ON THE SPECTRUM OF NATURAL FREQUENCIES OF CIRCULAR CYLINDRICAL SHELLS COMPLETELY FILLED WITH A FLUID

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The natural frequencies of cylindrical shells filled with a fluid and having the ends either simply supported or clamped are determined. Conditions are studied under which the natural frequencies of the shell are close or multiple

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Introduction. In designing fluid-filled elastic cylindrical shells against nonlinear forced and parametric vibration, one needs preliminary information on the spectrum of their natural frequencies that would account for the presence of the fluid. The possible closeness or multiplicity of these frequencies (internal resonances [1, 5]) creates prerequisites for strong energy coupling and interaction of different modes of the shells during vibration [4, 5]. Because of this, the uncoupled (single-mode) vibration of these shells becomes unstable, and, simultaneously, complex, coupled (multimode) vibration may occur. Occurrence of internal resonances, including combinational ones, in the shell–fluid system is usually the starting point for approximating the expected dynamic deflection of shells.

In the present paper, we study the frequency spectrum of cylindrical shells of finite length completely filled with a fluid. We will examine the influence of the geometry of the shell and fluid on the feasibility of internal resonances that most often occur in real shell–fluid systems vibrating with large deflections.

1. To describe the dynamic behavior of a shell filled with a fluid, we use the well-known medium-deflection equations in mixed form [2, 3]:

$$\frac{D}{h}\nabla^4 w=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2\Phi}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\Phi}{\partial x\partial y}+\frac{1}{R}\frac{\partial^2\Phi}{\partial x^2}+\frac{q}{h}-\frac{P_г}{h}-\rho\frac{\partial^2 w}{\partial t^2},$$

$$\frac{1}{E}\nabla^4\Phi=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\frac{1}{R}\frac{\partial^2 w}{\partial x^2}, \tag{1.1}$$

where $w = w(x, y, t)$ is the radial deflection (positive when directed toward the center of curvature); x and y are the longitudinal and circumferential coordinates, respectively, the Ox-axis being reckoned from one of the ends of the shell; $D = Eh^3/[12(1-\mu^2)]$ is cylindrical stiffness (E is the elastic modulus, h is the shell thickness, and μ is Poisson's ratio); $\Phi = \Phi(x, y, t)$ is the function of stresses in the mid-surface of the shell; ρ is density; $q = q_0(x, y)\cos\Omega t$ is the external transverse pressure on the shell ($q_0(x, y)$ is the function of pressure distribution over the lateral surface); P_h is the hydrodynamic pressure due to the motion of the fluid; and $\nabla^4 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^2$ is a differential operator.

Since the shell is closed [2], its dynamic deflection can always be approximated by a two-parameter series:

$$w=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(f_1^{nm}(t)\cos s_n y+f_2^{nm}(t)\sin s_n y)X_m(x), \tag{1.2}$$

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where $f_{1,2}^{nm}$ are unknown functions of time having the sense of generalized coordinates of the shell; $s_n = n/R$ is the circumferential wave number; and $X_m(x)$ are axial coordinate functions that satisfy prescribed end conditions.

To determine the hydrodynamic pressure exerted by the fluid on the shell (the static pressure of the fluid is hereafter neglected [6]), we will use the following well-known relation [3, 6]:

$$P_h = P_h(x, r, \Theta, t) = -\rho_0 \left.\frac{\partial \varphi}{\partial t}\right|_{r=R}, \tag{1.3}$$

where ρ_0 is the density of the fluid; and x, r, and Θ are cylindrical coordinates. The velocity potential φ can be found from the solution of the boundary-value problem [3, 5, 6]

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\Theta^2} = 0 \quad \text{in } Q,$$

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=0} < \infty, \qquad \left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = -\frac{\partial w}{\partial t}, \qquad \varphi\big|_{x=0} = 0, \qquad \varphi\big|_{x=l} = 0, \tag{1.4}$$

where l is the length of the shell; and Q is the domain occupied by the fluid ($0 \le x \le l$; $0 \le r \le R$; and $0 \le \Theta \le 2\pi$).

Substituting the potential φ (in view of (1.3)) into the first equation in (1.1) and applying the Bubnov–Galerkin method, we obtain a system of ordinary differential equations for f_i^{nm}:

$$\ddot{f}_1^{nm} + \omega_{nm}^2 f_1^{nm} = F_1^{nm}(\{f_1\}, \{f_2\}) + Q_1^{nm}\cos\Omega t,$$

$$\ddot{f}_2^{nm} + \omega_{nm}^2 f_2^{nm} = F_2^{nm}(\{f_1\}, \{f_2\}) + Q_2^{nm}\cos\Omega t \qquad (n = 0, 1, 2, \ldots; m = 1, 2, \ldots), \tag{1.5}$$

where ω_{nm} are the natural frequencies of the shell with fluid; F_1^{nm} and F_2^{nm} are functions nonlinear with respect to the generalized displacements $\{f_i\} = \{f_i^{01}, f_i^{02}, \ldots, f_i^{11}, f_i^{12}, \ldots\}$ ($i = 1, 2$) (up to the third power inclusively [5, 7–12]); and $Q_{1,2}^{nm}$ are constants dependent on the form of the function $q_0(x, y)$.

It follows from (1.5) that there are always internal resonances [5]. Indeed, each pair of the generalized displacements f_1^{nm} and f_2^{nm}, and, hence, each pair of the corresponding natural modes $X_m \cos s_n y$ and $X_m \sin s_n y$, is associated with the same natural frequencies ω_{nm}.

In addition to these resonances, more complex resonances are possible, such as $\omega_{n_1m_1} \approx (p/q)\omega_{n_2m_2}$ $(n_1 \approx n_2)$, where p and q are some coprime numbers. However, such resonances occur only with certain geometries of the shell. Specific values of p and q depend on the type of nonlinear relations among the functions f_i^{nm} in Eqs. (1.5) [1, 5].

2. To ascertain whether one internal resonance or another is feasible in (1.5), let us consider a shell with boundary conditions of two most popular types at the shell ends $x = 0$ and $x = l$: simply supported and clamped. In the former case, we have [2]

$$w = 0, \qquad \frac{\partial^2 w}{\partial x^2} = 0. \tag{2.1}$$

In the latter case, we have

$$w = 0, \qquad \frac{\partial w}{\partial x} = 0. \tag{2.2}$$

The corresponding axial modes $X_m(x)$ can be represented as

1) $$X_m = \sin\frac{m\pi x}{l},$$

$$2) \qquad X_m = \sin^2 \frac{m\pi x}{l}. \tag{2.3}$$

On the whole, the dynamic deflection w is approximated by a five-term series:

$$w = (f_1 \cos s_1 y + f_2 \sin s_1 y)\sin \lambda_m x + (f_3 \cos s_2 y + f_4 \sin s_2 y)\sin \lambda_m x + f_5 \sin^4 \lambda_m x \tag{2.4}$$

for the simply supported shell and

$$w = (f_1 \cos s_1 y + f_2 \sin s_1 y)\sin^2 \lambda_m x + (f_3 \cos s_2 y + f_4 \sin s_2 y)\sin^2 \lambda_m x + f_5 \sin^4 \lambda_m x \tag{2.5}$$

for the clamped shell. Here, $s_1 = n_1/R$; $s_2 = n_2/R$; and $\lambda_m = m\pi/l$.

When $q \equiv 0$ (free vibration), the governing equations (1.5) for both cases take the following form [6, 9–12]:

$$\ddot{f}_1 + \omega_1^2 f_1 + k_{11}(f_1^2 + f_2^2) f_1 + k_{12}(f_3^2 + f_4^2) f_1 + k_{13} f_1 f_5 + k_{14} f_1 f_5^2 = Q_{10} \cos \Omega t,$$

$$\ddot{f}_2 + \omega_1^2 f_2 + k_{11}(f_1^2 + f_2^2) f_2 + k_{12}(f_3^2 + f_4^2) f_2 + k_{13} f_2 f_5 + k_{14} f_2 f_5^2 = Q_{20} \cos \Omega t,$$

$$\ddot{f}_3 + \omega_2^2 f_3 + k_{21}(f_1^2 + f_2^2) f_3 + k_{22}(f_3^2 + f_4^2) f_3 + k_{23} f_3 f_5 + k_{24} f_3 f_5^2 = Q_{30} \cos \Omega t,$$

$$\ddot{f}_4 + \omega_2^2 f_4 + k_{21}(f_1^2 + f_2^2) f_4 + k_{22}(f_3^2 + f_4^2) f_4 + k_{23} f_4 f_5 + k_{24} f_4 f_5^2 = Q_{40} \cos \Omega t,$$

$$\ddot{f}_5 + \omega_3^2 f_5 + k_{31}(f_1^2 + f_2^2) + k_{32}(f_3^2 + f_4^2) + k_{33}(f_1^2 + f_2^2) f_5 + k_{34}(f_3^2 + f_4^2) f_5 = Q_{50} \cos \Omega t, \tag{2.6}$$

where ω_k are the natural frequencies of the shell with fluid. In the case of the boundary conditions (2.1), we have

$$\omega_k^2 = \frac{1}{\rho m_{0k}} \left[\frac{D}{h} \Delta(\lambda_m, s_k) + \frac{E\lambda_m^4}{R^2 \Delta(\lambda_m, s_k)} \right] \qquad (k = 1, 2), \tag{2.7}$$

where [7, 11]

$$m_{0k} = 1 + \frac{\rho_0}{\rho} \frac{I_{n_k}(\lambda_m R)}{\lambda_m h I'_{n_k}(\lambda_m R)} \tag{2.8}$$

(I_n are modified Bessel functions). In the case of the boundary conditions (2.2), we have

$$\omega_k^2 = \frac{1}{3\rho m_{0k}} \left[\frac{D}{h} (\Delta(2\lambda_m, s_k) + 2 s_k^4) + \frac{16 E\lambda_m^4}{R^2 \Delta(2\lambda_m, s_k)} \right] \qquad (k = 1, 2), \tag{2.9}$$

where

$$m_{0k} = 1 + \frac{2}{3} \frac{\rho_0}{\rho h} \left(\frac{R}{n_k} + \frac{N_k}{4\lambda_m} \right), \qquad N_k = \frac{I_{n_k}(2\lambda_m R)}{I'_{n_k}(2\lambda_m R)}, \tag{2.10}$$

Q_{j0} ($j = 1$–5) are constants obtained by applying the Bubnov–Galerkin method and taking series (2.4) and (2.5) into account.

The frequency ω_3 corresponding to an axisymmetric vibration mode is expressed as follows for both types of boundary conditions:

$$\omega_3^2 = \frac{64}{35 \rho m_{03}} \left[\frac{8 D \lambda_m^4}{h} + \frac{35}{64} \frac{E}{R^2} \right], \tag{2.11}$$

and

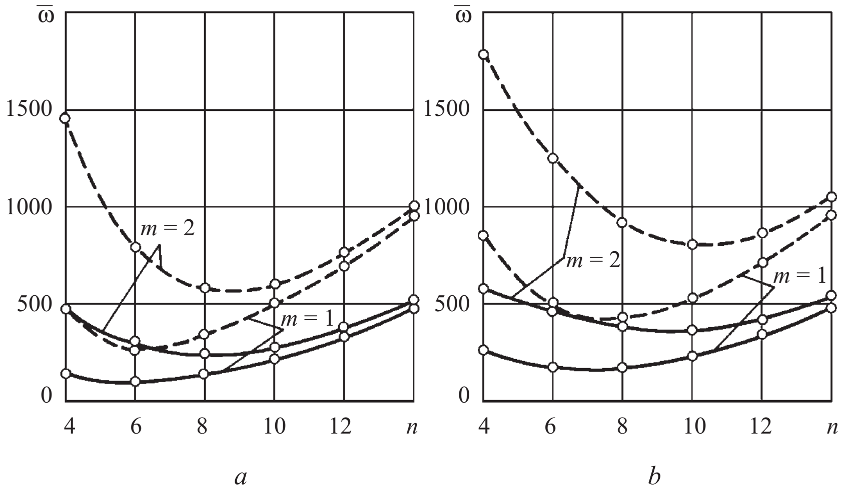


Fig. 1

$$m_{03} = 1 + \frac{16}{35}\frac{\rho_0}{\rho}\frac{1}{hl^2}\sum_{k=1}^{\infty}\frac{I_0(\lambda_k R)M_k^2(1-(-1)^k)}{\lambda_k I_0'(\lambda_k R)},$$

$$M_k = \frac{192\lambda_m^4}{\lambda_k(\lambda_k^2 - 4\lambda_m^2)(\lambda_k^2 - 16\lambda_m^2)}, \quad \lambda_k = \frac{k\pi}{l}. \tag{2.12}$$

The constant coefficients k_{ij} of the nonlinear terms in Eqs. (2.6) depend on the material and geometrical parameters of the shell, the wave numbers, and the added masses of the fluid [11].

Analyzing the system of nonlinear equations (2.6) in a well-known way [1], we establish the following five internal resonances in the first approximation [5, 10]:

$$\omega_1 \approx \omega_2, \quad \omega_1 \approx \omega_2/2, \quad \omega_1 \approx 2\omega_2, \quad \omega_{1,2} \approx \omega_3, \quad \omega_{1,2} \approx \omega_3/2. \tag{2.13}$$

Of prime practical interest are the first three resonances, since the frequency ω_3 in shells of medium length [2] is usually much greater than the frequencies ω_1 and ω_2 [12]. The frequency ω_3 can become comparable to ω_1 and ω_2 only in modes with a sufficiently large wave number n.

Let us consider in more detail the conditions under which the first three resonances (2.13) occur.

3. To derive specific analytic relations among geometrical parameters of the shell (at which the frequencies ω_1 and ω_2 "resonate" in the above-mentioned sense), we require that $\omega_1 = k\omega_2$, where $k = 1, 2, 1/2$. In the case of the simply supported boundary conditions (2.7), (2.8), we get the equation

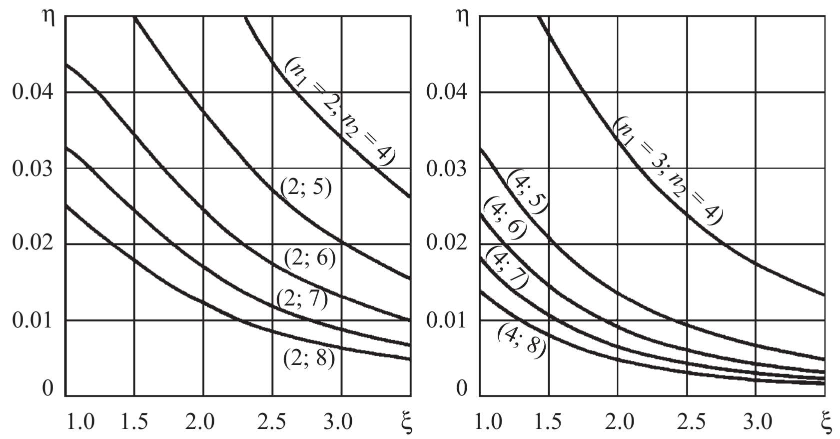
$$\frac{h^2}{12(1-\mu^2)}(M_1^2 - k^2\alpha M_2^2) = \frac{\lambda^4}{R^2}\frac{\alpha k^2 M_1^2 - M_2^2}{M_1^2 M_2^2},$$

where $\alpha = \frac{m_{01}}{m_{02}}$, $M_1^2 = \frac{1}{R^4\xi^4}(n_1^2\xi^2 + m^2\pi^2)^2$, $M_2^2 = \frac{1}{R^4\xi^4}(n_2^2\xi^2 + m^2\pi^2)^2$, and $\xi = \frac{l}{R}$.

This equation yields a relation between the dimensionless parameters ξ and $\eta = h/R$:

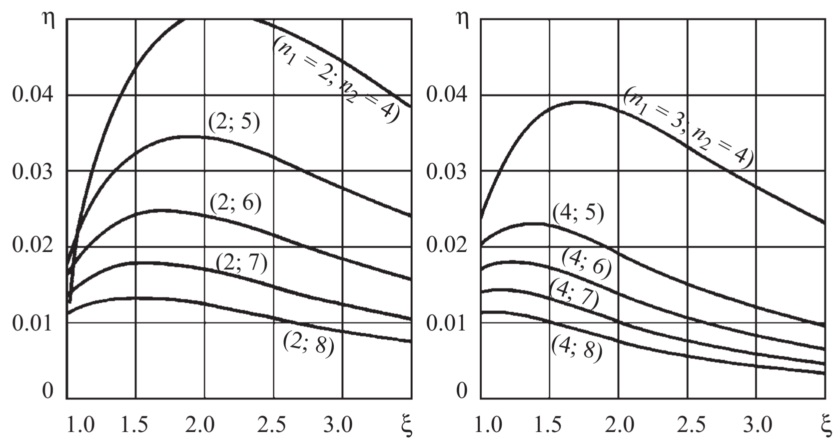
$$F(\xi,\eta) = c_0(\xi)\eta^3 + c_1(\xi)\eta^2 + c_2(\xi)\eta + c_3(\xi) = 0, \tag{3.1}$$

where $c_0(\xi) = A - Bk^2$, $c_1(\xi) = Ak_2 - Bk^2k_1$, $c_2(\xi) = M(\xi)(B - Ak^2)$, $c_3(\xi) = M(\xi)(Bk_2 - Ak_1k^2)$, $M(\xi) = 12(1-\mu^2)\frac{(m\pi\xi)^4}{AB}$, $A = (m^2\pi^2 + n_1^2\xi^2)^2$, $B = (m^2\pi^2 + n_2^2\xi^2)^2$, and $k_i = \frac{\rho_0}{\rho}\frac{\xi I_{n_i}(m\pi/\xi)}{m\pi I_{n_i}'(m\pi/\xi)}$ $(i = 1, 2)$.



a *b*

Fig. 2



a *b*

Fig. 3

Thus, we have derived an expression for selecting values of the length l, radius R, or thickness h of the filled shell at which it has equal or multiple natural frequencies. Note that if the shell is empty, then relation (3.1) becomes more simple [5], $\eta^2 = -c_2(\xi)/c_0(\xi)$, and leads to the condition $\omega_1 = k\omega_2$.

In the case of the clamped boundary conditions (2.2), for the "resonance" condition $\omega_1 = k\omega_2$ to hold, it is necessary to set

$$c_0(\xi) = A - Bk^2, \quad c_1(\xi) = Ak_2 - Bk^2 k_1, \quad c_2(\xi) = M(\xi)(B_1 - A_1 k^2), \quad c_3(\xi) = M(\xi)(B_1 k_2 - A_1 k_1 k^2)$$

in Eq. (3.1), where $A = 2n_1^4 + A_1$, $B = 2n_2^4 + B_1$, $A_1 = (n_1^2 + 4m^2\pi^2/\xi^2)^2$, $B_1 = (n_2^2 + 4m^2\pi^2/\xi^2)^2$, $k_1 = \dfrac{2}{3}\dfrac{\rho_0}{\rho}\left(\dfrac{1}{n_1} + \dfrac{Q_1(\xi)\xi}{4m\pi}\right)$,

$k_2 = \dfrac{2}{3}\dfrac{\rho_0}{\rho}\left(\dfrac{1}{n_2} + \dfrac{Q_2(\xi)\xi}{4m\pi}\right)$, $M(\xi) = \dfrac{192(1-\mu^2)m^4\pi^4}{\xi^4 A_1 B_1}$, and $Q_i(\xi) = \dfrac{I_{n_i}(2m\pi/\xi)}{I'_{n_i}(2m\pi/\xi)}$ ($i = 1, 2$).

4. As a numerical example, let us consider a shell ($E = 2 \cdot 10^{11}$ Pa; $\rho = 7.8 \cdot 10^3$ kg/m^3; and $\mu = 0.3$) filled with water ($\rho_0 = 1 \cdot 10^3$ kg/m^3).

Figure 1 shows the fundamental natural frequencies $\overline{\omega}$ (in Hz) of this shell for the following geometry: $h/R = 3.125 \cdot 10^{-3}$, $l/R = 2.45$, and $R = 0.16$ m, and the following wave numbers: $m = 1, 2$; $4 \le n \le 14$. Figure 1*a* corresponds to the shell with simply supported ends, and Fig. 1*b* to the shell with clamped ends. The dashed lines represent the natural frequencies of the dry shell.

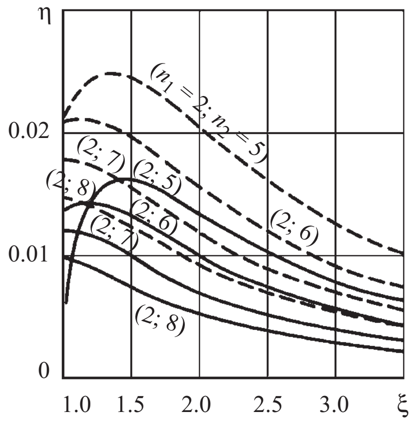


Fig. 4

It follows from the figures that the fluid not only significantly reduces (by a factor of 2 to 4) the natural frequencies of the cylindrical shell, but also causes spectrum "crowding" (in the range of *n* under consideration), compared with the empty shell. Thus, the probability that internal resonances (especially the principal resonance $\omega_1 \approx \omega_2$) occur in the filled shell is much higher than in the corresponding "dry" shell.

Figures 2 and 3 show the graphs of $\eta = \eta(\xi)$ plotted from Eq. (3.1) for simply supported (Fig. 2) and clamped (Fig. 3) fluid-filled cylindrical shell for $k = 1$, $m = 1$, and different values of n_i (indicated in the figures).

As is seen, almost every filled cylindrical shell can have close natural frequencies. Two to three and more modes with different wave numbers may "resonate" (in nonlinear sense). For example, Fig. 3*a*, *b* suggests that for $\xi = 1.35$ and $\eta = 0.023$, the following modes (m, n_i) satisfy the resonance relation $\omega_1 \approx \omega_2$: (1, 2), (1, 6), (1, 4), and (1, 5), which, in particular, correspond to the following frequencies: $\omega_{1,2} = 1391.5$ Hz and $\omega_{1,6} = 1390.9$ Hz, $\omega_{1,4} = 1264.1$ Hz, and $\omega_{1,5} = 1262.6$ Hz. During free or forced vibration, these modes are strongly energetically coupled with one another; hence, they should be accounted for in groups in selecting an approximation for the dynamic deflection *w*.

Figure 4 illustrates the graphs of $\eta = \eta(\xi)$ for a simply supported shell with (solid curves) and without (dashed curves) the fluid at $k = 2$, i.e., at resonance $\omega_1 = 2\omega_2$.

It follows from Fig. 4 that multiple frequencies ($\omega_1 = 2\omega_2$) occur in the "dry" shell at larger values of the thickness *h* than in the filled shell. This is also true of the resonance $\omega_1 = \omega_2$ considered earlier.

Conclusions. We have studied the problem of natural frequencies of cylindrical shells completely filled with a fluid and simply supported or clamped at the ends. From the results obtained we conclude that the effect of the fluid on the frequencies of the shell is stronger for modes with smaller wave numbers *n*. The more complex the modes, i.e., the greater the wave number *n*, the weaker the effect of the fluid on the frequency spectrum. The fluid may change the flexural vibration mode corresponding to the minimum natural frequency ω_{min}. From Fig. 1, for example, it follows that the mode $n = 9$, $m = 2$ of a simply supported empty shell is observed at $\omega = \omega_{min}$. If this shell is filled with a fluid, then the frequency ω_{min} corresponds to the mode $n = 8$, $m = 2$. A similar effect is observed in the clamped shell. The vibration mode at $\omega = \omega_{min}$ changes at $m = 1$. This mode corresponds to $n = 8$ in the "dry" shell and to $n = 7$ in the filled shell.

The analytic relation (3.1) derived here is the key to determining whether the vibrating shell–fluid system has internal resonances or not.

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