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STRESS–STRAIN ANALYSIS OF CONICAL SHELLS WITH DIFFERENT BOUNDARY CONDITIONS AND THICKNESS VARYING IN TWO DIRECTIONS AT CONSTANT MASS

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UDC 539.3

A method developed for solving two-dimensional problems in the theory of conical shells is used to analyze the stress–strain state of shells with different boundary conditions and thickness varying in two directions at constant mass. Numerical results are given in the form of plots and tables

Keywords: conical shell, varying thickness, discrete Fourier series, discrete-orthogonalization method

Shells of varying thickness, as well as shells of constant thickness, are widely used as structural members. By changing the law of variation in thickness at constant mass, it is possible to provide a more rational stress–strain state of shell elements [1–3]. This particularly true of conical shells used as components of machines, aircraft, and devices [4].

1. We will address the class of stress–strain problems for a conical shell with thickness varying in two coordinate directions. Use will be made of the exact equations of the moment theory of shells. The boundary conditions at the ends are arbitrary. The shell is subjected to a surface load. The midsurface of the shell is referred to an orthogonal coordinate system s, θ, where s is the arc length along the generatrix, and θ is the central angle in the cross section. If the radius of the cross-sectional circle is expressed as

$$r(s)=r_0+\cos\varphi\cdot s, \tag{1}$$

where r_0 is the radius of the datum plane circle along the generatrix and φ is the angle between the normal to the midsurface and the z-axis of revolution, then the governing equations can be written as follows [1, 2]:

$$\frac{\partial N_r}{\partial s}=-\frac{(1-\nu)\cos\varphi}{r}N_r+\frac{\nu\sin\varphi}{r}N_z-\frac{\cos\varphi}{r}\frac{\partial\hat{S}}{\partial\theta}-\frac{\nu\sin\varphi}{r^2}\frac{\partial^2 M_s}{\partial\theta^2}+\frac{(1-\nu^2)\sin^2\varphi}{r^4}\frac{\partial^2}{\partial\theta^2}\left(D_M\frac{\partial^2 u_r}{\partial\theta^2}\right)$$

$$+\frac{(1-\nu^2)D_N}{r^2}u_r-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\frac{\partial^2}{\partial\theta^2}\left(D_M\frac{\partial^2 u_z}{\partial\theta^2}\right)+\frac{(1-\nu^2)}{r^2}D_N\frac{\partial v}{\partial\theta}$$

$$-\frac{(1-\nu^2)\sin^2\varphi}{r^4}\frac{\partial^2}{\partial\theta^2}\left(D_M\frac{\partial v}{\partial\theta}\right)-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^2}\frac{\partial^2}{\partial\theta^2}\left(D_M\vartheta_s\right)-q_r,$$

$$\frac{\partial N_z}{\partial s}=-\frac{\cos\varphi}{r}N_z-\frac{\sin\varphi}{r}\frac{\partial\hat{S}}{\partial\theta}+\frac{4\sin\varphi}{r^3}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0 D_N}\hat{S}\right)$$

$$+\frac{\nu\cos\varphi}{r^2}\frac{\partial^2 M_s}{\partial\theta^2}-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\frac{\partial^2}{\partial\theta^2}\left(D_M\frac{\partial^2 u_r}{\partial\theta^2}\right)+\frac{(1-\nu^2)\cos^2\varphi}{r^4}\frac{\partial^2}{\partial\theta^2}\left(D_M\frac{\partial^2 u_z}{\partial\theta^2}\right)$$

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$$
-\frac{2(1-\nu)}{r^4}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0}\frac{\partial u_z}{\partial\theta}\right)+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\frac{\partial^2}{\partial\theta^2}\left(D_M\frac{\partial v}{\partial\theta}\right)
$$

$$
+\frac{(1-\nu^2)\cos^2\varphi}{r^3}\frac{\partial^2}{\partial\theta^2}\left(D_M\vartheta_s\right)+\frac{2(1-\nu)}{r^3}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0}\frac{\partial\vartheta_s}{\partial\theta}\right)-q_z,
$$

$$
\frac{\partial\hat{S}}{\partial s}=-\frac{\nu\cos\varphi}{r}\frac{\partial N_r}{\partial\theta}-\frac{\nu\sin\varphi}{r}\frac{\partial N_z}{\partial\theta}-\frac{2\cos\varphi}{r}\hat{S}-\frac{\nu\sin\varphi}{r^2}\frac{\partial M_s}{\partial\theta}+\frac{(1-\nu^2)\sin^2\varphi}{r^4}\frac{\partial}{\partial\theta}\left(D_M\frac{\partial^2 u_r}{\partial\theta^2}\right)
$$

$$
-\frac{(1-\nu^2)}{r^2}\frac{\partial}{\partial\theta}\left(D_N u_r\right)-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\frac{\partial}{\partial\theta}\left(D_M\frac{\partial^2 u_z}{\partial\theta^2}\right)-\frac{1-\nu^2}{r^2}\frac{\partial}{\partial\theta}\left(D_N\frac{\partial v}{\partial\theta}\right)
$$

$$
-\frac{(1-\nu^2)\sin^2\varphi}{r^4}\frac{\partial}{\partial\theta}\left(D_M\frac{\partial v}{\partial\theta}\right)-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\frac{\partial}{\partial\theta}\left(D_M\vartheta_s\right)-q_\theta,
$$

$$
\frac{\partial M_s}{\partial s}=\sin\varphi N_r-\cos\varphi N_z-\frac{4\sin\varphi}{r^2}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0 D_N}\hat{S}\right)-\frac{(1-\nu)\cos\varphi}{r}M_s-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\left(D_M\frac{\partial^2 u_r}{\partial\theta^2}\right)
$$

$$
+\frac{2(1-\nu)}{r^3}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0}\frac{\partial u_z}{\partial\theta}\right)+\frac{(1-\nu^2)\cos^2\varphi}{r^3}\left(D_M\frac{\partial^2 u_z}{\partial\theta^2}\right)+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\left(D_M\frac{\partial v}{\partial\theta}\right)
$$

$$
-\frac{2(1-\nu)}{r^2}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0}\frac{\partial\vartheta_s}{\partial\theta}\right)+\frac{(1-\nu^2)\cos^2\varphi}{r^2}\left(D_M\vartheta_s\right),
$$

$$
\frac{\partial u_r}{\partial s}=\frac{\cos^2\varphi}{D_N}N_r+\frac{\sin\varphi\cos\varphi}{D_N}N_z-\frac{\nu\cos\varphi}{r}u_r-\frac{\nu\cos\varphi}{r}\frac{\partial v}{\partial\theta}-\sin\varphi\vartheta_s,
$$

$$
\frac{\partial u_z}{\partial s}=\frac{\sin\varphi\cos\varphi}{D_N}N_r+\frac{\sin^2\varphi}{D_N}N_z-\frac{\nu\sin\varphi}{r}u_r-\frac{\nu\sin\varphi}{r}\frac{\partial v}{\partial\theta}+\cos\varphi\vartheta_s,
$$

$$
\frac{\partial v}{\partial s}=\frac{2}{(1-\nu)p_0 D_N}\hat{S}-\frac{\cos\varphi}{r}\frac{\partial u_r}{\partial\theta}+\frac{\sin\varphi}{r}\left(\frac{4D_M}{r^2 p_0 D_N}-1\right)\frac{\partial u_z}{\partial\theta}+\frac{\cos\varphi}{r}v-\frac{4\sin\varphi}{r^2 p_0}\frac{D_M}{D_N}\frac{\partial\vartheta_s}{\partial\theta},
$$

$$
\frac{\partial\theta_s}{\partial s}=\frac{1}{D_M}M_s+\frac{\nu\sin\varphi}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{\nu\cos\varphi}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}-\frac{\nu\sin\varphi}{r^2}\frac{\partial v}{\partial\theta}-\frac{\cos\varphi}{r}\vartheta_s,
$$

$$
p_0=1+\frac{4\sin^2\varphi}{r^2}\frac{D_M}{D_N}, \tag{2}
$$

where ($0\le s\le L$, $0\le\theta\le\pi$)

$$
N_r=N_s\cos\varphi+\hat{Q}_s\sin\varphi, \qquad N_z=N_s\sin\varphi-\hat{Q}_s\cos\varphi,
$$

$$
u_r=u\cos\varphi+w\sin\varphi, \qquad u_z=u\sin\varphi-w\cos\varphi,
$$

$$
q_r=q_s\cos\varphi+q_\gamma\sin\varphi, \qquad q_z=q_s\sin\varphi-q_\gamma\cos\varphi,
$$

$$
\hat{Q}_s=Q_s+\frac{1}{r}\frac{\partial H}{\partial\theta}, \qquad \hat{S}=S+\frac{2\sin\varphi}{r}H, \qquad \vartheta_s=-\frac{\partial w}{\partial s}. \tag{3}
$$

In (2) and (3): N_s, Q_s, S, M_s, and H are forces and moments; u, v, and w are displacements; q_s, q_θ, and q_γ are the load components; $D_N = \dfrac{E\cdot h(s,\theta)}{1-\nu^2}$ and $D_M = \dfrac{E\cdot h^3(s,\theta)}{12(1-\nu^2)}$ are the tangential and flexural stiffnesses; $h = h(s, \theta)$ is the shell thickness; E is Young's modulus; and ν is Poisson's ratio. End conditions are prescribed.

Thus, the boundary-value stress–strain problem for a conical shell is described by a system of partial differential equations with variable coefficients and boundary conditions at $s = 0$ and $s = L$.

2. In the general case, the coefficients of Eqs. (2) depend on the variables s and θ, which makes it impossible to separate variables. Therefore, we will solve the two-dimensional boundary-value problem by making it one-dimensional and expanding functions defined on a discrete point set into Fourier series (for brevity, we will call them discrete Fourier series [5, 6]). With this aim in mind, we introduce the following additional functions:

$$\psi_1^j = D_M\left\{\frac{\partial^2 u_r}{\partial\theta^2}, \frac{\partial^2 u_z}{\partial\theta^2}, \frac{\partial v}{\partial\theta}, \vartheta_s\right\} \quad (j=1,2,3,4), \qquad \psi_2^j = D_N\left\{u_r, \frac{\partial v}{\partial\theta}\right\} \quad (j=1,2),$$

$$\psi_3^j = \frac{D_M}{p_0}\left\{\frac{\partial u_z}{\partial\theta}, \frac{\partial\vartheta_s}{\partial\theta}\right\} \quad (j=1,2), \qquad \psi_4^j = \frac{D_M}{p_0 D_N}\left\{\hat{S}, \frac{\partial u_z}{\partial\theta}, \frac{\partial\vartheta_s}{\partial\theta}\right\} \quad (j=1,2,3),$$

$$\psi_5^j = \frac{1}{D_N}\{N_r, N_z\} \quad (j=1,2), \qquad \psi_6 = \frac{1}{p_0 D_N}\hat{S}, \qquad \psi_7 = \frac{1}{D_M}M_s. \tag{4}$$

Let us substitute the functions (4) into the governing equations (2) and expand the unknown and additional functions into Fourier series in θ:

$$X(s,\theta) = \sum_{n=0}^{\infty} X_n(s)\cos\lambda_n\theta, \qquad Y(s,\theta) = \sum_{n=1}^{\infty} Y_n(s)\sin\lambda_n\theta, \qquad \lambda_n = \frac{\pi n}{2\pi} = \frac{n}{2},$$

$$X = \left\{N_r, N_z, M_s, u_r, u_z, \vartheta_s, \psi_1^j, \psi_2^j, \psi_5^j, \psi_7, q_r, q_z\right\}, \qquad Y = \left\{\hat{S}, v, \psi_3^j, \psi_4^j, \psi_6, q_\theta\right\}. \tag{5}$$

After substitution of the series (5) into Eqs. (2) and some manipulations, we arrive at a coupled system of ordinary differential equations for amplitude values of (5):

$$\frac{dN_r^n}{ds} = -\frac{(1-\nu)\cos\varphi}{r}N_r^n + \frac{\nu\sin\varphi}{r}N_z^n - \frac{\cos\varphi}{r}\lambda_n\hat{S}^n + \frac{\nu\sin\varphi}{r^2}\lambda_n^2 M_s^n - \frac{(1-\nu^2)\sin^2\varphi}{r^4}\lambda_n^2\psi_{1,n}^1 + \frac{1-\nu^2}{r^2}\psi_{2,n}^1$$

$$+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\lambda_n^2\psi_{1,n}^2 + \frac{1-\nu^2}{r^2}\psi_{2,n}^2 + \frac{(1-\nu^2)\sin^2\varphi}{r^4}\lambda_n^2\psi_{1,n}^3 + \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^2}\lambda_n^2\psi_{1,n}^4 - q_r,$$

$$\frac{dN_z^n}{ds} = -\frac{\cos\varphi}{r}N_z^n - \frac{\sin\varphi}{r}\lambda_n\hat{S}^n + \frac{4\sin\varphi}{r^3}\lambda_n\psi_{4,n}^1 - \frac{\nu\cos\varphi}{r^2}\lambda_n^2 M_s^n$$

$$+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\lambda_n^2\psi_{1,n}^1 - \frac{(1-\nu^2)\cos^2\varphi}{r^4}\lambda_n^2\psi_{1,n}^2$$

$$-\frac{2(1-\nu)}{r^4}\lambda_n\psi_{3,n}^1 - \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\lambda_n^2\psi_{1,n}^3 - \frac{(1-\nu^2)\cos^2\varphi}{r^3}\lambda_n^2\psi_{1,n}^4 + \frac{2(1-\nu)}{r^3}\lambda_n\psi_{3,n}^2 - q_z,$$

$$\frac{d\hat{S}^n}{ds} = \frac{\nu\cos\varphi}{r}\lambda_n N_r^n + \frac{\nu\sin\varphi}{r}\lambda_n N_z^n - \frac{2\cos\varphi}{r}\hat{S}^n + \frac{\nu\sin\varphi}{r^2}\lambda_n M_s^n - \frac{(1-\nu^2)\sin^2\varphi}{r^4}\lambda_n\psi_{1,n}^1 + \frac{(1-\nu^2)}{r^2}\lambda_n\psi_{2,n}^1$$

$$+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^4}\lambda_n\psi_{1,n}^2 + \frac{1-\nu^2}{r^2}\lambda_n\psi_{2,n}^2 + \frac{(1-\nu^2)\sin^2\varphi}{r^4}\lambda_n\psi_{1,n}^3 + \frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\lambda_n\psi_{1,n}^4 - q_\theta,$$

$$\frac{dM_s^n}{ds}=\sin\varphi N_r^n-\cos\varphi N_r^n-\frac{4\sin\varphi}{r^2}\lambda_n\psi_{4,n}^1-\frac{(1-\nu)\cos\varphi}{r}M_s^n-\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\psi_{1,n}^1+\frac{2(1-\nu)}{r^3}\lambda_n\psi_{3,n}^1$$

$$+\frac{(1-\nu^2)\cos^2\varphi}{r^3}\psi_{1,n}^2+\frac{(1-\nu^2)\sin\varphi\cos\varphi}{r^3}\psi_{1,n}^3-\frac{2(1-\nu)}{r^2}\lambda_n\psi_{3,n}^2+\frac{(1-\nu^2)\cos^2\varphi}{r^2}\psi_{1,n}^4,$$

$$\frac{du_r^n}{ds}=\cos^2\varphi\cdot\psi_{5,n}^1+\sin\varphi\cos\varphi\cdot\psi_{5,n}^2-\frac{\nu\cos\varphi}{r}u_r^n-\frac{\nu\cos\varphi}{r}\lambda_n v^n-\sin\varphi\vartheta_s^n,$$

$$\frac{du_z^n}{ds}=\sin\varphi\cos\varphi\cdot\psi_{5,n}^1+\sin^2\varphi\cdot\psi_{5,n}^2-\frac{\nu\sin\varphi}{r}u_r^n-\frac{\nu\sin\varphi}{r}\lambda_n v^n+\cos\varphi\vartheta_s^n,$$

$$\frac{dv^n}{ds}=\frac{2}{1-\nu}\psi_{6,n}+\frac{\cos\varphi}{r}\lambda_n u_r^n+\frac{4\sin\varphi}{r^3}\psi_{4,n}^2+\frac{\sin\varphi}{r}\lambda_n u_z^n+\frac{\cos\varphi}{r}v^n-\frac{4\sin\varphi}{r^2}\psi_{4,n}^3,$$

$$\frac{d\theta_s^n}{ds}=\psi_{7,n}-\frac{\nu\sin\varphi}{r^2}\lambda_n^2u_r^n+\frac{\nu\cos\varphi}{r^2}\lambda_n^2u_z^n-\frac{\nu\sin\varphi}{r^2}\lambda_n v^n-\frac{\cos\varphi}{r}\vartheta_s^n. \quad (6)$$

Boundary conditions for (6) are prescribed in the form

$$B_1\bar{Z}(0)=\bar{b}_1,\qquad B_2\bar{Z}(L)=\bar{b}_2, \quad (7)$$

where $\bar{Z}(s)=\{N_{r,n},N_{z,n},\hat{S}_n,M_{s,n},u_{r,n},u_{z,n},v_n,\vartheta_{s,n}\}^T$ is the column vector of unknown functions; B_1 and B_2 are rectangular matrices; and $\bar{b}_1$ and $\bar{b}_2$ are vectors.

The functions $\psi_{1,n}^j\ (j=1,2,3,4)$, $\psi_{2,n}^j\ (j=1,2)$, $\psi_{3,n}^j\ (j=1,2)$, $\psi_{4,n}^j\ (j=1,2,3)$, $\psi_{5,n}^j\ (j=1,2)$, $\psi_{6,n}$, and $\psi_{7,n}$ appearing in (6) are amplitude values of the additional functions (4) and are not expressed explicitly in terms of the Fourier coefficients of the series for the unknown functions. Rather, these functions are calculated during the integration of system (6) using discrete Fourier series at each step s = const and are dependent on the amplitude values of the unknown functions [7–10].

Equations (6) are integrated simultaneously for all harmonics of series (5) using the discrete-orthogonalization method. During the integration, the amplitude values of the additional functions are calculated from the current amplitude values of the unknown functions for $s=s_k\ (k=\overline{0,K})$ at some points θ_r of the interval $0\le\theta\le\pi$, and a Fourier series is formed for a discrete function using the Runge scheme [11]. At the beginning of the integration, the associated boundary conditions are taken into account. The integration over s is continued after the substitution of the found amplitude values of the additional functions into Eqs. (6). The following quantities are calculated at the points θ_r $(1\le r\le R)$:

$$D_M^r,\quad D_N^r,\quad \frac{D_M^r}{p_0^r},\quad \frac{D_M^r}{p_0^rD_N^r},\quad \frac{1}{D_N^r},\quad \frac{1}{p_0^rD_N^r},\quad \frac{1}{D_M^r}.$$

Using the values obtained and the additional functions (4) expressed in terms of unknown functions, we get the equalities

$$\psi_1^1(s_k,\theta_r)=-D_M(s_k,\theta_r)\sum_{n=1}^N u_r^n(s_k)\lambda_n^2\cos\lambda_n\theta_r,$$

$$\psi_1^2(s_k,\theta_r)=-D_M(s_k,\theta_r)\sum_{n=1}^N u_z^n(s_k)\lambda_n^2\cos\lambda_n\theta_r,$$

$$\psi_1^3(s_k,\theta_r)=D_M(s_k,\theta_r)\sum_{n=1}^N v^n(s_k)\lambda_n\cos\lambda_n\theta_r,$$

$$\psi_1^4(s_k,\theta_r)=D_M(s_k,\theta_r)\sum_{n=1}^N \vartheta_s^n(s_k)\cos\lambda_n\theta_r. \quad (8)$$

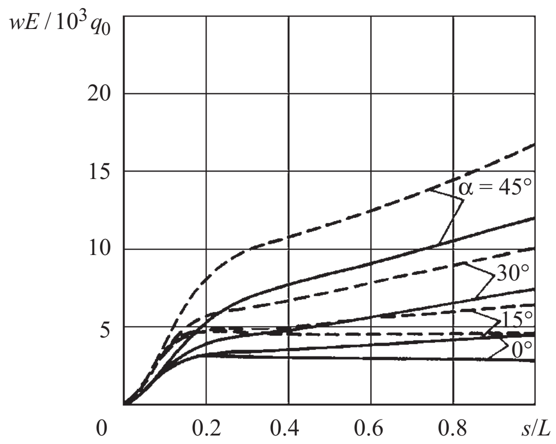

Fig. 1

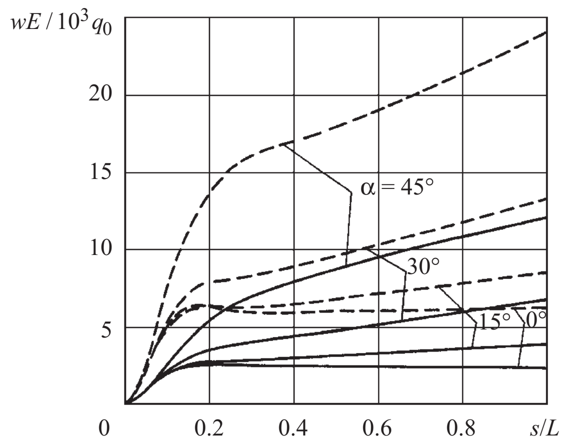

Fig. 2

The values of all the additional functions are calculated in much the same way.

Using the found values of the additional functions, we construct discrete Fourier series in the form (5) whose coefficients are the missing amplitude values of the additional functions at $s = s_k$ for Eqs. (6). Their values can be found in a standard way [5, 11]. After substitution of these values into Eqs. (6), we continue the integration over s, proceeding from the point s_k to the point s_{k+1}. The boundary-value problem (6), (7) is solved by the stable discrete-orthogonalization method, which produces a highly accurate solution.

The values of the unknown functions can be used to determine all the stress/strain characteristics of the shell.

Following this approach, we will analyze the displacement and stress fields in conical shells with thickness varying in two coordinate directions at constant mass. Let the thickness vary according to the formula

$$h(s,\theta) = h_0\left[1-\gamma\left(\frac{2s}{L}-1\right)^2\right]\cdot(1+\beta\cos\theta) \qquad (0 \le s \le L,\ 0 \le \theta \le 2\pi). \tag{9}$$

We will show that the thickness of the shell $\bar{h} = \int_0^L \int_0^{2\pi} h(s,\theta)\,ds\,d\theta$ is independent of γ and β.

Since s does not appear in the expression $1+\beta\cos\theta$, we can use the following expression when h changes with s:

$$h(s) = h_0\left[1-\gamma\left(\frac{2s}{L}-1\right)^2\right]. \tag{10}$$

If the shell has constant thickness, we assume that $h = H = \text{const}$. Then, requiring that the mass of the shell be independent of γ, we get

$$\int_0^L h(s)\,ds = \int_0^L H\,ds, \tag{11}$$

whence

$$h_0\left(1-\frac{\gamma}{3}\right) = H \quad \text{or} \quad h_0 = \frac{H}{1-\gamma/3}, \tag{12}$$

and

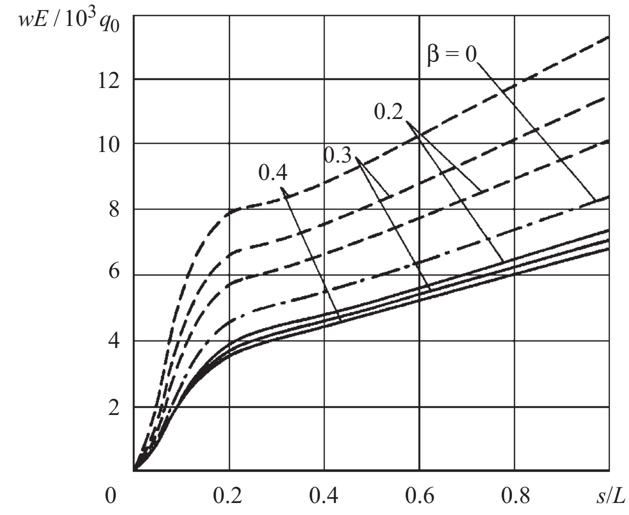

Fig. 3

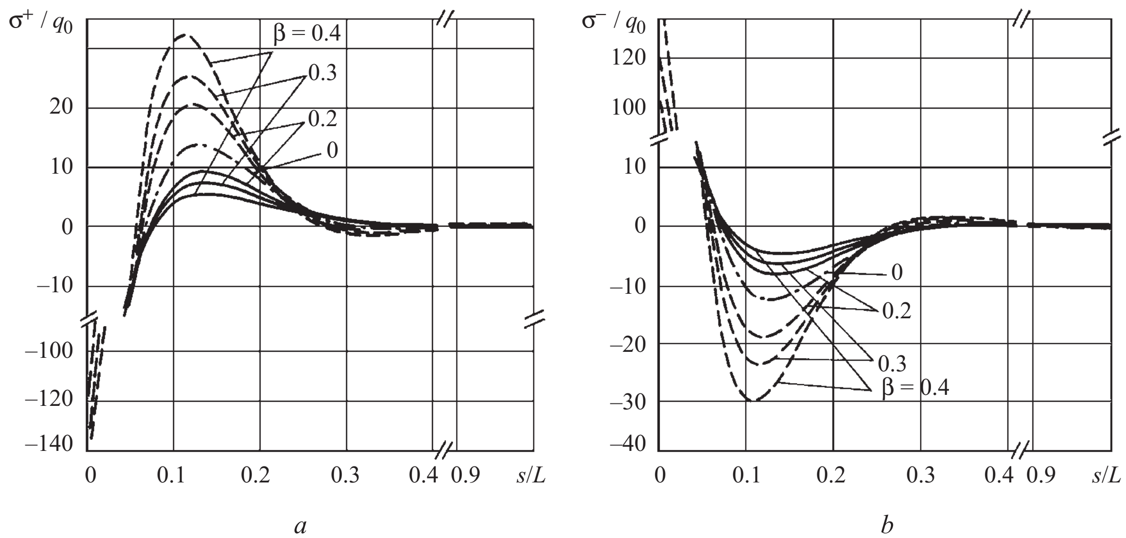

Fig. 4

$$\int_0^{2\pi} (1+\beta\cos\theta)d\theta = 2\pi. \tag{13}$$

Formulas (11)–(13) allow us to conclude that the thickness of the shell remains constant while γ and β change, i.e., the requirement of constant mass is met.

Let us first solve the stress–strain problem for a conical shell with circumferentially varying thickness, i.e., $\gamma = 0$ in (9). The shell is rigidly fixed at $s = 0$, is free from loads at $s = L$, and is subjected to a normal load $q = q_0 = \text{const}$. The problem has been solved for $L = 30$; $H = 0.25$; $\beta = 0.2, 0.4$; mean radius $R = 30$; $\alpha = 90°$; and $\varphi = 0°, 15°, 30°, 45°$.

Figure 1 shows the longitudinal distribution of deflection for $\beta = 0.2$ depending on the apex angle α (the solid lines correspond to $\theta = 0$, and the dashed lines to $\theta = \pi$). Similar curves for $\beta = 0.4$ are shown in Fig. 2. It can be seen from Fig. 1 that when the thickness varies along the circumference at constant mass and $\beta = 0.2$, the ratios of the deflections (at $s = L$) for $\theta = 0$ and $\theta = \pi$ are: 1.53, 1.43, 1.37, and 1.33 for $\alpha = 0°, 15°, 30°, 45°$, respectively. When $\beta = 0.4$, we have 2.7, 2.2, 1.9, and 2.0, according to Fig. 2. Hence, the ratio of the deflections at $\theta = 0$ and $\theta = \pi$ increases with β, the mass of the shell remaining constant.

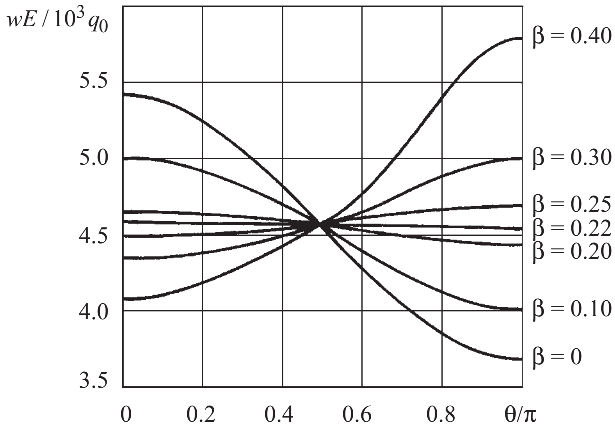

Fig. 5

TABLE 1

s/L	θ/π	$wE/10^3 q_0$								
		$\gamma = 0$			$\gamma = 0.4$			$\gamma = 0.4$		
		$\beta = 0$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0$	$\beta = 0.2$	$\beta = 0.4$
0.2	0	3.518	2.963	2.594	3.659	3.112	2.883	3.412	2.876	2.556
0.4		4.196	3.604	3.132	3.706	3.215	3.002	4.668	4.013	3.529
0.6		5.047	4.318	3.743	4.458	3.842	3.487	5.608	4.796	4.191
0.8		5.442	4.557	4.078	5.800	4.871	4.399	5.196	4.350	3.926
0.2	0.5	3.518	3.510	3.484	3.659	3.641	3.564	3.412	3.407	3.386
0.4		4.196	4.189	4.168	3.706	3.694	3.640	4.668	4.663	4.637
0.6		5.047	5.038	5.009	4.458	4.446	4.400	5.608	5.599	5.571
0.8		5.442	5.458	5.511	5.800	5.806	5.825	5.196	5.217	5.287
0.2	1.0	3.518	4.433	6.081	3.659	4.548	5.855	3.412	4.309	5.877
0.4		4.196	5.112	6.783	3.706	4.491	5.709	4.668	5.677	7.432
0.6		5.047	6.116	7.932	4.458	5.376	6.786	5.608	6.797	8.753
0.8		5.442	6.958	9.597	5.800	7.344	9.865	5.196	6.672	9.260

TABLE 2

s/L	θ/π	σ+								
		γ = 0			γ = 0.4			γ = 0.4		
		β = 0	β = 0.2	β = 0.4	β = 0	β = 0.2	β = 0.4	β = 0	β = 0.2	β = 0.4
0	0	–64.20	–47.49	–32.33	–87.33	–65.00	–49.31	–54.00	–40.17	–27.99
0.2		6.35	4.66	2.46	4.76	4.01	3.54	6.14	4.25	2.33
0.4		–0.72	–0.48	–0.53	–1.50	–1.09	–0.07	–0.06	0.11	0.08
0.6		–0.87	–0.08	0.08	–2.51	–1.42	–0.36	0.47	1.03	0.95
0.8		15.69	11.44	7.13	13.39	10.28	6.63	15.05	10.61	6.58
1.0		–82.28	–60.33	–41.68	–113.0	–82.71	–58.37	–68.73	–50.57	–35.18
0	0.5	–64.20	–62.78	–58.08	–87.33	–85.34	–78.33	–54.00	–52.80	–48.70
0.2		6.35	5.95	4.81	4.76	4.42	3.24	6.14	5.74	4.47
0.4		–0.72	–0.74	–0.80	–1.50	–1.53	–1.72	–0.06	–0.09	–0.22
0.6		–0.87	–0.79	–0.53	–2.51	–2.40	–2.12	0.47	0.53	0.69
0.8		15.69	15.08	13.13	13.39	12.83	11.11	15.05	14.46	12.60
1.0		–82.28	–80.86	–76.01	–113.0	–111.0	–104.5	–68.73	–67.54	–63.51
0	1.0	–64.20	–87.40	–126.3	–87.33	–118.8	–165.9	–54.00	–73.07	–103.9
0.2		6.35	7.55	7.63	4.76	4.61	1.88	6.14	7.88	8.68
0.4		–0.72	–0.69	0.00	–1.50	–1.49	–1.10	–0.06	–0.04	0.49
0.6		–0.87	–1.90	–2.41	–2.51	–3.57	–4.03	0.47	–0.46	–1.16
0.8		15.69	19.64	21.85	13.39	15.23	13.75	15.05	19.84	23.95
1.0		–82.28	–112.3	–159.9	–113.0	–155.0	–220.6	–68.73	–93.50	–132.3

Figure 3 demonstrates how the circumferential variation of the thickness affects the longitudinal distribution of the deflection (the solid lines correspond to $\theta = 0$, and the dashed lines to $\theta = \pi$). When $\theta = 0$, the deflection at the end $s = L$ has the following values depending on β: 1.22; 1.38; and 1.60 (divided by the deflection value at $\beta = 0$). Hence, it is possible to influence the deflection by changing the thickness in the circumferential direction and not affecting the mass of the shell.

Figure 4*a*, *b* shows, for the same shell, the longitudinal distributions of stresses on the outside and inside surfaces ($\sigma^{\pm}$) for different values of β. It is seen that the stresses peak at the fixed end and then damp after a local extremum at some distance from this end.

TABLE 3

s/L	θ/π	σ^-								
		$\gamma = 0$			$\gamma = 0.4$			$\gamma = 0.4$		
		$\beta = 0$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0$	$\beta = 0.2$	$\beta = 0.4$
0	0	59.06	42.97	28.76	82.44	60.65	45.48	48.71	35.49	24.23
0.2		–5.84	–4.21	–2.19	–4.38	–3.63	–3.16	–5.64	–3.84	–2.07
0.4		0.66	0.43	0.47	1.36	0.98	0.06	0.06	–0.10	–0.07
0.6		0.80	0.07	–0.07	2.28	1.26	0.31	–0.44	–0.94	–0.85
0.8		–14.44	–10.35	–6.35	–12.33	–9.31	–5.91	–13.83	–9.59	–5.85
1.0		75.70	54.58	37.09	106.7	77.18	53.83	62.00	44.68	30.45
0	0.4	59.06	57.76	53.43	82.44	80.56	73.94	48.71	47.63	43.93
0.2		–5.84	–5.48	–4.42	–4.38	–4.07	–2.98	–5.64	–5.27	–4.11
0.4		0.66	0.68	0.73	1.36	1.39	1.57	0.06	0.08	0.20
0.6		0.80	0.72	0.49	2.28	2.18	1.93	–0.44	–0.49	–0.64
0.8		–14.44	–13.87	–12.08	–12.33	–11.82	–10.23	–13.83	–13.30	–11.58
1.0		75.70	74.39	69.93	106.7	104.8	98.64	62.00	60.93	57.29
0	1.0	59.06	81.76	120.1	82.44	113.4	160.2	48.71	67.29	97.75
0.2		–5.84	–7.06	–7.26	–4.38	–4.32	–1.79	–5.64	–7.36	–8.26
0.4		0.66	0.65	–0.00	1.36	1.38	1.04	0.061	0.04	–0.47
0.6		0.80	1.77	2.29	2.28	3.31	3.80	–0.44	0.44	1.11
0.8		–14.44	–18.37	–20.78	–12.33	–14.26	–13.09	–13.83	–18.54	–22.78
1.0		75.70	105.1	152.1	106.7	148.0	213.1	62.00	86.11	124.4

Figure 5 presents results for the following problem: for a shell with rigidly fixed ends under a varying normal load $q = q_0\,(1 + 0.1\cos\theta)$, determine a value of β such that the circumferential distribution of deflection in the cross section $s = L/2$ is most uniform. As is seen from the figure, such is the case when β = 0.22.

Tables 1–3 summarize the values of the deflections and stresses in a conical shell under a uniform normal load $q = q_0 = \text{const}$. These results have been obtained for different values of γ and β in the longitudinal and circumferential directions. The thickness of the shell varies in two directions, its mass is constant, and both of its ends are rigidly fixed.

Thus, from the results presented in Figs. 1–5 and Tables 1–3 it follows that the coefficients γ and β appearing in the law of variation in thickness (9) can be chosen so as to obtain the most rational distribution of deflections and stresses, with the mass of the shell remaining constant.

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