)\},$$

for each n, whence we find the pressure on the cavity surface:

$$p(t)=\sum_{n=0}^{\infty}p_n(t)\cos n\theta, \quad p_0^L(s)=-\frac{1}{\pi}V_0^L(s)\frac{K_0(s)}{K_0'(s)},$$

$$p_1^L(s)=-\frac{1}{2}V_0^L(s)\frac{K_1(s)}{K_1'(s)}, \quad p_n^L(s)=-\xi_n V_0^L(s)\frac{K_n(s)}{K_n'(s)}. \tag{4.6}$$

Again, the task is to recover the original time function from $p_n^L(s)$, $n=0,1,\ldots,\infty$. As in Sec. 3, we multiply the left- and right-hand sides of (4.6) by $e^{-s}K'_n(s)$, use the recurrence formulas (3.15) for $K'_n(s)$, and apply the convolution theorem, using formulas (3.19). As a result, we obtain the following Volterra equations for $p_n(t)$:

$$\int_0^t p_n(\tau)R_n(t-\tau)d\tau = F_n(t), \quad n=0,1,\ldots,\infty, \tag{4.7}$$

where the kernel $R_n(t)$ is determined from (3.19), and the right-hand side $F_n(t)$ is the convolution of the original functions for $V_{0n}(s)$ and $K_n(s)$,

$$F_n(t)=\xi_n\int_0^t V_0(t-\tau)\frac{z_1^n(\tau)+z_2^n(\tau)}{2\sqrt{(\tau+1)^2-1}}d\tau. \tag{4.8}$$

Equation (4.7) has a kernel with a weak singularity and can be solved numerically, by using, for example, the method from [5] to reduce it to a system of algebraic equations.

Expressions (4.6) can also be transformed back to the time domain by reducing them to a Volterra equation of the second kind. To this end, considering that $K_n(s)/K'_n(s)\to -1$ as $s\to\infty$, we rearrange (4.6) as

$$p_n^L\frac{K_n(s)-K'_n(s)+K'_n(s)}{K'_n(s)}=-\xi_n V_0^L(s)$$

or

$$p_n^L\left(1-\frac{K'_n(s)+K_n(s)}{K'_n(s)}\right)=\xi_n V_0^L(s). \tag{4.9}$$

Applying the convolution theorem to (4.9), we obtain the following system of integral equations:

$$p_n(t)-\int_0^t p_n(\tau)R_n^*(t-\tau)d\tau=\xi_n V_0(t), \quad n=0,1,\ldots,\infty. \tag{4.10}$$

It remains to recover the time-domain kernel. However, the original functions for $[K'_n(s)+K_n(s)]/K'_n(s)$ are much smoother than the original functions for the ratio $K_n(s)/K'_n(s)$ appearing in (4.6), and, thus, recovering them should not be a problem. Associated methods have already been discussed in Sec. 3.

Note that Eqs. (4.7) (or (4.10)) should be solved simultaneously with the equation of motion of the body.

The formulation of the problem for $D=0$ can be somewhat refined by determining the polar angle θ^* of the contact area from relation (2.10). If $D=0$, then this relation yields

$$\theta^*=\arccos\frac{w}{2}. \tag{4.11}$$

In closing this section, it should be pointed out that a system of integral equations similar to (4.10) can easily be derived for small values of D. To this end, it is enough to keep only diagonal terms in (3.11):

$$p_n(t)-\beta_{nn}(\theta^*)\int_0^t p_n(\tau)R_n(t-\tau)d\tau=V_{0n}(t), \quad n=0,1,\ldots,\infty. \tag{4.12}$$

5. Nonsmall Gap. This case appears to be the most frequent. With a nonsmall gap, the contact area is small compared with the typical dimension of the cavity. Moreover, since the cross-sectional radius of the cavity is much greater than that of the body, the undeformed surface of the cavity contacting with the body is only slightly curved. This makes it possible to model the contact interaction between a body and a supercavity by an impact of a rigid body on a plane infinite liquid surface (Fig. 1*c*).

Such a formulation was already used earlier to study processes accompanying the water landing of aircraft and the slamming of high-speed ships (see, e.g., the review [15] for references to relevant approaches and publications). Here we briefly outline an approach to solving the problem of transverse shock motions of a body within a supercavity. The hydrodynamic part of this problem reduces to an infinite system of Volterra equations of the second kind. It is clear that this system is similar to that derived in the previous sections.

The problem is briefly formulated as follows. An infinitely long rigid circular cylinder moves at a right angle to a plane liquid surface. At some time $t = 0$, the body reaches this surface and begins to submerge with velocity $V_0(t)$, the initial velocity of submersion being known: $V|_{t=0} = V_0$. The generatrix of the body is assumed to remain parallel to the originally undisturbed liquid surface. The liquid, as before, is assumed weightless and perfectly compressible. The task is to determine the hydrodynamic characteristics of the process of body–liquid interaction and the law of motion of the body.

We introduce a Cartesian coordinate system *Oxyz* in the half-space occupied by the liquid: the *Ox*- and *Oy*-axes lie on the undisturbed liquid surface, the *Oy*-axis being parallel to the generatrix of the body, and the *Oz*-axis is directed into the liquid. Thus, the undisturbed liquid surface coincides with the plane $z = 0$ (Fig. 1*c*). Since the hydrodynamic pattern of the submergence process is similar in all cross sections, it is enough to study motion in one of them—the plane *xOz*. The cross-section of the submerging cylinder in the plane *xOz* is a circle of radius R_b. Note that we are examining the early stage of shock interaction between a cylindrical body and a liquid, i.e., the submersion velocity is small compared with the speed of sound in the liquid c ($V_0 / c \ll 1$) and the submersion depth is small compared with the typical linear dimensions of the body.

We will use the same dimensionless notation (2.1). The wave potential describing the motion of the liquid satisfies the equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{\partial^2\Phi}{\partial t^2} = 0. \tag{5.1}$$

The hydrodynamic velocity and the pressure on the liquid surface are defined by the formulas

$$V(t,x) = \left.\frac{\partial\Phi}{\partial z}\right|_{z=0}, \qquad p(t,x) = -\left.\frac{\partial\Phi}{\partial t}\right|_{z=0}.$$

Denote by $-x^*$ and x^* the boundaries of the contact area (Fig. 1*c*). Since the submersion depth is assumed small, boundary conditions can be prescribed on the undisturbed liquid surface. Then the boundary conditions within the contact area are impermeability of the body and non-negativity of pressure:

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = V_0(t), \qquad -\left.\frac{\partial\Phi}{\partial t}\right|_{z=0} \geq 0, \qquad |x| < x^*. \tag{5.2}$$

The following dynamic condition (constant pressure (set at zero for simplicity)) must be prescribed on the free liquid surface:

$$-\left.\frac{\partial\Phi}{\partial t}\right|_{z=0} = 0, \qquad |x| > x^*. \tag{5.3}$$

The initial conditions for the liquid at rest are zero:

$$\Phi|_{t=0} = \left.\frac{\partial\Phi}{\partial t}\right|_{t=0} = 0. \tag{5.4}$$

The disturbances of the liquid caused by the body decay at infinity:

$$\Phi \to 0 \quad \text{as} \quad x^2 + z^2 \to \infty. \tag{5.5}$$

The equation of motion of the body interacting with the liquid and the initial condition are

$$M\frac{dV_0(t)}{dt}=Q(t), \quad V_0(t)\big|_{t=0}=V_0. \tag{5.6}$$

Due to the symmetry about the z-axis, the hydrodynamic drag $Q(t)$ can be evaluated by the formula

$$Q(t)=2\int_0^{x^*} p(t,x)dx. \tag{5.7}$$

Relations (5.1)–(5.7) should be supplemented with a condition for the contact boundary x^*. A natural and elementary way of finding it is to use the law of motion of the body to find the intersection of the moving body and the undisturbed liquid surface. The following obvious relation holds:

$$x^*(t)=\arccos\left[1-w(t)\right]=\arccos\left(1-\int_0^t V_0(t)dt\right), \tag{5.8}$$

where w denotes the submersion depth.

Since the law of motion of the body is unknown beforehand and is determined during the solution of the impact problem, the contact boundary is, as a rule, found by successive approximations. A more sophisticated and accurate method is to evaluate the change in the contact area due to the displacement of the liquid by the submerging body. To this end, not only the law of motion of the body is determined at each time step, but also the displacement of the free liquid surface near the body, which is then used in determining the contact boundary.

Note that despite the linearity of the liquid model adopted, the mixed initial–boundary-value problem (5.1)–(5.8) is nonlinear because of several factors. Nonlinearity appears in the second boundary condition in (5.2) and in Eq. (5.6), where the force $Q(t)$ depends on the unknown velocity $V_0(t)$ of the body in a complex manner indicated by relations (5.7) and (5.8).

To solve the problem formulated, we recognize that the liquid region involved in disturbed motion is finite at each finite time due to the short duration of the process in question and due to the finite velocity of disturbances in a compressible liquid. This allows us to model the domain occupied by the liquid as a half-strip of width $2l$ rather than a half-space. The choice of l is dictated by the duration of the process and the requirement of good convergence. Here, it is natural to set $l=\pi$. Boundary conditions on the lateral faces of the half-strip are chosen so as to make separation of variables easier. The unknown and given quantities are expanded into Fourier series on the interval $[-\pi;\pi]$:

$$\varphi(t,x,z)=\sum_{n=0}^{\infty}\varphi_n(t,x,z)\cos x, \quad V(t,x)=\sum_{n=0}^{\infty}V_n(t)\cos\pi x, \quad p(t,x)=\sum_{n=0}^{\infty}p_n(t)\cos\pi x. \tag{5.9}$$

Taking the Laplace transform of Eq. (5.1) yields

$$\frac{\partial^2\Phi^L}{\partial x^2}+\frac{\partial^2\Phi^L}{\partial z^2}-s^2\Phi^L=0. \tag{5.10}$$

This equation has the following general solution decaying at infinity:

$$\Phi^L=\sum_{n=0}^{\infty}e^{-z\sqrt{s^2+n^2}}A_n\cos nx, \tag{5.11}$$

where A_n are unknown coefficients.

Solving an auxiliary problem where the pressure on the liquid surface is given by a function $P(t,x)$ expandable into Fourier series:

$$-\frac{\partial}{\partial t}\Phi\bigg|_{r=1}=P(t,x),$$

we obtain a formula linking the coefficients $V_n^L(s)$ and $P_n^L(s)$:

$$V_n^L(s)=\frac{\sqrt{s^2+n^2}}{s}P_n^L(s), \quad n=0,1,\dots . \tag{5.12}$$

Next, applying the convolution theorem [2] yields

$$V_n(t)=P_n(t)-\int_0^t P_n(\tau)r_n(t-\tau)d\tau, \quad n=0,1,\dots, \tag{5.13}$$

which relates the coefficients $V_n^{(i)}(t)$ and $P_n^{(i)}(t)$. The function $r_n(t)$ is defined by the integral

$$r_n(t)=-n\int_0^t \frac{J_1(n\xi)}{\xi}d\xi, \tag{5.14}$$

where $J_1(t)$ is a Bessel function [3].

Noteworthy is the obvious similarity of the function (5.14) and the function $R_n(t)$, which has been derived in Sec. 3 based on asymptotic expansion of Bessel functions of high order for small gap.

Using (5.9) and (5.13), we arrive at the relationship between the velocity $V(t,x)$ and the pressure $p(t,x)$ on the liquid surface:

$$V(t,x)=p(t,x)+\sum_{n=0}^{\infty}\left(\int_0^t p_n(\tau)r_n(t-\tau)d\tau\cos nx\right). \tag{5.15}$$

Substituting (5.9) and (5.15) into the boundary conditions (5.2) and (5.3), we get

$$p(t,x)=H(x-|x^*|)\left\{V_0(t)-\sum_{n=0}^{\infty}\left(\int_0^t p_n(\tau)r_n(t-\tau)d\tau\cos nx\right)\right\}, \tag{5.16}$$

where $H(x)$ is the Heaviside function.

Expanding the left- and right-hand sides of (5.16) into Fourier series and equating the coefficients of like cosines, we arrive at an infinite system of linear Volterra equations of the second kind for $p_n(t)$:

$$p_n(t)=V_{0n}(t)-\sum_{m=0}^{\infty}\beta_{mn}(x^*)\int_0^t p_m(\tau)r_m(t-\tau)d\tau, \quad n=0,1,\dots . \tag{5.17}$$

The functions $V_{0n}(t)$ and $\beta_{mn}(x^*)$ are given by the formulas

$$V_{0n}(t)=\begin{cases}\pi^{-1}V_0(t)x^*(t), & n=0,\\ 2\pi^{-1}n^{-1}V_0(t)\sin nx^*, & n=1,2,\dots,\end{cases}$$

$$\beta_{mn}(x)=\begin{cases}\pi^{-1}x, & m=0, \quad n=0,\\ \pi^{-1}\sin mx, & m\neq n, \quad n=0,\\ -1+\pi^{-1}\left(x+\dfrac{\sin 2mx}{2m}\right), & m=n, \quad n\neq 0,\\ \pi^{-1}\left(\dfrac{\sin(m-n)x}{m-n}+\dfrac{\sin(m+n)x}{m+n}\right), & m\neq n, \quad n\neq 0.\end{cases}$$

The infinite system of equations (3.11) differs from (5.17) only by the algorithm of determining the functions $R_n(t)$ for small n (see formulas (3.12) and (3.14)).

In view of (5.11), the hydrodynamic drag is expressed as

$$Q(t) = 2\left[p_0(t)x^* + \sum_{m=1}^{\infty} p_m(t)\frac{\sin mx^*}{m} \right]. \tag{5.18}$$

Thus, problem (5.2)–(5.8) has been reduced to the infinite system of integral equations (5.17) and the equation of motion (5.6), with $Q(t)$ defined by (5.18).

6. Solution Methods. The governing systems of integral equations derived in the previous sections can be solved numerically. To this end, they are reduced to algebraic equations by truncating and using some quadrature formulas or other. Here, we first obtain an elementary solution for the earliest stage of interaction based on the asymptotic properties of Bessel functions and then find a numerical solution for a nonsmall gap in the general case.

6a. Zero Gap (Asymptotic Solution). As follows from Sec. 4, for a zero gap the problem reduces to the system of Volterra equations of the second kind (4.7) or (4.10) and the differential equation (2.8). If the period of interaction is very small, then this problem can be solved using the following asymptotic representation of the modified Bessel functions of the second kind of high order [3]:

$$K_n(s) \approx \sqrt{\frac{\pi}{2s}}e^{-s}(1+\dots), \qquad K'_n(s) \approx -\sqrt{\frac{\pi}{2s}}e^{-s}(1+\dots), \tag{6.1}$$

which, according to the final-value theorem, allows recovering time functions for small initial time intervals. Then, the time-domain relation $p_n(t) = \xi_n V_0(t)$, $n = 0, 1, \dots, \infty$, follows from (4.6), whence

$$p = V = H\left(\frac{\pi}{2} - \theta\right) V_0(t)\cos\theta, \tag{6.2}$$

i.e., the pressure is proportional to the velocity (plane-wave approximation well known in hydroelasticity and acoustics and valid only for short periods of interaction). This relation together with the equation of motion yields

$$M\frac{dV}{dt} = -2V(t)\int_0^{\pi/2}\cos^2\theta d\theta, \quad V|_{t=0} = V_0$$

or

$$M\frac{dV}{dt} = -\frac{\pi}{2}V(t), \quad V|_{t=0} = V_0. \tag{6.3}$$

Equation (6.3) has a solution in the form

$$V(t) = V_0 e^{-\frac{\pi}{2M}t}.$$

According to Eq. (6.2), the pressure depends on time in a similar manner. Then, the hydrodynamic drag is defined by the formula

$$Q = \frac{\pi}{2}V_0 e^{-\frac{\pi}{2M}t}. \tag{6.4}$$

Formula (6.4) suggests that the drag is finite at $t = 0$ and decreases with time. Note that this initial value of drag is substantially (severalfold) different from that predicted by the incompressible liquid model. It should be born in mind, however, that the above results have been obtained using the asymptotic relations (6.1) and are valid only on a short time interval, beginning with the moment of initial contact. The solution does not describe the backward motion of the body and possible repeated impacts. To improve this solution, it is necessary to solve the integral equations (4.7) (or (4.10)) or, better still, the infinite system of Volterra equations.

If the contact boundary is defined by (4.11), then the hydrodynamic drag is given by

$$Q(t) = -2V(t) \int_0^{\arccos\frac{W}{2}} \cos^2\theta d\theta, \tag{6.5}$$

and the penetration velocity can be determined from the following two differential equations of the first order:

$$\frac{dW}{dt} = V, \quad M\frac{dV}{dt} = -V(t)\left(\arccos\frac{W}{2} + \frac{1}{2}\sqrt{1 - \frac{W^2}{4}}\right), \quad W|_{t=0} = 0, \quad V|_{t=0} = V_0. \tag{6.6}$$

6b. Small Gap (Asymptotic Solution). Applying asymptotics (6.1) to relations (3.6), we again obtain equality $p_n(t) = V_n(t)$, $n = 0, 1, ...$, which yields (6.2). Here, the hydrodynamic drag is given by

$$Q(t) = 2V_0(t) \int_0^{\theta^*} \cos^2\theta d\theta, \quad \theta^* = \arccos\frac{(1+W)D + W^2/2}{D+W},$$

and the penetration velocity can be determined from

$$\frac{dW}{dt} = V, \quad M\frac{dV}{dt} = -V(t)(\theta^* + \frac{1}{2}\sin 2\theta^*), \quad W|_{t=0} = 0, \quad V|_{t=0} = V_0. \tag{6.7}$$

6c. Nonsmall Gap (Asymptotic Solution). If the liquid boundary is flat, then it is obvious (Fig. 1*c*) that

$$x^* = \sin\theta^* = \sin\arccos(1-W) = \sqrt{2W - W^2}.$$

As above, we use the asymptotics of cylinder functions to obtain relation (6.2) and the following expression for the hydrodynamic drag:

$$Q(t) = 2V_0(t) \int_0^{x^*} \cos^2 x dx, \quad x^* = \sqrt{2W - W^2}.$$

Then, the penetration velocity can be determined from

$$\frac{dW}{dt} = V, \quad M\frac{dV}{dt} = -V(t)(\sqrt{2W - W^2} + \frac{1}{2}\sin 2(\sqrt{2W - W^2})), \quad W|_{t=0} = 0, \quad V|_{t=0} = V_0. \tag{6.8}$$

Thus, the systems of differential equations (6.3), (6.6), (6.7), and (6.8) allow us to calculate the velocity and drag at the earliest stage of body–liquid interaction for different typical relationships between the radii of the body and the cavity. We have solved these equations numerically with the help of the Maple 6 analytical computation software system.

6d. Nonsmall Gap (Numerical Solution). The boundary-value problem was reduced in the general case to the infinite system of integral equations (5.17) and the equation of motion (5.6) (where $Q(t)$ is defined by (5.18)) on the contact boundary determined at each instant t from relation (5.8). The system was solved on a finite time interval $[0; T]$. This interval was divided into equal subintervals Δt to evaluate all the unknown quantities at the nodes.

The infinite system of integral equations and the series in (5.18) were truncated. The degree of truncation was selected from the considerations of practical convergence. All the integrals in (2.9) were evaluated using the trapezoidal and Simpson's rules. The Gibbs σ-factors were used to improve the convergence of the Fourier series.

The input parameters: time interval [0; 4]; the initial velocity of penetration $V_0(0) = 0.05$; the mass of the body $M = 0.5$, 1.0, 2.0, 5.0; gap size $D = 0$, 0.1, 0.2; time step $\Delta t = 0.01$–0.001; the number of terms $m = 50$–500 in the finite sum approximating the series in (5.18); and the number of equations $n = 50$–200 in the truncated system of equations (5.17).

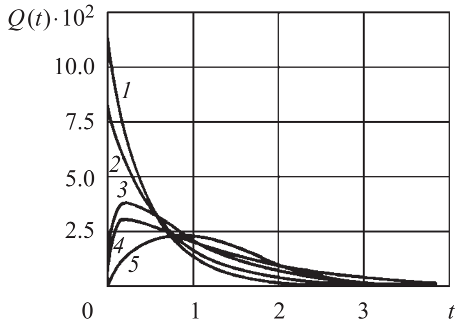

Fig. 2

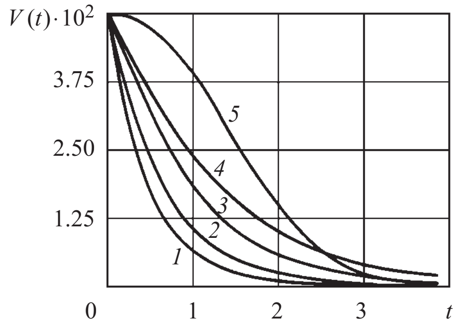

Fig. 3

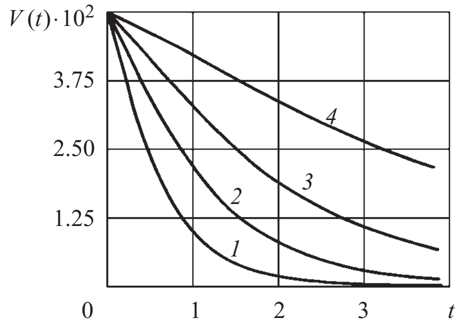

Fig. 4

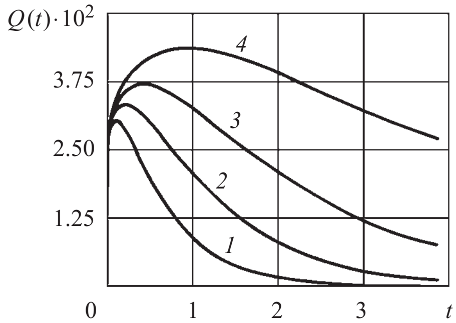

Fig. 5

7. Numerical Results.

7a. Asymptotic Solution. The basic parameters characterizing the process in question are the penetration velocity $V_0(t)$ and the hydrodynamic drag $Q(t)$. Figure 2 shows the time-dependence of the hydrodynamic drag obtained from the asymptotic solution (Sec. 6*a–c*) for a body with $M = 1$ and $V_0(0) = 0.05$. Curves *1* and *2* correspond to zero gap ($D = 0$), curve *2* being plotted from formula (6.4), i.e., for $\theta^* = \pi/2$, and curve *2* from (6.6), where the contact boundary is given by (4.11). It can be seen that the drag is finite and maximum at the initial instant and abruptly decreases with time. Curves *3* and *4* correspond to small gaps $D = 0.1$ and $D = 0.2$, respectively. When the gap is small, the initial magnitude of drag is finite, yet decreases with increase in D. The drag peaks in some time after the initial contact. Curve *5* corresponds to $D \to \infty$ (nonsmall gap). In this case, the drag begins to increase from a nonzero magnitude and peaks even later, the maximum being less than at small and zero gaps.

Figure 3 illustrates the time-dependence of the penetration velocity for a body with $M = 1$ and $V_0(0) = 0.05$. Curves *1–5* correspond to the same values of D as in Fig. 2: $D = 0$ (formula (4.11)); $D = 0$ ($\theta^* = \pi/2$); $D = 0.1$; $D = 0.2$; and $D \to \infty$. It can be seen that the greater the gap, the less the rate of decrease in the penetration velocity.

Figures 4 and 5 show the penetration velocity and the hydrodynamic drag as functions of the mass of the body for the same small gap ($D = 0.2$) and initial velocity ($V_0(0) = 0.05$). Curves *1–4* correspond to $M = 0.5$; 1.0; 2.0; 5.0, respectively. It can easily be seen that the greater the mass, the less the rate of decrease in the velocity. In turn, the heavier the body, the greater the maximum drag, this maximum being reached later. The behavior is similar for zero and nonsmall gaps.

7b. Numerical Solution (Nonsmall Gap). Figure 6 shows the hydrodynamic drag for a body with $M = 0.5$; 1.0; 2.0; 5.0 (curves *1–4*, respectively) and $V_0(0) = 0.05$ in the case of nonsmall gap ($D \to \infty$). It is seen that the time- and mass-dependences of the drag are qualitatively similar to those predicted by the asymptotic solution. The quantitative difference, however, is significant. For example, the maximum magnitudes of drag differ by a factor of 2 to 3.

Figure 7 shows the penetration velocity as a function of time for a body with the same masses as in the previous figure (curves *1–4*, respectively). In contrast to the drag, the velocity decreases less than predicted by the asymptotic solution. As a result, no backward motion is observed within the time interval under consideration. Note that we have neglected here the hydrodynamic effect due to the high-speed longitudinal motion of the body. This effect is significant even at small deviations of the longitudinal axis of the body from the course vector.

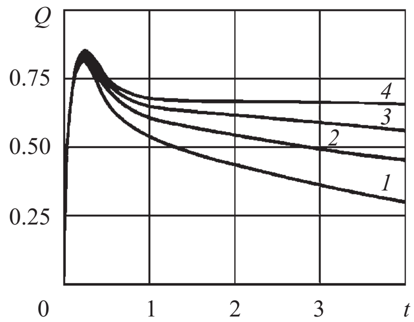

Fig. 6

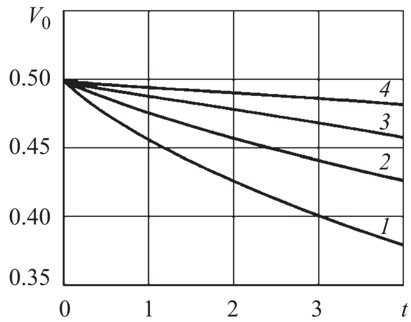

Fig. 7

8. Conclusions. The approach developed here allows us to study, in a plane formulation, the shock interaction between a long thin cylindrical body and a cylindrical cavity in a perfect compressible liquid. The approach has several modifications, depending on the relationship between the typical dimensions of the body and the cavity. In all cases, a mixed nonstationary boundary-value problem with an unknown moving boundary is formulated and then reduced to an infinite system of Volterra equations of the second kind by expanding the unknown quantities into Fourier series. This system of equations is a part of the governing system of equations, which also includes the equation of transverse motion of the body and the equation of the contact boundary. For all cases, we have obtained an approximate solution for the initial stage of interaction using asymptotic representations of cylinder functions. In the case of a nonsmall gap, we have additionally found a numerical solution. The asymptotic and numerical solutions are in qualitative agreement. The quantitative difference, however, is significant, which necessitates a numerical solution of the governing system of equations. The results obtained here lead us to the conclusion that the longitudinal motion of a supercavitating body may have a significant effect on the behavior of the impact processes.

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