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NATURAL VIBRATION OF A SANDWICH BEAM ON AN ELASTIC FOUNDATION

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The natural vibration of an elastic sandwich beam on an elastic foundation is studied. Bernoulli's hypotheses are used to describe the kinematics of the face layers. The core layer is assumed to be stiff and compressible. The foundation reaction is described by Winkler's model. The system of equilibrium equations is derived, and its exact solution for displacements is found. Numerical results are presented for a sandwich beam on an elastic foundation of low, medium, or high stiffness

Keywords: free vibration, elastic sandwich beam, Winkler foundation, exact solution, numerical analysis

Introduction. Widespread use of laminated thin-walled members in modern structures has aroused considerable scientific interest in their physical and mechanical properties and behavior under various external forces. The inhomogeneity and anisotropy of such structures call for special methods of their analysis. Of special interest is the behavior of laminated members under dynamic loads typical primarily of modern high-speed vehicles. The vibration of orthotropic shells was studied, for example, in the monograph [3], which, as well as the review [9], provides an extensive bibliography on the subject. Techniques and examples of studying the behavior of laminated bodies under dynamic loads were considered in [1, 4–6, 8, 10]. The vibration of sandwich beams not lying on an elastic foundation was studied in [2, 7, 11]. The present paper addresses the natural vibration of a sandwich beam asymmetric across the thickness. It lies on an elastic inertialess foundation.

1. Formulation of the Problem. Suppose that Bernoulli's hypotheses hold in the isotropic face layers, and the exact relations of the theory of elasticity with linear approximation of displacements in the transverse coordinate z hold in the stiff core layer. The displacements are assumed to be continuous at the interfaces between the layers. The face layers are transversely incompressible, and the core is subject to the corresponding reduction. Strains are assumed to be small.

A coordinate system x, y, z is fixed to the mid-surface of the core. A distributed surface load $q(x)$ is applied to the outer surface of the upper face layer at a right angle (Fig. 1). The foundation acts on the outer surface of the lower face layer $q_r(x, t)$. Let $w_k(x)$ and $u_k(x)$ denote the deflections and longitudinal displacements of the mid-surfaces of the face layers; h_k and ρ_k denote the thickness and density of the kth layer (k = 1, 2, 3 is the layer number); $h_3 = 2c$; and b_0 denotes the width of the beam. All displacements and linear dimensions of the beam are referred to its length l.

The longitudinal and transverse displacements $u^{(k)}(x, z)$ and $w^{(k)}(x, z)$ can be expressed in terms of four unknown functions $w_1(x, t)$, $u_1(x, t)$, $w_2(x, t)$, and $u_2(x, t)$ as follows:

$$u^{(1)} = u_1 - \left(z - c - \frac{h_1}{2}\right) w_{1,x}, \qquad w^{(1)} = w_1 \qquad (c \le z \le c + h_1),$$

$$u^{(2)} = u_2 - \left(z + c + \frac{h_2}{2}\right) w_{2,x}, \qquad w^{(2)} = w_2 \qquad (-c - h_2 \le z \le -c)$$

in the face layers and

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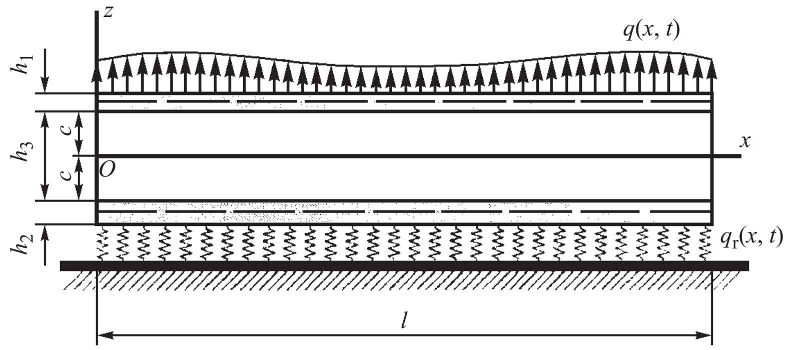

Fig. 1

$$u^{(3)} = \left(1+\frac{z}{c}\right)\left(\frac{1}{2}u_1 + \frac{h_1}{4}w_{1,x}\right) + \left(1-\frac{z}{c}\right)\left(\frac{1}{2}u_2 - \frac{h_2}{4}w_{2,x}\right),$$

$$w^{(3)} = \frac{1}{2}\left(1+\frac{z}{c}\right)w_1 + \frac{1}{2}\left(1-\frac{z}{c}\right)w_2 \qquad (-c \le z \le c)$$

in the core layer, where z is the distance from a filament of interest to the mid-line of the core; and the comma in the subscript denotes differentiation with respect to the coordinate indicated next to the comma.

The equations of motion of the sandwich beam follow from Lagrange's principle in view of the work done by inertial forces:

$$\delta A - \delta W = \delta A_I, \tag{1.1}$$

where δA is the variation of the work done by external forces; δW is the variation of the work done by internal elastic forces; and δA_I is the variation of the work done by inertial forces.

To determine the work done by external forces, we assume that an arbitrary transverse load $q(x, t)$ is applied to the outer surface of the upper face layer (Fig. 1), and some forces and moments are applied to the ends of the beam. Then we have

$$\delta A = b_0 \int_0^l \left[p\left(\delta u_1 - \frac{h_1}{2}\delta w_{1,x}\right) + q\delta w_1 - q_{\mathrm{r}}\delta w_2 \right] dx. \tag{1.2}$$

The variation of the work done by elastic forces is defined by

$$\delta W = b_0 \int_0^l \left[\sum_{k=1}^{3} \int_{h_k} \sigma_{xx}^{(k)} \delta\varepsilon_{xx}^{(k)} dz + 2\int_{h_3} \sigma_{xz}^{(3)} \delta\varepsilon_{xz}^{(3)} dz + \int_{h_3} \sigma_{zz}^{(3)} \delta\varepsilon_{zz}^{(3)} dz \right] dx, \tag{1.3}$$

and the variation of the work done by inertial forces is given by

$$\delta A_I = b_0 \sum_{k=1}^{3} \int_0^l \int_{h_k} \left[\rho_k \left(\ddot{w}^{(k)} \delta w^{(k)} + \ddot{u}^{(k)} \delta u^{(k)} \right) \right] dz dx, \tag{1.4}$$

where the double overdot denotes the second derivative of displacement with respect to time.

Substituting (1.2)–(1.4) into (1.1) yields an equation that must identically hold at arbitrary values of the varied quantities. This will be possible if we equate the coefficients of independent variations to zero. As a result, we arrive at the following system of equations of motion written for forces:

$$
\begin{cases}
F_1 + \dfrac{1}{b_0}\left(H_1 - P_{1,x}\right) = 0, \quad F_3 + \dfrac{1}{b_0}\left(S_{1,xx} + H_2 - T_{1,x}\right) = q, \\
F_2 - \dfrac{1}{b_0}\left(H_1 + P_{2,x}\right) = 0, \quad F_4 + \dfrac{1}{b_0}\left(S_{2,xx} - H_2 - T_{2,x}\right) = -q_{\mathrm{r}},
\end{cases} \tag{1.5}
$$

where the inertial terms are given by

$$
\begin{aligned}
F_1 &= m_1 \ddot{u}_1 + m_8 \ddot{u}_2 + 2m_5 \ddot{w}_{1,x} - m_7 \ddot{w}_{2,x}, \\
F_2 &= m_8 \ddot{u}_1 + m_2 \ddot{u}_2 + m_5 \ddot{w}_{1,x} - 2m_7 \ddot{w}_{2,x}, \\
F_3 &= -2m_5 \ddot{u}_{1,x} - m_5 \ddot{u}_{2,x} + m_1 \ddot{w}_1 + m_8 \ddot{w}_2 - m_3 \ddot{w}_{1,xx} + m_6 \ddot{w}_{2,xx}, \\
F_4 &= m_7 \ddot{u}_{1,x} + 2m_7 \ddot{u}_{2,x} + m_8 \ddot{w}_1 + m_2 \ddot{w}_2 + m_6 \ddot{w}_{1,xx} - m_4 \ddot{w}_{2,xx}.
\end{aligned} \tag{1.6}
$$

Depending on the properties of the elastic foundation, the deflection and the reaction may be related in different ways. Let the foundation be inertialess (Winkler's model); then

$$
q_{\mathrm{r}} = \kappa_0 w_2, \tag{1.7}
$$

where κ_0 is the foundation modulus.

Applying Hooke's law, the Cauchy relations, and expression (1.7) to Eqs. (1.5), we obtain a system of partial differential equations for the four unknown functions $w_1(x, t)$, $u_1(x, t)$, $w_2(x, t)$, and $u_2(x, t)$. Retaining only those inertial terms (1.6) that describe the inertia of motion along the coordinate axes and the rotary inertia of normals in the face layers, we get equations of motion in the form

$$
\begin{aligned}
& a_1 u_1 - a_1 u_2 - a_4 u_{1,xx} - a_5 u_{2,xx} + a_2 w_{1,x} + a_3 w_{2,x} - 2a_6 w_{1,xxx} + a_7 w_{2,xxx} + m_1 \ddot{u}_1 = 0, \\
& -a_1 u_1 + a_1 u_2 - a_5 u_{1,xx} - a_9 u_{2,xx} - a_{10} w_{1,x} - a_{17} w_{2,x} - a_6 w_{1,xxx} + 2a_7 w_{2,xxx} + m_2 \ddot{u}_2 = 0, \\
& -a_2 u_{1,x} + a_{10} u_{2,x} + 2a_6 u_{1,xxx} + a_6 u_{2,xxx} + a_{11} w_{1,xx} - a_{12} w_{2,xx} \\
& \quad + a_{15} w_{1,xxxx} - a_{16} w_{2,xxxx} + a_8 w_1 - a_8 w_2 + m_1 \ddot{w}_1 - m_3 \ddot{w}_{1,xx} = q, \\
& -a_3 u_{1,x} + a_{17} u_{2,x} - a_7 u_{1,xxx} - 2a_7 u_{2,xxx} - a_{12} w_{1,xx} + a_{14} w_{2,xx} \\
& \quad - a_{16} w_{1,xxxx} + a_{13} w_{2,xxxx} - a_8 w_1 + \left(a_8 + \kappa_0\right) w_2 + m_2 \ddot{w}_2 - m_4 \ddot{w}_{2,xx} = 0,
\end{aligned} \tag{1.8}
$$

where K_k and G_k are the bulk and shear moduli of elasticity of the layers;

$$
a_1 = \frac{G_3}{2c}, \quad a_2 = \frac{G_3}{2}\left(1 + \frac{h_1}{2c}\right) - \frac{K_3^-}{2}, \quad a_3 = \frac{G_3}{2}\left(1 + \frac{h_2}{2c}\right) + \frac{K_3^-}{2}, \quad a_4 = K_1^+ h_1 + \frac{2K_3^+ c}{3},
$$

$$
a_5 = \frac{K_3^+ c}{3}, \quad a_6 = \frac{K_3^+ c h_1}{6}, \quad a_7 = \frac{K_3^+ c h_2}{6}, \quad a_8 = \frac{K_3^+}{2c}, \quad a_9 = K_2^+ h_2 + \frac{2K_3^+ c}{3},
$$

$$
a_{10} = \frac{G_3}{2}\left(1 + \frac{h_1}{2c}\right) + \frac{K_3^-}{2}, \quad a_{11} = \frac{K_3^- h_1}{2} - \frac{G_3 c}{2}\left(1 + \frac{h_1}{2c}\right)^2 - \frac{G_3 c}{6},
$$

$$
a_{12} = \frac{K_3^- (h_1 + h_2)}{4} + \frac{G_3 c}{2}\left(1 + \frac{h_1}{2c}\right)\left(1 + \frac{h_2}{2c}\right) - \frac{G_3 c}{6}, \quad a_{13} = \frac{K_2^+ h_2^3}{12} + \frac{K_3^+ c h_2^2}{6},
$$

$$
a_{14} = \frac{K_3^- h_2}{2} - \frac{G_3 c}{2}\left(1 + \frac{h_2}{2c}\right)^2 - \frac{G_3 c}{6}, \quad a_{15} = \frac{K_1^+ h_1^3}{12} + \frac{K_3^+ c h_1^2}{6},
$$

$$a_{16}=\frac{K_3^+ ch_2 h_1}{12}, \quad a_{17}=\frac{G_3}{2}\left(1+\frac{h_2}{2c}\right)-\frac{K_3^-}{2}.$$

Let the beam be simply supported at the ends. The supports are rigid and fixed. The corresponding constraints for the displacements to the sections $x = 0$; l are given by

$$w_k = u_{k,x} = w_{k,xx} = 0 \quad (k = 1, 2). \tag{1.9}$$

The initial conditions ($t = 0$) are

$$u_k(x,0)=u_{k0}(x), \quad \dot{u}_k(x,0)=\dot{u}_{k0}(x), \quad w_k(x,0)=w_{k0}(x), \quad \dot{w}_k(x,0)=\dot{w}_{k0}(x) \quad (k=1,2), \tag{1.10}$$

where $u_0^k(x)$, $\dot{u}_0^k(x)$, $w_0^k(x)$, and $\dot{w}_0^k(x)$ are given initial displacements and velocities of the mid-surfaces of the face layers.

The initial–boundary-value problem (1.8)–(1.10) is solved by the Bubnov–Galerkin method. To this end, the unknown displacements $u_1(x)$, $u_2(x)$, $w_1(x)$, and $w_2(x)$ and the load $q(x, t)$ are expanded into series in terms of basis functions satisfying the boundary conditions (1.9):

$$u_1(x,t)=\sum_{m=0}^{\infty}\cos\frac{\pi mx}{l}T_{m1}(t), \quad u_2(x,t)=\sum_{m=0}^{\infty}\cos\frac{\pi mx}{l}T_{m2}(t),$$

$$w_1(x,t)=\sum_{m=1}^{\infty}\sin\frac{\pi mx}{l}T_{m3}(t), \quad w_2(x,t)=\sum_{m=1}^{\infty}\sin\frac{\pi mx}{l}T_{m4}(t),$$

$$q(x,t)=\sum_{m=1}^{\infty}\sin\frac{\pi mx}{l}q_m(t), \tag{1.11}$$

where $q_m(t)$ are the expansion coefficients, $q_m(t)=\frac{2}{l}\int_0^l q(x,t)\sin\frac{\pi mx}{l}dx.$

Substituting (1.11) into (1.8) yields a system of equations for the time functions $T_{mi}(t)$ ($i = 1, 2, 3, 4$). We write it in matrix form:

$$[B]\{T\}+[M]\{\ddot{T}\}=\{Q\}, \tag{1.12}$$

where $[B]$ is a fourth-order square matrix composed of the coefficients B_{mij}; $[M]$ is a fourth-order diagonal matrix with elements M_{mij}; $\{T\}$ and $\{\ddot{T}\}$ are formed of the unknown functions T_{mi} and their second derivatives; and $\{Q\}$ is a vector whose elements Q_{mk} are composed of the expansion coefficients of the load;

$$[M]=\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_1+m_3\left(\frac{\pi m}{l}\right)^2 & 0 \\ 0 & 0 & 0 & m_2+m_4\left(\frac{\pi m}{l}\right)^2 \end{bmatrix}, \quad \{\ddot{T}\}=\begin{bmatrix} \ddot{T}_{m1} \\ \ddot{T}_{m2} \\ \ddot{T}_{m3} \\ \ddot{T}_{m4} \end{bmatrix},$$

$$[B]=\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_2 & b_5 & b_6 & -b_7 \\ b_3 & b_6 & b_8 & b_9 \\ b_4 & -b_7 & b_9 & b_{10} \end{bmatrix}, \quad \{T\}=\begin{bmatrix} T_{m1} \\ T_{m2} \\ T_{m3} \\ T_{m4} \end{bmatrix}, \quad \{Q\}=\begin{bmatrix} 0 \\ 0 \\ q_m \\ 0 \end{bmatrix},$$

where the coefficients b_i are dependent on the parameter m and are expressed in terms of the coefficients a_i as follows:

$$b_1 = a_1 + a_4\left(\frac{\pi m}{l}\right)^2, \quad b_2 = -a_1 + a_5\left(\frac{\pi m}{l}\right)^2, \quad b_3 = a_2\frac{\pi m}{l} + 2a_6\left(\frac{\pi m}{l}\right)^3,$$

$$b_4 = a_3\frac{\pi m}{l} - a_7\left(\frac{\pi m}{l}\right)^3, \quad b_5 = a_1 + a_9\left(\frac{\pi m}{l}\right)^2, \quad b_6 = -a_{10}\frac{\pi m}{l} + a_6\left(\frac{\pi m}{l}\right)^3,$$

$$b_7 = a_{17}\frac{\pi m}{l} + 2a_7\left(\frac{\pi m}{l}\right)^3, \quad b_8 = -a_{11}\left(\frac{\pi m}{l}\right)^2 + a_{15}\left(\frac{\pi m}{l}\right)^4 + a_8,$$

$$b_9 = a_{12}\left(\frac{\pi m}{l}\right)^2 - a_{16}\left(\frac{\pi m}{l}\right)^4 - a_8, \quad b_{10} = -a_{14}\left(\frac{\pi m}{l}\right)^2 + a_{13}\left(\frac{\pi m}{l}\right)^4 + a_8 + \kappa_0.$$

Equations (1.12) can be written as

$$\sum_{j=1}^{4} B_{mkj}T_{mj} + M_{mkk}\ddot{T}_{mk} = Q_{mk} \qquad (k = 1, \ldots, 4). \tag{1.13}$$

Since the matrix $[M]$ is diagonal, only the kth term remains in the second sum. To close the problem, the initial conditions (1.10) should be added to Eqs. (1.13).

2. Analytic Solution. Assume that there is no external load: $q(x, t) = 0$. Then the initial–boundary-value problem (1.8)–(1.10) will describe the natural vibration of the sandwich beam on elastic foundation. The equations of motion (1.13) take the following form ($Q_{mk} = 0$):

$$\sum_{j=1}^{4} B_{mkj}T_{mj} + M_{mkk}\ddot{T}_{mk} = 0 \qquad (k = 1, \ldots, 4). \tag{2.1}$$

The solution can be represented as

$$T_{mk}(t) = A_{mk}\sin(\omega_m t + \alpha_{mk}), \tag{2.2}$$

where A_{mk}, ω_m, and α_{mk} are the amplitude, frequency, and initial phase of vibration, respectively.

Substituting (2.2) into (2.1) leads to the generalized eigenvalue problem

$$[B]\{A\} = \omega^2 [M]\{A\}, \tag{2.3}$$

where $\{A\}$ is the vector of amplitudes A_{mk}.

Inverting the matrix $[M]$, which is not degenerate, in Eq. (2.3) leads to the standard eigenvalue problem

$$[R]\{A\} = \omega^2\{A\}, \quad [R] = [M]^{-1}[B], \quad ([R] - \omega^2 E)\{A\} = 0. \tag{2.4}$$

The system of equations (2.4) is homogeneous in A_{mk}. A trivial solution would mean no vibration. To find it, it is necessary to equate the determinant to zero. Doing this yields an algebraic equation of the fourth order for ω_m^2. Solving it, we obtain four real non-negative roots. Thus, the vibratory process for each value of m appears four-frequency. Hence, instead of (2.2) it is necessary to set

$$T_{mk}(t) = \sum_{i=1}^{4} A_{mki}\sin(\omega_{mi} t + \alpha_{mi}). \tag{2.5}$$

The unknown displacements are defined by (1.11) in view of (2.5). The twenty constants of integration A_{mki}, α_{mi} for each m are determined as follows. Substituting ω_{mi}^2 into Eqs. (2.1), we get four equations for each k = 1, 2, 3, 4. Of these equations, three are independent. These twelve independent homogeneous equations should be supplemented with eight

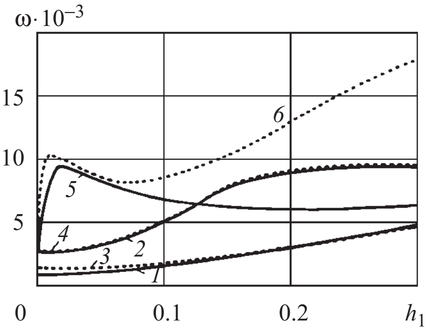

Fig. 2

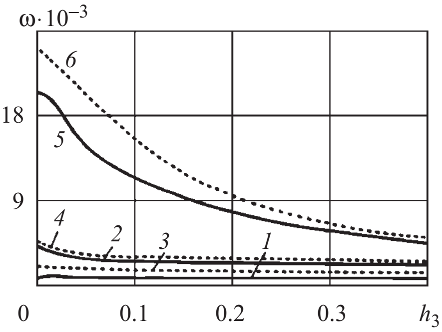

Fig. 3

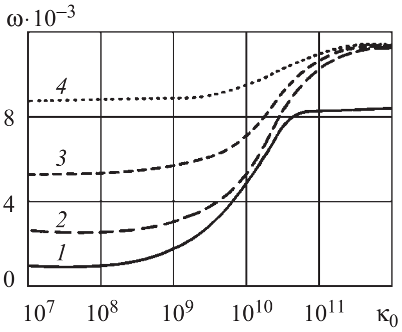

Fig. 4

inhomogeneous equations that follow from the initial conditions (1.10). Note that if all the initial conditions are zero, then all A_{mki} and α_{mi} will also be equal to zero because the solution of the homogeneous algebraic system of equations is unique.

3. Numerical Results. We have determined the natural frequencies of a D16T–PTFE–D16T beam.

Figure 2 shows the minimum frequencies ω_{m1} of the beam as functions of the thickness of the upper face layer h_1 for different stiffnesses of the elastic foundation ($c = 0.09$, $h_2 = 0.05$) and different values of m: $m = 1$, $\kappa_0 = 1$ MPa/m (curve *1*); $m = 2$, $\kappa_0 = 1$ MPa/m (curve *2*); $m = 1$, $\kappa_0 = 500$ MPa/m (curve *3*); $m = 2$, $\kappa_0 = 500$ MPa/m (curve *4*); $m = 1$, $\kappa_0 = 10^5$ MPa/m (curve *5*); and $m = 2$, $\kappa_0 = 10^5$ MPa/m (curve *6*). With foundations of low and medium stiffnesses ((1, 2) and (3, 4)), the first two frequencies are almost equal. If the beam lies on a foundation of high stiffness (5, 6), then as the layer increases in thickness, the first frequency decreases and the second frequency abruptly increases. With further increase in the foundation modulus, these curves move farther apart.

As the thickness of the relatively soft core ($h_2 = 0.05$; $h_1 = 0.01$) increases, the natural frequencies of the beam decrease (Fig. 3). The curves in Fig. 3 are numbered in the same way as in Fig. 2.

Figure 4 shows the frequencies ω_{m1} as functions of the foundation modulus κ_0 ($c = 0.09$, $h_2 = 0.05$, and $h_1 = 0.01$). The value of m is indicated near each curve. With foundations of low and high stiffnesses ($\kappa_0 < 10^8$ Pa/m and $\kappa_0 > 10^{11}$ Pa/m), this dependence is weak and frequencies vary a little. In the range $10^8 < \kappa_0 < 10^{11}$ Pa/m, the frequencies significantly increase with the stiffness of the foundation. It should be noted that the natural frequencies of the beam not lying on elastic foundation are scarcely different from those of the beam on the foundation of low stiffness.

Conclusions. We have studied the natural vibration of a sandwich beam. The analytic and numerical results obtained allow us to conclude that elastic foundations of medium and high stiffness have a significant effect on the natural frequencies. The influence of foundations of low stiffness may in some cases be neglected.

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