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STRESS ANALYSIS OF FLEXIBLE NONCIRCULAR CYLINDRICAL SHELLS WITH HINGED EDGES FOR DIFFERENT CRITICAL LOADS

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An analytic nonlinear boundary-value solution is found and used in analysis of the precritical and postcritical stress states of a flexible long cylindrical shell with variable curvature and hinged longitudinal edges under nonuniform loading

Keywords: noncircular cylindrical shell, geometrical nonlinearity, exact solution

Geometrically nonlinear stress problems for shells assume describing their behavior within both precritical and postcritical deformation ranges [2, 5, 8, 10–14], with particular emphasis on their behavior under loads of critical levels. Along with approximate solutions to this class of problems [1, 3, 6], of interest are exact analytic solutions [4, 7, 9], which would make it possible to analyze the entire deformation range.

In the present paper, we use the exact analytic solution of a nonlinear boundary-value problem to analyze the stress state of a noncircular long cylindrical shell with hinged longitudinal edges for different critical loads.

1. Let us examine the deformation of a noncircular infinite cylindrical shell of constant thickness for different curvatures of its cross section and different degrees of nonuniformity of the load. The original equations [2, 4, 7] include

$$D_N \frac{d^2 v}{dy^2} = D_N \left[-\frac{dw}{dy}\frac{d^2 w}{dy^2} + k\frac{dw}{dy} + \frac{dk}{dy} w \right], \tag{1}$$

$$D_M \frac{d^4 w}{dy^4} = D_N \left[\frac{dv}{dy} - kw + \frac{1}{2}\left(\frac{dw}{dy}\right)^2 \right] \left(\frac{d^2 w}{dy^2} + k \right) + q, \tag{2}$$

where v and w are the tangential and normal displacements; $k = k(y)$ is the curvature of the shell's cross section; $D_N = \frac{Eh}{1-\nu^2}$; $D_M = \frac{Eh^3}{12(1-\nu^2)}$; $h = \text{const}$ is the thickness of the shell; E is the elastic modulus; ν is Poisson's ratio; and the coordinate y is measured along the directrix of the mid-surface ($-b \le y \le b$).

The longitudinal edges of the shell are hinged. The boundary conditions along these edges are given by

$$w(\pm b) = 0, \qquad \left.\frac{d^2 w}{dy^2}\right|_{y=\pm b} = 0, \qquad v(\pm b) = 0. \tag{3}$$

The curvature and load in Eqs. (1) and (2) are defined by

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$$k=k_0(1+\gamma_0 y^2), \qquad q=q_0(1+\delta_0 y^2), \tag{4}$$

$\gamma_0=\mathrm{const}$, $\delta_0=\mathrm{const}$, $k_0=\mathrm{const}$ is the curvature of a circle when $\gamma_0=0$, i.e., $k_0=1/r_0$ (r_0 is its radius), $q_0=\mathrm{const}$. In Eq. (1), we can isolate the first integral and write it as

$$\frac{d}{dy}\left\{D_N\left[\frac{dv}{dy}-kw+\frac{1}{2}\left(\frac{dw}{dy}\right)^2\right]\right\}=0. \tag{5}$$

Hence,

$$D_N\left[\frac{dv}{dy}-kw+\frac{1}{2}\left(\frac{dw}{dy}\right)^2\right]=C=N_y, \tag{6}$$

where N_y is the force along the directrix.

With (6), Eq. (2) takes on the form

$$D_M\frac{d^4w}{dy^4}=N_y\left(k+\frac{d^2w}{dy^2}\right)+q \tag{7}$$

or

$$\frac{d^4w}{dy^4}-\frac{N_y}{D_M}\frac{d^2w}{dy^2}=\frac{N_y}{D_M}k+\frac{q}{D_M}. \tag{8}$$

By assuming that $N_y<0$ (compression) and denoting $-N_y/D_M=\lambda^2$, we reduce Eqs. (6) and (8) to

$$\frac{dv}{dy}-kw+\frac{1}{2}\left(\frac{dw}{dy}\right)^2=-\lambda^2\frac{D_M}{D_N}, \tag{9}$$

$$\frac{d^4w}{dy^4}+\lambda^2\frac{d^2w}{dy^2}=-\lambda^2k+\frac{q}{D_M}. \tag{10}$$

Thus, we arrive at a nonlinear boundary-value problem for the system of equations (9), (10) with the boundary conditions (3). In (3), one of the equalities $v(\pm b)=0$ may be replaced by

$$\Delta=\int_{-b}^{b}\frac{dv}{dy}dy=v(b)-v(-b)=0, \tag{11}$$

where $\Delta=0$ means that the edges $y=\pm b$ do not converge.

Let us introduce the following dimensionless quantities:

$$\eta=\frac{y}{b}\quad(-1\le\eta\le1),\quad w^*=\frac{w}{k_0b^2},\quad v^*=\frac{v}{k_0^2b^3},\quad \mu=b\lambda,\quad q^*=\frac{b^2}{k_0D_M}q_0,$$

$$k^*=\frac{4k_0b^2}{h}=\frac{4b^2}{r_0h},\quad \gamma=\gamma_0b^2,\quad \delta=\delta_0b^2. \tag{12}$$

Then Eqs. (9) and (10) take on the form

$$\frac{dv^*}{d\eta}-\varphi(\eta)w^*+\frac{1}{2}\left(\frac{dw^*}{d\eta}\right)^2=-\frac{4}{3}\frac{\mu^2}{(k^*)^2}, \tag{13}$$

$$\frac{d^4 w^*}{d\eta^4}+\mu^2\frac{d^2 w^*}{d\eta^2}=-\mu^2\varphi(\eta)+q^*\psi(\eta), \tag{14}$$

where

$$\varphi(\eta)=1+\gamma\eta^2,\quad \psi(\eta)=1+\delta\eta^2, \tag{15}$$

and the boundary conditions (3) and (11) become

$$w^*(\pm 1)=0,\quad \left.\frac{d^2 w^*}{d\eta^2}\right|_{\eta=\pm 1}=0,\quad v^*(1)=0,\quad \Delta^*=0, \tag{16}$$

where

$$\Delta^*=\int_{-1}^{1}\frac{dv^*}{d\eta}d\eta . \tag{17}$$

Equation (14) is a linear differential equation with a constant coefficient μ^2; therefore, w^* is expressed linearly in terms of q^*. However, $\mu^2=b^2\lambda^2=-b^2 N_y / D_M$, where N_y depends nonlinearly on q, which leads to nonlinear dependence of w^* on q^*. The exact solution of the nonlinear boundary-value problem (13), (14), (16) is obtained as two dependences. One follows from the solution of Eq. (14) as an explicit expression for the deflection

$$w^*=(\alpha+\beta)\frac{1}{\mu^2}\left(\frac{\cos\mu\eta}{\cos\mu}-1\right)+\frac{\alpha}{2}(\eta^2-1)+\frac{\beta}{12}(\eta^4-1), \tag{18}$$

where

$$\alpha=\left(1-\frac{2\delta}{\mu^2}\right)\frac{q^*}{\mu^2}-\left(1-\frac{2\gamma}{\mu^2}\right),\qquad \beta=\frac{\delta q^*}{\mu^2}-\gamma. \tag{19}$$

The other dependence as the following quadratic equation for the function q^* with the coefficients dependent on μ follows from Eq. (13) in view of (18) when the boundary condition $\Delta^*=0$ (16), (17) is satisfied:

$$\frac{a}{\mu^4}(q^*)^2+\frac{b}{\mu^2}q^*+c=0, \tag{20}$$

where

$$a=N_1\cos^2\mu+N_2\sin\mu\cdot\cos\mu-\frac{A^2}{2\mu^2}\sin^2\mu,$$

$$b=N_3\cos^2\mu+N_4\sin\mu\cdot\cos\mu+\frac{AB}{\mu^2}\sin^2\mu,$$

$$c=N_5\cos^2\mu-\frac{8}{3}\left(\frac{\mu}{k^*}\right)^2\cos^2\mu+N_6\sin\mu\cdot\cos\mu-\frac{B^2}{2\mu^2}\sin^2\mu, \tag{21}$$

$$N_1=-\frac{1}{3}\left(1+\frac{2}{5}\delta+\frac{\delta^2}{21}\right)-\frac{2}{\mu^2}\left(1+\frac{2}{3}\delta+\frac{1}{5}\delta^2\right)+\frac{12}{\mu^4}\left(\delta+\frac{2}{3}\delta^2\right)-\frac{16\delta^2}{\mu^6}-\frac{A^2}{2\mu^2},$$

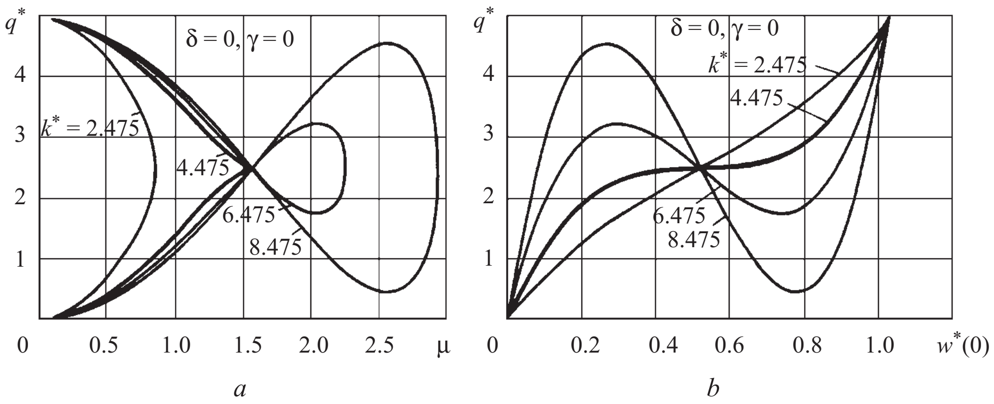


Fig. 1

$$N_2 = \frac{A}{\mu^3}\left(\frac{5}{2}A - \frac{4\delta}{\mu^2}\right),$$

$$N_3 = \frac{2}{\mu^2}\left(1+\frac{1}{3}\delta+\frac{1}{3}\gamma+\frac{1}{5}\delta\gamma\right)-\frac{8}{\mu^4}\left(\gamma+\delta+\frac{4}{3}\gamma\delta\right)+\frac{24\delta\gamma}{\mu^6}+\frac{AB}{\mu^2},$$

$$N_4 = \frac{1}{\mu^3}\left(\frac{4\gamma}{\mu^2}A - 3AB + \frac{4\delta}{\mu^2}B\right), \tag{22}$$

$$N_5 = \frac{1}{3}\left(1+\frac{2}{5}\gamma+\frac{\gamma^2}{21}\right)+\frac{4}{\mu^4}\left(\gamma+\frac{2}{3}\gamma^2\right)-\frac{8\gamma^2}{\mu^6}-\frac{B^2}{2\mu^2}, \quad N_6 = \frac{B}{\mu^3}\left(\frac{B}{2}-\frac{4\gamma}{\mu^2}\right),$$

$$A = 1 - \frac{2\delta}{\mu^2} + \delta, \quad B = 1 - \frac{2\gamma}{\mu^2} + \gamma. \tag{23}$$

Solving the quadratic equation (20), we obtain the relationship between μ^* and q^*. After that, fixing η and eliminating μ, we find the dimensionless deflection $w^*(q^*)$. Equation (20) has a solution $q^* = q^*(\mu)$ if the discriminant D is greater than or equal to zero. When $D = 0$, Eq. (20) has equal roots. As follows from Eq. (20), when $\mu = \pi/2$ the variable q^* is independent of the geometrical parameter k^*. This means that all the curves $q^*(\mu)$ intersect at one point (Fig. 1*a*). When $\mu = \pi/2$, Eq. (20) yields

$$16A^2(q^*)^2 - 8\pi^2 ABq^* + \pi^4 B^2 = 0. \tag{24}$$

This equation has multiple roots

$$q^*(\pi/2) = \frac{\pi^2 B(\gamma)}{4A(\delta)}. \tag{25}$$

Let us determine the value of the generalized geometrical parameter $\tilde{k}^*$ at which the curve $q^*(w^*)$ for $\mu = \pi/2$ and $q^* = \pi^2 B(\gamma)/(4A(\delta))$ does not have maxima and minima and has only an inflection point at which the tangent line to the curve $q^*(w^*)$ is parallel to the Ow^*-axis. This curve divides the domain $q^* \times w^*$ so that the curves $q^*(w^*)$ do not have critical points under it (up to the inflection point) and have critical points corresponding to snap buckling above it.

To that end, let us analyze the expression of the discriminant D equated to zero:

$$\left[N_3^2 - 4N_1N_5 + \frac{32}{3}\left(\frac{\mu}{k^*}\right)^2 N_1\right]\cos^2\mu + \left[N_3N_4 - 2N_1N_6 - 2N_2N_5 + \frac{16}{3}\left(\frac{\mu}{k^*}\right)^2 N_2\right]\cdot 2\cos\mu\sin\mu$$

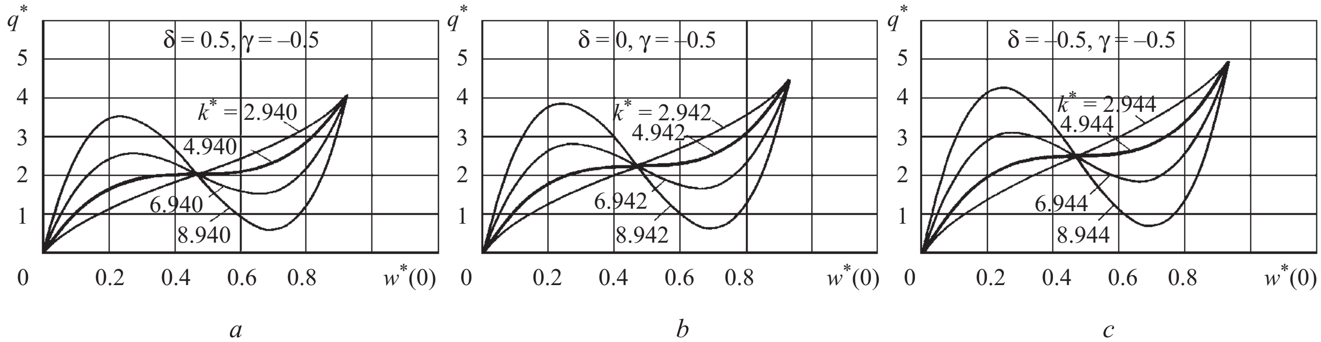


Fig. 2

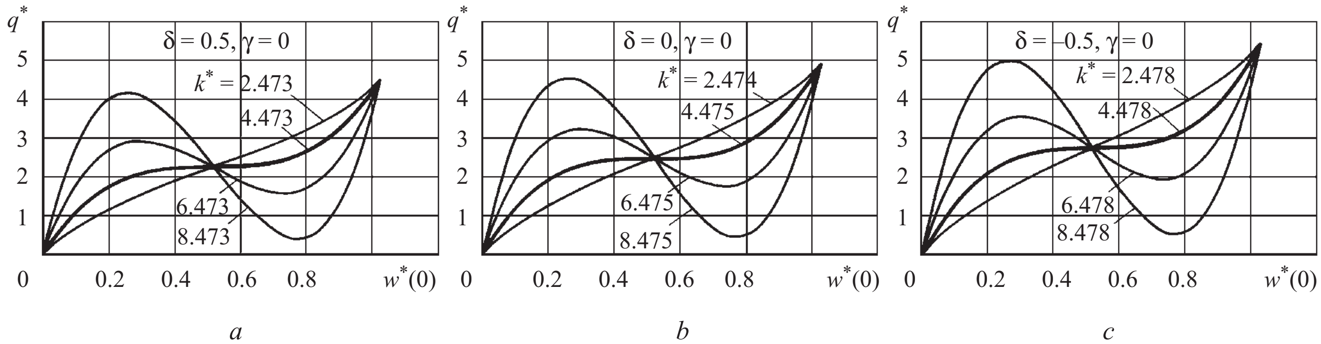


Fig. 3

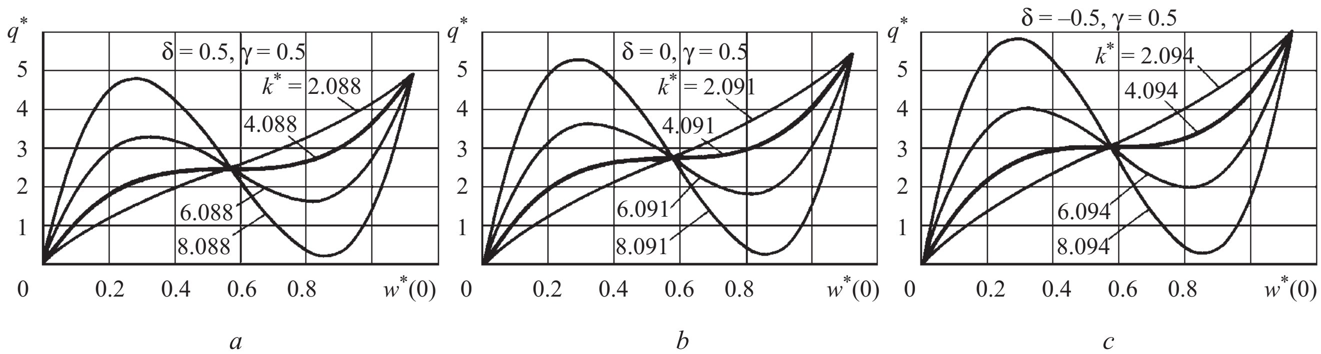


Fig. 4

$$+\left[N_4^2 - 4N_2N_6 + \frac{2}{\mu^2}(B^2N_1 + A^2N_5 + ABN_3) - \frac{16}{3}\frac{A^2}{(k^*)^2}\right]\sin^2\mu = 0. \tag{26}$$

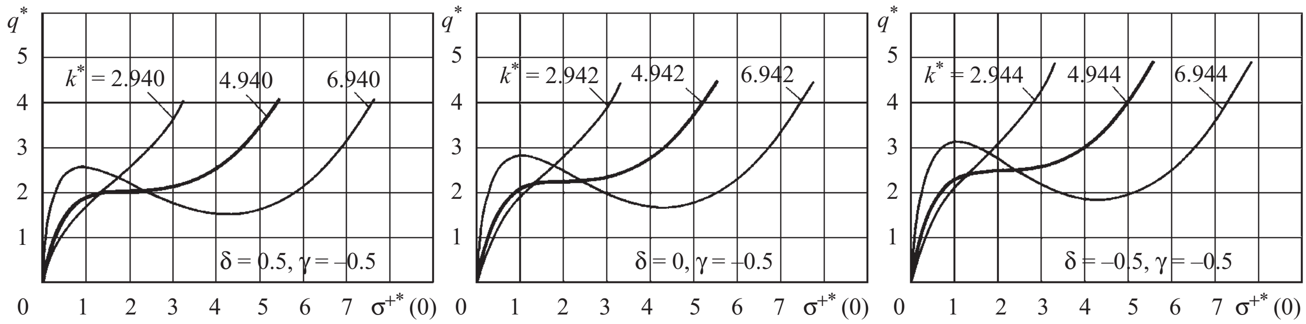
Assuming that $\mu = \pi/2$ in (26), we obtain the following equation for $\tilde{k}^*$:

$$24(A^2N_5 + B^2N_1 + ABN_3) + 3\pi^2(N_4^2 - 4N_2N_6) - \frac{16\pi^2A^2}{(\tilde{k}^*)^2} = 0, \tag{27}$$

whence

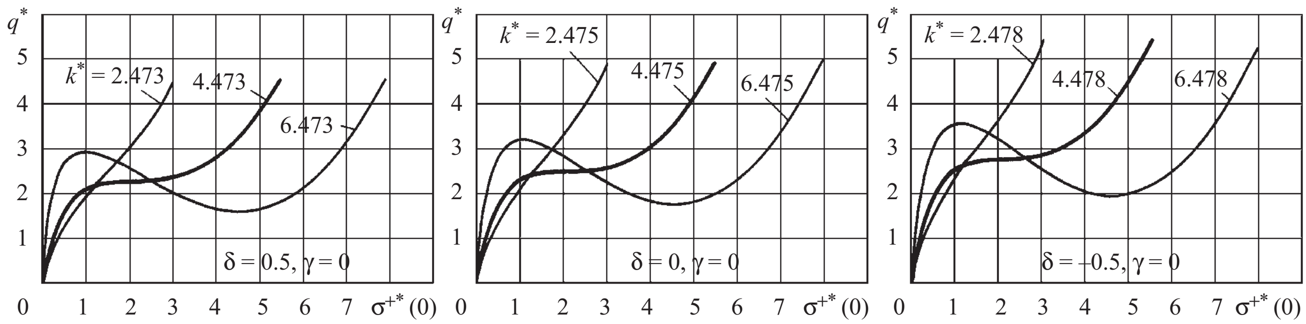
$$\tilde{k}^* = \frac{4\pi A}{\left[24(A^2N_5 + B^2N_1 + ABN_3) + 3\pi^2(N_4 - 4N_2N_6)\right]^{1/2}}. \tag{28}$$

We use formula (28) to determine $\tilde{k}^*$ for different values of γ and δ. After that, we plot graphs of the function $q^*(w^*)$, using formulas (18) and (20), for different values of $\tilde{k}^*$ and for a fixed value of η.



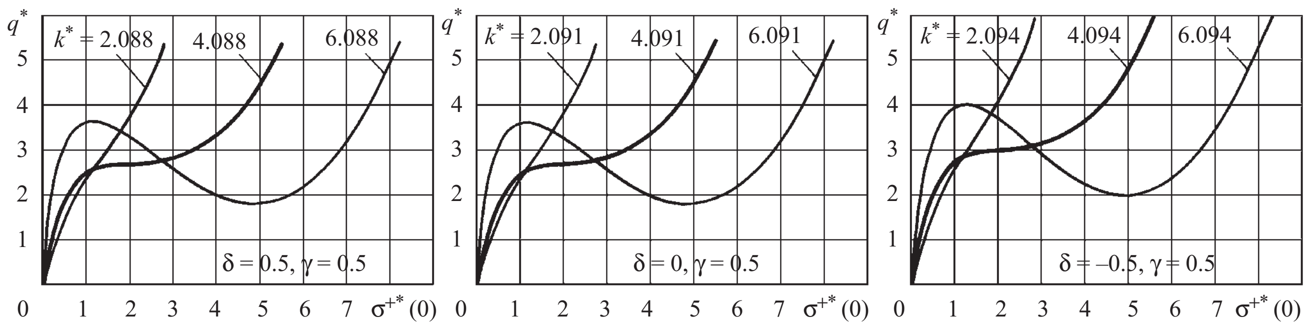
a *b* *c*

Fig. 5



a *b* *c*

Fig. 6



a *b* *c*

Fig. 7

2. Let us use the above solution to analyze the stress–strain state of noncircular cylindrical shells for different values of curvature and load.

We will illustrate the solution for a circular shell ($\gamma = 0$) under a uniform load ($\delta = 0$). The value of the parameter $\tilde{k}^* = 4.475$ is determined from formula (28). Figure 1*a* shows graphs of the function $q^*(\mu)$, the heavy curve corresponding to $\tilde{k}^*$. The values of k^* for the other curves for which $D \neq 0$ are indicated in the same figure. These curves have two branches that coincide when $\mu = \pi/2$ and take on the values $q^*(\pi/2) = \pi^2/4$. Figure 1*b* shows graphs of the function $q^*(w^*)$ for $\eta = 0$. The heavy line in this figure corresponds to the heavy line in Fig. 1*a* and has an inflection point when $q^* = \pi^2/4$ at which the tangent line to it is parallel to the Ow^*-axis. As is seen from Fig. 1, when $k^* < \tilde{k}^*$, $\tilde{k}^* = 2.475$, the curve $q^*(\mu)$ does not loop and the curve $q^*(w^*)$ does not have maxima and minima. Once $k^* > \tilde{k}^*$, the curves $q^*(\mu)$ loop, the curves $q^*(w^*)$ acquire upper and lower critical points, and the values of q^* at these points coincide with the extreme values of the curves $q^*(\mu)$ on the loop. Thus, the curve $q^*(w^*)$ corresponding to $\widetilde{k}^*$ divides the deformation range into two parts: $k^* < \widetilde{k}^*$ where there are no critical points and $k^* > \widetilde{k}^*$ where there are critical points. Figure 1*b* demonstrates how the upper and lower critical loads change with increase in the geometrical parameter k^*.

3. Let us use the solution obtained above to analyze the behavior of a flexible noncircular cylindrical shell over the entire deformation range under a nonuniform normal load for different values of k^*.

Figures 2–4 present graphs of the function $q^*(w^*)$ for different values of k^*, γ, and δ, the heavy line corresponding to $\widetilde{k}^*$. It is seen from Fig. 3 how the upper and lower critical loads of a circular shell $(\gamma = 0)$ change for $\eta = 0$, different values of k^*, and $\delta = 0.5, 0, -0.5$. As δ decreases, the value of $\widetilde{k}^*$, as well as the upper and lower loads, increases. Figure 2 shows that the upper values of q^* increase, compared with Fig. 3, with decrease in γ (shells that are more shallow) for the same values of δ as in Fig. 3.

Figure 4 demonstrates how the behavior of the curves $q^*(w^*)$ changes when γ increases for the same values of δ as in Fig. 3, i.e., for deeper shells. The upper and lower critical values of q^* increase compared with Fig. 3.

Figures 5–7 show graphs of the functions $q^* = q^*(\sigma^{+*})$ for $\eta = 0$, where σ^{+*} is the stress on the outside surface of the shell. Figure 6 presents curves for a circular shell for different values of δ. It is seen from Fig. 6 that the maximum critical loads increase with decrease in δ. In contrast to Fig. 6, the curves in Figs. 5 and 7 illustrate the behavior of the stresses σ^{+*} with decrease and increase in γ, i.e., with variation in curvature for the same values of δ.

Thus, exact analytic solutions of a nonlinear boundary-value problem make it possible to analyze the behavior of a flexible cylindrical shell in the precritical and postcritical deformation ranges during variation in the curvature of the cross section and in the degree of nonuniformity of the load.

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