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# ON STATIC OUTPUT-FEEDBACK STABILIZATION OF A PERIODIC SYSTEM

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**Relations are derived that allow standard MATLAB routines to be used to solve the static output-feedback control problem for a periodic discrete-time system. The efficiency of the approach proposed to design optimal static output-feedback controllers is demonstrated by examples**

**Keywords:** periodic discrete-time system, static output feedback, objective function

**Introduction.** The optimization problem for linear periodic systems has a wide range of applications [2, 3, 6, 9–12, 14, 15, 18, 19, 30–32]. One of such is the synthesis of stabilization systems for walking and hopping robots [20–25, 27–29]. It is known (see, e.g., [13]) that for a nonstationary linear system whose phase vector can only partially be observed, the feedback design procedure reduces to solving two matrix differential Riccati equations.

One of these equations describes a filter that generates an estimate of the whole phase vector, and the other equation produces a controller matrix that relates this estimate and the control. This is the so-called case of dynamic feedback design. Designing static feedback is a more complicated problem because it requires determining constant matrices that form the control directly from the observable portion of the phase vector. Even if the system is stationary, this problem is very complicated [16, 33, 35]. To solve it, numerical algorithms based on the gradient of the objective function were proposed in [7, 8, 33–35].

Moreover, if the system is unstable, these algorithms additionally require an initial approximation, i.e., a stabilizing matrix. Forming this matrix is an independent problem [33]. Various approaches to its solution were proposed in [4, 8, 26, 33, 36].

Here, we outline an algorithm for design of the optimal (i.e., minimizing a quadratic performance criterion) static output-feedback controller for a periodic discrete(-time) system.

This algorithm generalizes the approach from [4, 26] to periodic discrete-time systems, which substantially simplifies the procedure of selecting an initial approximation. The relations describing the objective function and its gradient are also generalized appropriately. We will show that standard MATLAB routines can be used to implement this algorithm.

**1. Problem Formulation.** Consider a $p$-periodic discrete-time system described by the following difference relations:

$$x_{i+1} = A_i x_i + B_i u_i, \quad i = 0, 1, \ldots, \quad x(0) = x_0, \quad y_i = C_i x_i, \tag{1.1}$$

where $x_i$, $u_i$, and $y_i$ are the phase vector, control vector, and observable vector, respectively; and $A_i$, $B_i$, and $C_i$ are $p$-periodic matrices, i.e.,

$$A_{i+p} = A_i, \quad B_{i+p} = B_i, \quad C_{i+p} = C_i, \quad \forall\, i.$$

Let an output feedback be applied to system (1.1). In other words, the dynamics of system (1.1) with feedback

$$u_i = K_i y_i, \quad K_{i+p} = K_i, \quad \forall\, i, \tag{1.2}$$

is described by the following periodic difference equations:

$$x_{i+1} = \left(A_i + B_i K_i C_i\right) x_i = \overline{A}_i x_i, \quad x(0) = x_0, \tag{1.3}$$

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where $K_i$ are the feedback matrices.

Let us optimize (by selecting the corresponding values of the matrices $K_i$ in (1.2)) system (1.3) using the following quadratic functional:

$$J=\sum_{i=0}^{\infty}\left(x_i'Q_ix_i+u_i'R_iu_i\right)=\sum_{i=0}^{\infty}x_i'\left(Q_i+C_i'K_i'R_iK_iC_i\right)x_i=\sum_{i=0}^{\infty}x_i'\,\overline{Q}_ix_i,$$

$$\overline{Q}_i=Q_i+C_i'K_i'R_iK_iC_i,\qquad Q_{i+p}=Q_i,\qquad R_{i+p}=R_i,\qquad \forall i. \tag{1.4}$$

Here and later on, the prime denotes transposition.

The task is to minimize (by selecting $p$ matrices $K_i$ in (1.2)) the functional (1.4) on the class of asymptotically stable ($\lim\limits_{i\to\infty}x_i=0$) closed-loop systems (1.3).

Let us reduce this problem to the optimization problem for a stationary system. Using relations (1.3), we examine the change in the phase vector $x_i$ over one period:

$$x_{i+p}=\overline{A}_{i+p-1}\overline{A}_{i+p-2}\ldots\overline{A}_i=\psi_ix_i,\qquad \psi_i=\overline{A}_{i+p-1}\overline{A}_{i+p-2}\ldots\overline{A}_i, \tag{1.5}$$

whence it follows that

$$x_{i+lp}=\psi_ix_{i+(l-1)p}. \tag{1.6}$$

Assuming that $i=0$ in (1.6) and denoting $x_{lp}=x_l$, we associate the periodic system (1.3) with the following stationary system:

$$x_{l+1}=\psi x_l,\qquad \psi=\overline{A}_{p-1}\overline{A}_{p-2}\ldots\overline{A}_0. \tag{1.7}$$

In this connection, it is also necessary to modify the functional (1.4). Let us consider the sum of the first $p$ terms of this functional:

$$\sum_{i=0}^{p-1}x_0'\left(\overline{Q}_0+\overline{A}_0'\overline{Q}_1\overline{A}_0+\ldots+\overline{A}_{p-2}'\ldots\overline{A}_0'\overline{Q}_{p-1}\overline{A}_0\ldots\overline{A}_{p-2}\right)x_0=x_0'Tx_0,$$

$$T=\overline{Q}_0+\overline{A}_0'\overline{Q}_1\overline{A}_0+\ldots+\overline{A}_{p-2}'\ldots\overline{A}_0'\overline{Q}_{p-1}\overline{A}_0\ldots\overline{A}_{p-2}. \tag{1.8}$$

With (1.7) and (1.8), the functional (1.4) can be rearranged as

$$J=\sum_{l=0}^{\infty}x_l'Tx_l. \tag{1.9}$$

Thus, the optimization of the periodic system (1.3) according to the criterion (1.4) has been reduced to the optimization of the stationary system (1.7) according to the criterion (1.9).

Assume that the vector of initial conditions $x_0$ is a random vector with zero expectation and the following covariance matrix:

$$S=\langle x_0x_0'\rangle, \tag{1.10}$$

where "< >" denotes the averaging operator.

According to [34], the (averaged) value of the functional (1.9) to be minimized can be written as

$$J=\mathrm{tr}(TP), \tag{1.11}$$

where the matrix $P$ is a solution of the discrete Lyapunov equation

$$P=\psi P\psi'+S \tag{1.12}$$

and "tr" denotes the trace of a matrix.

Thus, the task is to minimize the functional (1.11) on a set of matrices $K_i$, $i = 0, \ldots, p-1$, appearing in (1.2) provided that the closed-loop system is stable, i.e., all the eigenvalues of the matrix $\psi$ fall within a unit circle,

$$|\lambda(\psi)| < 1. \tag{1.13}$$

**2. Optimization Procedure.** The use of one numerical procedure or another to minimize the functional (1.11) on the set of matrices $K_i$ requires an initial approximation (matrices $K_i$) that satisfies condition (1.13). As is known, (see, e.g., [33]), even if the system is stationary, it is very difficult to select such an initial approximation.

We will use, as in [4, 26], a modification of the original problem that would substantially simplify the selection of an initial approximation. This modification implies introducing an additional variable $\mu$ to replace the matrices $A_i$ in (1.1) by the matrices $A_{i\mu} = (1-\mu)A_i$.

Now relation (1.3) takes on the form

$$x_{i+1} = \overline{A}_{i\mu} x_i, \qquad \overline{A}_{i\mu} = (A_{i\mu} + B_i K_i C_i).$$

After the modification (replacement of $\overline{A}_i$ by $\overline{A}_{i\mu}$), the matrices $\psi$ and $T$ appearing in (1.7) and (1.8) are denoted by $\psi_\mu$ and $T_\mu$.

The functional (1.11) is replaced by

$$J_\mu = \mathrm{tr}(T_\mu P_\mu) + r\mu^2, \qquad r > 0, \tag{2.1}$$

where the matrix $P_\mu$ satisfies the Lyapunov equation

$$P_\mu = \psi_\mu P_\mu \psi'_\mu + S. \tag{2.2}$$

Thus, the original problem (1.11), (1.12) has been reduced to the modified problem (2.1), (2.2). It should be expected that if $r$ in (2.1) is sufficiently large, then the value of the parameter $\mu$ in the solution of problem (2.1), (2.2) (result of minimization of (2.1) in $K_i$ and $\mu$) is small. Therefore, with a sufficiently large $r$ the solution of the modified problem may be used as an acceptable approximation for the original problem (1.11), (1.12).

In the problem thus modified, it is expedient to select an initial value of $\mu$ such that

$$\left|\lambda\left((1-\mu)^p A_{p-1} A_{p-2} \ldots A_0\right)\right| < 1. \tag{2.3}$$

Such a choice of an initial value for $\mu$ makes it simple to select an initial value for the matrix $K_i$.

Indeed, in this case $K_i = 0$ can be chosen as an initial approximation.

**3. Finding the Gradient of the Objective Function.** In Sect. 2, we have derived the objective function (2.1) for output-feedback optimization of a periodic system and pointed out how to select an initial approximation ($K_i = 0$, the initial value of $\mu$ is selected according to (2.3)).

Thus, to solve this problem, we can use optimization algorithms that do not require calculating derivatives of the objective function (for example, the Nelder–Mead method [5] and the fmins.m MATLAB routine). However, the optimization methods that employ the gradient of the objective function are usually more efficient [5]. In this connection, we will use the relations from [1] to derive expressions for the gradient of the objective function (2.1) in the case $p = 2$.

To calculate the gradient of the objective function, we will use the relations from [1, formulas (6.541)–(6.544)]. These relations are presented below. Let the matrices $M$, $N$, and $Y$ depend on the parameter $\omega$. The matrix $X$ is a solution of the following discrete Lyapunov equation:

$$X = MXM' + N. \tag{3.1}$$

Consider the scalar

$$\mathrm{tr}(XY). \tag{3.2}$$

According to [1], its derivative with respect to $\omega$ is defined by

$$\frac{\partial\left(\operatorname{tr}(XY)\right)}{\partial\omega}=\operatorname{tr}\left(\frac{\partial Y}{\partial\omega}X+\frac{\partial N}{\partial\omega}U+2U\frac{\partial M}{\partial\omega}XM'\right), \tag{3.3}$$

where the matrix $U$ is a solution of the discrete Lyapunov equation

$$U=M'UM+Y. \tag{3.4}$$

Let us also present some relations used to differentiate the trace of a matrix product with respect to a matrix. Let $a=\operatorname{tr}(AX)$. As in [32, 33], the derivative of a scalar $a$ with respect to a matrix $X$ is a matrix $g$ with elements $g_{ij}=\frac{\partial a}{\partial X_{ij}}$, where $X_{ij}$ are the elements of the matrix $X$.

Note that Bryson and Ho [13] define $g$ differently, namely $g_{ij}=\frac{\partial a}{\partial X_{ji}}$.

Using the relation $\frac{\partial}{\partial X}\operatorname{tr}(AX)=A'$, which can easily be verified, we obtain the expressions

$$\frac{\partial}{\partial X}\operatorname{tr}(AXB)=A'B',\quad \frac{\partial}{\partial X}\operatorname{tr}(AX'B)=BA,\quad \frac{\partial}{\partial X}\operatorname{tr}(X'AXB)=AXB+A'XB',$$

$$\frac{\partial}{\partial X}\operatorname{tr}\left((A+BXC)'R(A+BXC)F\right)=B'R(A+BXC)(F+F')C'.$$

Considering that

$$\psi_\mu=\overline{A}_{1\mu}\overline{A}_{0\mu},\qquad T_\mu=\overline{Q}_0+\overline{A}'_{0\mu}\overline{Q}_1\overline{A}_{0\mu}$$

for $p$ = 2 and using relations (3.1)–(3.4) and the expressions for the differentiation of matrix trace, we obtain the following relations for the gradient of the objective function (2.1):

$$\frac{\partial J_\mu}{\partial\mu}=2r\mu-\operatorname{tr}\left((A'_0\overline{Q}_1\overline{A}_{0\mu}+\overline{A}'_{0\mu}\overline{Q}_1A_0)P_\mu+2U(A_{1\mu}\overline{A}_{0\mu}+\overline{A}_{1\mu}A_0)P_\mu\psi'\right), \tag{3.5}$$

$$\frac{\partial J_\mu}{\partial K_0}=2\left(R_0K_0C_0+B'_0(\overline{Q}_1\overline{A}_{0\mu}+\overline{A}'_{1\mu}U\psi_\mu)\right)P_\mu C'_0, \tag{3.6}$$

$$\frac{\partial J_\mu}{\partial K_1}=2\left(R_1K_1C_1\overline{A}_{0\mu}+B'_0U\psi_\mu\right)P_\mu\overline{A}'_{0\mu}C'_1, \tag{3.7}$$

$$U=\psi'_\mu U\psi_\mu+T_\mu. \tag{3.8}$$

Thus, we have all the necessary relations to solve the optimization problem, namely, relations (2.1) and (2.2) for the objective function and relations (3.5)–(3.8) for its gradient in the variables $\mu$, $K_0$, and $K_1$.

As already mentioned, a value of $\mu$ that satisfies (2.3) should be taken as an initial approximation, i.e., if the maximum absolute value of the eigenvalue ($\lambda_m$) of the matrix $A_{p-1}\dots A_0$ is less than unity, then we can set $\mu=0$. Otherwise ($\lambda_m>1$),

$$\mu>1-\frac{1}{\sqrt[p]{\lambda_m}}. \tag{3.9}$$

For such a choice of $\mu$, zero values of the matrices $K_0$ and $K_1$ can be taken as an initial approximation.

Note that the above relations allow us to use the fminu.m routine to solve the output-feedback optimization problem for a periodic system.

**4. Examples.** Let us illustrate our algorithm by way of the following examples.

*Example 1.* Consider a system described by the differential equation

$$\dot{x} = \psi x + \Gamma u, \tag{4.1}$$

where $\psi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x$ is the phase vector and $u$ is the control. System (4.1) is controlled at discrete time intervals $\tau = 0.2$.

Assume that the control $u$ is constant during each time interval. In this case, it is expedient to replace the continuous-time system (4.1) by a discrete-time system similar to (1.1),

$$x_{i+1} = Ax_i + Bu_i, \tag{4.2}$$

where $x_i$ and $u_i$ are the values of the phase vector and control at the beginning of the $i$th interval. The c2d.m routine helps to pass from (4.1) to (4.2) with the result that

$$A = \begin{bmatrix} 1.0201 & 0.2013 \\ 0.2013 & 1.0201 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0201 \\ 0.2013 \end{bmatrix}. \tag{4.3}$$

It is assumed that the first and second components of the vector $x_i$ are observed one after the other. Namely, the matrices $C_i$ appearing in (1.1) are given by

$$C_i = [1 \quad 0] \quad \text{for} \quad i = 0, 2, 4, \ldots,$$

$$C_i = [0 \quad 1] \quad \text{for} \quad i = 1, 3, 5, \ldots. \tag{4.4}$$

Thus, relations (4.2)–(4.4) describe the periodic discrete-time system (1.1) ($A_i = A$, $B_i = B$, $\forall i$) with period $p = 2$. Let the matrices appearing in (1.4) and (2.1) have the following values:

$$Q_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_i = 0, \quad \forall i, \quad r = 10^8. \tag{4.5}$$

We use the fminu.m routine to optimize this system, considering that relations (2.1) and (2.2) define the function to be minimized and relations (3.5)–(3.8) define its gradient.

In selecting initial conditions for the optimization procedure, we consider that the system is unstable ($\lambda_m > 1$). The right-hand side of (3.9) is equal to 0.1813; therefore, $\mu = 0.2$, $K_0 = 0$, and $K_1 = 0$ are taken as initial conditions. We also assume that $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in (1.10).

Solving the optimization problem, we obtain the following values of $\mu$, $K_0$, and $K_1$:

$$\mu = 2.75 \cdot 10^{-7}, \quad K_0 = -6.9521, \quad K_1 = -3.8123. \tag{4.6}$$

These values correspond to the following value of the functional (2.1): $J_\mu = 10.4185$.

Note that the obtained value of $\mu$ ($2.75 \cdot 10^{-7}$) is rather small, which is indicative of close approximation of the solution of the original problem ($\mu = 0$). The matrix $\psi$ defined by (1.7) and (4.6) has the following eigenvalues: 0.6185 and 0.4283.

Thus, condition (1.13) is satisfied.

Note that the solution obtained will change if the index $i$ in (1.1) is "shifted" by a value less than $p$. For example, if it is shifted by 1, i.e.,

$$C_i = [0 \quad 1] \quad \text{for} \quad i = 0, 2, \ldots,$$

$$C_i = [1 \quad 0] \quad \text{for} \quad i = 1, 3, 5, \ldots,$$

then $J_\mu = 10.1810$, $K_0 = -3.7515$, and $K_1 = -7.1625$.

The matrix $\psi$ has the following eigenvalues: $0.5241 \pm 0.0722i$.

*Example 2.* With increase in the period $p$, the relations defining the gradient of the objective function (an analog of relations (3.5)–(3.8)) become more awkward. In this kind of problems (with large $p$) it may appear reasonable to use optimization procedures (for example, fmins.m) that do not require calculating the gradient of the objective function.

Let us illustrate the possibility of using fmins.m in the following example. Consider a system described by the differential equations (4.1). Let its output be defined by

$$y = [\sin \omega t \quad \cos \omega t], \quad \omega = \pi. \tag{4.7}$$

Setting $\tau = 0.2$ as in Example 1, we approximate system (4.1) by the following system of difference equations:

$$x_{i+1} = Ax_i + Bu_i, \quad i = 0, 1, 2, \ldots, \tag{4.8}$$

where the matrices $A$ and $B$ are defined by (4.3).

The observation process (4.7) is approximated as follows:

$$y_i = [\sin(\omega \tau i) \quad \cos(\omega \tau i)], \quad i = 0, 1, 2, \ldots. \tag{4.9}$$

As a result, we have obtained the periodic discrete-time system (4.8), (4.9) with period $p = 10$.

As in Example 1, the matrices and scalars appearing in (1.4), (1.10), and (2.1) are defined as

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_i = 0, \quad \forall i, \quad r = 10^8.$$

The use of the fmins.m routine to minimize (2.1) has produced the following results:

$$J_\mu = 8.6123, \quad r = 2.3751 \cdot 10^{-7}, \quad K_0 = -5.0055, \quad K_1 = -8.8772, \quad K_2 = -0.0211,$$

$$K_3 = 0.0010, \quad K_4 = -0.0007, \quad K_5 = -0.0182, \quad K_6 = 0.3013,$$

$$K_7 = 0.1191, \quad K_8 = -0.0045, \quad K_9 = -0.0339. \tag{4.10}$$

The matrix $\psi$ of the closed-loop system (1.7) whose feedback gains are given in (4.10) has the following eigenvalues: 0.1380 and 0.0182. Thus, condition (1.13) is satisfied.

**Conclusions.** We have derived relations that make it possible to use the standard MATLAB routines fmins.m and fminu.m to solve the output-feedback optimization problem for a periodic discrete-time system.

The efficiency of the approach to the design of the optimal output-feedback controller has been illustrated by examples.

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