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DYNAMICS OF ELLIPSOIDAL CAVITIES IN FLUID

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A mathematical model of the hydrodynamics of free closed surfaces in a fluid is expounded. It is used for studying the dynamics of ellipsoidal cavities during their development. The model is based on a system of differential equations that accounts for the influence exerted on the dynamics of cavities by various perturbations such as gravity, surface tension, viscosity, and geometrical features of the cavity. Solving this system makes it possible to determine the hydrodynamic characteristics of the flow around the cavity and to plot cavity shapes depending on time and flow regimes. Characteristic features of the development of such cavities under gravity and surface tension are established

Keywords: hydrodynamics, cavity, gravity, surface tension, viscosity, hydrodynamic characteristics, development of cavity

Introduction. The behavior of cavities in fluid is known to depend on its properties and the cavitation conditions. Among the conditions strongly affecting the behavior of cavities are primarily gravity (which gives rise to the buoyancy force), surface tension, and viscosity. If the influence of these factors was insignificant, then we could design a mathematical model that would make it possible to determine all the basic dynamic characteristics of the cavity and its shape depending on flow regime and time. In reality, however, this can be done only within the framework of a linear or linearized theory and the superposition principle; therefore, the greater the effect of these forces, the less reliable the results. When these forces are strong, the mathematical model of the process is rather involved and not easy to implement, and the cavity no longer has well-defined boundaries. Therefore, even if a linearized model may not be used for a flow highly disturbed by these forces, it can still be applicable to a moderately disturbed flow.

The behavior of a spherical cavity is well understood. There are a number of mathematical models [3, 11–13, etc.] that account, to some extent, for the forces mentioned above. The mathematical model of such a cavity includes nothing but the density of the fluid and the difference of the pressures inside and outside (far from) the cavity. If, however, the cavity is not spherical, then its dynamic analysis involves considerable difficulties [6, 8]. This is why the search is still under way for simpler mathematical models that are capable of describing the deformation of nonspherical cavities with adequate accuracy and simplicity [2, 5, 7]. One is a linearized model used in acoustics [9], particle dynamics [10], surface impact theory [12], and other problems. A version of the linearized theory, which was repeatedly used and tested against hydrodynamic problems for flows with free boundaries, was proposed in [1]. It is based on using a system of differential equations for the deformation modes of the cavity and determining the initial perturbations of the cavity shape.

1. Problem Formulation. Mathematical Model. First, note that the model is based on the hydrodynamics of thin axisymmetric bodies and spherical cavities in a perfect fluid. Let a free closed surface at time zero be a nonspherical surface of revolution. The general equation of a disturbed surface S reads $F(t, r, \vartheta) = 0$. The mathematical model of the potential flow around the cavity S can be expressed as

$$\nabla^2\Phi = 0 \quad \text{outside} \quad S,$$

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$$\frac{\partial F}{\partial t} + \nabla\Phi\nabla F = 0 \quad \text{on} \quad S,$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + \frac{\tilde{p} - p_\infty}{\rho} = 0 \quad \text{on} \quad S, \tag{1}$$

where $\Phi(t, r, \vartheta)$ is the velocity potential; $\tilde{p}$ is the generalized pressure on the cavity surface; p_∞ is the pressure at infinity; and ρ is the density of the fluid. It is expedient to place the origin of coordinates at the center of the cavity.

Thus, in this model it is necessary to find a solution of the Laplace equation for the potential Φ that would be valid outside the surface S and would satisfy the kinematic no-flow condition (the second equation in (1)) and the dynamic condition (in the form of the Lagrange–Cauchy integral) that the pressures are equal on the cavity surface S (the third equation in (1)). Moreover, the condition at infinity should also be met. Note that the kinematic and dynamic conditions have been written in a fixed coordinate system. If the coordinate system moves with a velocity $\vec{c}$, then $\nabla\Phi$ should be replaced by $\nabla\Phi - \vec{c}$ in the kinematic condition and $\frac{\partial \Phi}{\partial t}$ by $\frac{\partial \Phi}{\partial y} - \vec{c}\,\nabla\Phi$ in the dynamic condition.

If the free boundary kept its shape, then the problem could be solved quite easily. However, after moving to a region of lower pressure, the free boundary takes a specific shape ensuring minimum strain energy. This means that near the cavity a new flow with a potential φ, which will be called a perturbation, is superimposed on the main flow with a potential Φ_0. The next task is to determine the potential of the new flow.

If the velocity potential Φ_0 and the radius R_0 of the cavity (sphere) in these flows are known, then the velocity potentials Φ and the shapes (radii R) of cavities in disturbed flows can be sought in the form

$$\Phi = \Phi_0 + \varphi, \qquad R = R_0 + f, \tag{2}$$

as suggested by small-perturbation theory.

In this case, the problem arises of satisfying the conditions (1) on the disturbed surface S, which is still unknown. This could be done by linearizing all the equations in (1). Expanding the unknowns into Taylor series about the nonperturbed surface and neglecting the nonlinear terms, we reduce the problem (1) to the following problem for the perturbations themselves:

$$\nabla^2\varphi = 0,$$

$$-\frac{\partial f}{\partial t} = \left(\nabla\Phi_0 - \vec{c}\right)\nabla f - \nabla\varphi\nabla F_0 - f\frac{\partial\left(\nabla\Phi_0 - \vec{c}\right)\nabla F_0}{\partial n},$$

$$\frac{\partial\varphi}{\partial t} + \left(\nabla\Phi_0 - c\right)\nabla\varphi + f\left[\frac{\partial^2\Phi_0}{\partial n\partial t} + \left(\nabla\Phi_0 - c\right)\frac{\partial\nabla\Phi_0}{\partial n}\right] = -\frac{\partial\nabla\Phi_0}{\partial t} + c\nabla\Phi_0 - \frac{1}{2}(\nabla\Phi_0)^2 - \frac{\tilde{p}_k - p_\infty}{\rho}. \tag{3}$$

Note that $\vec{c}$ is the vector of velocity with which the coordinate system moves under the action of, for example, the buoyancy force, and $F_0(x, y, z, t) = 0$ is the equation of nondisturbed surface of the cavity. For flows not very different from spherically symmetric, the generalized pressure can be found, according to [4], by the formula $\tilde{p} = p_i + p_\tau + p_g + p_\mu$, where $p_g = \rho g Z$, $p_\tau = -2\tau H$, $p_\mu = -4\mu\dot{R}/R$, and τ and μ are the coefficients of surface tension and viscosity, respectively. In this case, the perturbations φ and f may be due to various physical or geometrical factors. These may include anything that makes a specific mathematical model different from the nondisturbed flow model (Φ_0, R_0). For example, among these factors are gravity, surface tension, compressibility and viscosity of the fluid, local pressure, and various geometrical features of flow (the shape and orientation of the nondisturbed cavity, asymmetry, finiteness, etc.). The velocity $\vec{c} = \{u\cos\vartheta, -u\sin\vartheta, 0\}$ of the cavity moving with no change of shape is expressed in terms of its vertical velocity u.

We will consider that a flow around a sphere is nondisturbed. The potential of the flow is given by

$$\Phi_0 = -\frac{R_0^2\dot{R}_0}{r} - \frac{uR_0^2}{2r^2}\cos\vartheta.$$

Here the first term describes a spherically symmetric flow; and the second, a flow caused by the upward motion of the cavity with the velocity u and with no change of shape.

The perturbation potential φ as a damped-at-infinity solution of the Laplace equation and the perturbation $f(t,\vartheta)$ can be written in terms of Legendre polynomials $P_n(\mu)$ as

$$\varphi=\sum_{n=0}^{\infty}\frac{a_n(t)}{r^{n+1}}P_n(\mu),\qquad f=\sum_{n=0}^{\infty}f_n(t)P_n(\mu)\qquad (\mu=\cos\vartheta). \tag{4}$$

Then the second equation in (3) yields relations between potential modes and deformation modes:

$$\frac{a_0}{R_0^2}=-\dot f_0-\frac{2f_0}{R_0}\dot R_0,\qquad \frac{2a_1}{3R_0^2}=-\left(f_0-\frac{f_2}{5}\right)u,$$

$$\frac{3a_2}{3R_0^4}=-\dot f_2-\frac{2f_2}{R_0}\dot R_0+\frac{9f_3}{7R_0}u,$$

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$$\frac{(n+1)a_n}{3R_0^{n+2}}=-\dot f_n-\frac{2f_n}{R_0}\dot R_0-\frac{3n(n+1)}{2R_0}\left(\frac{f_{n-1}}{2n-1}-\frac{f_{n+1}}{2n+1}\right)u,\quad n\ge 3. \tag{5}$$

Next, it is necessary to satisfy the dynamic condition in (3) to obtain a final system of differential equations for deformation modes $f_n(t)$ [2]:

$$R_0\ddot R_0+\frac{3}{2}\dot R_0^2=\sqrt{\frac{\Delta p}{\rho}}\,, \tag{6}$$

$$-R_0\ddot f_0-3\dot R_0\dot f_0-\ddot R_0 f_0+\frac{u^2}{4R_0}=-\frac{8H_0}{We}+\frac{Z_0}{2Fr^2}-\frac{8}{R_0\,\mathrm{Re}}\left(\frac{\dot R_0}{V_0}\right),$$

$$-\frac{\dot u}{2}\left(R_0+f_0-\frac{7}{5}f_2\right)-\frac{u}{2}\left(\dot R_0+\dot f_0-\frac{3}{5}\dot f_2-\frac{4\dot R_0}{5R_0}f_2\right)+\frac{27u^2}{70R_0}f_3=\tilde\sigma_1,$$

$$-\frac{R_0\ddot f_n+3\dot R_0\dot f_n}{n+1}+\left[\frac{n-1}{n+1}\ddot R_0+\frac{9nu^2}{2R_0}\,\frac{n^2(2n+1)-2(n-1)}{(2n-1)(2n+1)(2n+3)}\right]f_n$$

$$-\frac{3u}{2}\left(\dot f_{n-1}-\frac{2n+1}{2n+3}\dot f_{n+1}\right)-\left[\frac{n\dot u}{2}+(n-1)\frac{3u\dot R_0}{R_0}\right]\frac{f_{n-1}}{2n-1}+\left[\frac{5n+2}{2}\dot u+(n+1)\frac{3u\dot R_0}{R_0}\right]\frac{f_{n+1}}{2n+3}$$

$$+\delta_{n2}\left[\frac{3u^2}{4}+\left(\frac{\dot u}{2}+\frac{u\dot R_0}{R_0}\right)f_1\right]-\frac{9nu^2}{4R_0}\left[\frac{(n-1)(n-2)f_{n-2}}{(2n-1)(2n-3)}+\frac{(n+1)(n+2)f_{n+2}}{(2n+3)(2n+5)}\right]+\frac{9u^2f_1}{10R_0}\delta_{n3}=\tilde\sigma_n\qquad (n\ge 2), \tag{7}$$

where $\tilde\sigma_n=\sigma_n+\dfrac{8H_n}{We}-\dfrac{Z_n}{2Fr^2}$, δ_{nm} is the Kronecker delta; $We=\dfrac{\rho V_0^2 d_{\mathrm{in}\,0}}{\tau}$ and $Fr=\dfrac{V_0}{\sqrt{gd_{\mathrm{in}\,0}}}$ are the Weber number (τ is the surface tension coefficient) and the Froude number, where $R_{\mathrm{in}\,0}=\sqrt[3]{a^2c}$ is the diameter of an equal-area sphere at the initial time. Note that all the variables appearing in the equations are made nondimensional by dividing them by $R_{\mathrm{in}\,0}$ or $V_0=\sqrt{\Delta p/\rho}$. If $\tilde\sigma_n$, H_n, and Z_n are not constant, then these are the coefficients in the Legendre expansion series of the differential pressure, mean curvature, and the vertical distance from a point on the cavity boundary to the horizontal plane. It should also be noted that V_0^2 is not a physical velocity—it just has dimensions of velocity because it is given by the formula $V_0^2 = \Delta p / \rho$, where Δp is the difference of the pressures inside and outside the cavity.

2. Solution of Nonlinear Problems. Initial and Boundary Conditions. The system of equations (7) is quasilinear, since the higher derivative enters linearly into the equations, and the nonlinearity is due to the fact that the velocity u of the cavity is also an unknown function, which is to be determined. This nonlinearity is principal; it reflects the interaction of deformation modes. In linear theory, deformation modes are always independent. An analysis of Eqs. (7) shows that the interaction of the modes $n-1$ and $n+1$ is most significant, though the modes $n-2$ and $n+2$ have some effect too.

Solving the system of equations (7) and using formulas (4), we determine the potential of the disturbed flow and then the potential of the main flow:

$$\Phi = -\frac{\dot{R}_0 R_0^2}{r} + \sum_{n=0}^{\infty} \frac{a_n(t)}{r^{n+1}} P_n(\mu), \tag{8}$$

all characteristics of this flow, and the shape of the deformed cavity

$$R(t, \vartheta) = R_0(t) + \sum_{n=0}^{\infty} f_n(t) P_n(\mu). \tag{9}$$

However, to integrate the system (7), we need initial conditions, in addition to the initial shape of the cavity and the initial velocities of its change. These initial conditions can be determined considering an ellipsoid as a deformed sphere. To this end, we write the equation of an ellipsoid of revolution $\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$ using spherical coordinates $x = r\cos\varphi\cos\vartheta$, $y = r\sin\varphi\cos\vartheta$, and $z = r\sin\vartheta$:

$$r(\vartheta) = \frac{ac}{\sqrt{c^2\cos^2\vartheta + a^2\sin^2\vartheta}} = \frac{c}{\sqrt{1 - e^2\cos^2\vartheta}}, \tag{10}$$

where e is the eccentricity of the cross section of the ellipsoid; the squared eccentricity is given by the formula $e^2 = 1 - c^2/a^2$ ($c < a$ corresponds to a horizontally oblong ellipsoid). Note that the angle ϑ is measured from the major axis counterclockwise. In the case of a vertically oblong ellipsoid ($c > a$), we may use another coordinate system in which the angle ϑ is measured from the Oz-axis clockwise. Then the squared eccentricity in formula (10) will be defined by the formula $\varepsilon^2 = 1 - a^2/c^2$, and the numerator in (10) will include the factor a instead of c. Note that the orientation of the ellipsoid is important only in the presence of gravity.

With a small eccentricity, the expression under the radical sign in (10) can be expanded into a power series:

$$r = c\sum_{0}^{\infty} \frac{(2n-1)!!}{(2n)!!} e^{2n} \mu^{2n},$$

where the function $\mu^{2n} = (\cos\vartheta)^{2n}$ is also expanded into a Legendre series:

$$\mu^{2n} = \frac{1}{2n+1} P_0(\mu) + \sum_{k=1}^{\infty} (4k+1) \frac{2n(2n-2)\cdots(2n-2k+2)}{(2n+1)(2n+3)\cdots(2n+2k+1)} P_{2k}(\mu).$$

Then the function $r(\vartheta)$ is also represented as a Legendre series:

$$r = r(\vartheta) = r_0 + \sum_{n=1}^{\infty} r_{2n} P_{2n}(\mu).$$

To determine the coefficients of this expansion, we multiply this equality by $P_m(\mu)$,

$$r(\vartheta)P_m(\mu)=\sum_{n=0}^{\infty} r_n P_n(\mu)P_m(\mu)$$

and integrate the new equality over the interval [–1, 1] to obtain

$$\int_{-1}^{1} r(\vartheta)P_m(\mu)d\mu=\sum_{n=0}^{\infty} r_n \int_{-1}^{1} P_n(\mu)P_m(\mu)d\mu.$$

We will now use the fact that Legendre polynomials are orthogonal on this interval:

$$\int_{-1}^{1} P_n(\mu)P_m(\mu)d\mu=\begin{cases}\dfrac{2}{2n+1}, & m=n,\\ 0, & m\neq n.\end{cases}$$

This leads to the following formulas for the expansion coefficients:

$$r_n=\frac{2n+1}{2}\int_{-1}^{1} r(\mu)P_n(\mu)d\mu=\frac{(2n+1)a}{2}\int_{-1}^{1}\frac{P_n(\mu)d\mu}{\sqrt{1-e^2\mu^2}}.$$

Here are several terms of this expansion:

$$r_0=R_{\text{in}0}=c\left(1+\frac{e^2}{6}+\frac{3e^4}{40}+\frac{5e^6}{112}+\frac{35e^8}{1152}+\ldots\right),\qquad r_2=\frac{ce^2}{3}\left(1+\frac{9e^2}{14}+\frac{25e^4}{56}+\frac{4725e^6}{18304}+\ldots\right),$$

$$r_4=\frac{3ce^4}{35}\left(1+\frac{175}{154}e^2+\frac{1225}{1144}e^4+\frac{1505}{2288}e^6+\ldots\right),\qquad r_6=\frac{5ce^6}{231}\left(1+\frac{539}{330}e^2+2e^2+\ldots\right),$$

which are power series in the squared eccentricity.

Since all the terms in these series are positive, the truncated series used to calculate the coefficients r_n underestimate their values. Therefore, for example, the coefficient r_0 is expedient to calculate as the radius of a sphere equivalent to an ellipsoid: $r_0=R_{\text{in}0}=\sqrt[3]{a^2c}$. The coefficients r_n $(n=1, 2, 3, \ldots)$ can be considered the initial perturbations of a spherical cavity of radius $R_{\text{in}\,0}$. Therefore, the system of equations (7) should be supplemented with the initial conditions

$$f_n(0)=r_n,\qquad \dot{f}_n(0)=0. \tag{11}$$

The second equality in (11) means that the initial velocities are equal to zero (they can be specified in any other way).

Thus, the system of equations (7) and the initial conditions (11) completely define a Cauchy problem for the system. There is an obstacle here, however: the equations include the function $R_0(t)$ that is a solution of the Rayleigh equation (6). This equation is known not to have an analytic solution. Therefore, it is necessary either to incorporate the Rayleigh equation into system (7) or to use an approximate expression for this function:

$$R_0(t)=\sqrt{0.84t^2+0.11t+1}\quad (0<t<3),\qquad R_0(t)=0.82t+0.4\quad (t>3).$$

The exact expressions for the derivatives of this function are well-known:

$$\dot{R}_0(t)=\sqrt{\frac{2}{3}\left(1-\frac{1}{R_0^3}\right)},\qquad \ddot{R}_0(t)=\frac{1}{R_0^4}.$$

Note that all variables here are relative:

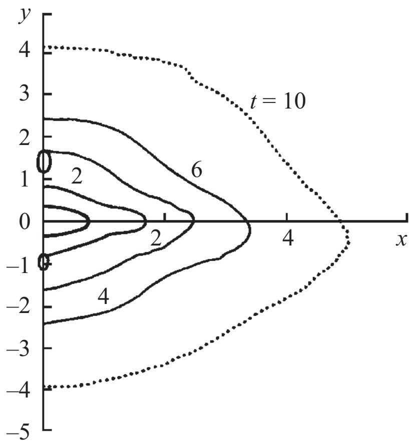

Fig. 1

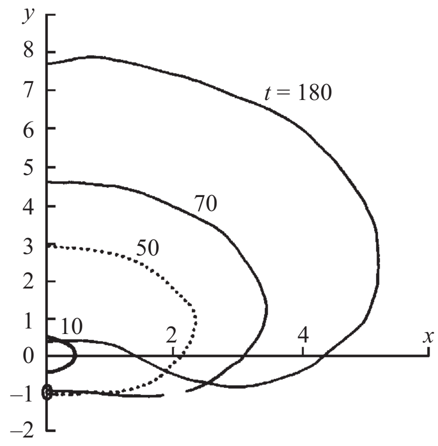

Fig. 2

$$R_0(t)=\frac{R_0^*(t)}{R_{\text{in}\,0}^*}, \qquad t=\frac{V_0^* t^*}{R_{\text{in}\,0}^*}, \qquad V_0^*=\sqrt{\frac{\Delta p}{\rho}}.$$

Now the problem can be solved by using the Runge-Kutta method (or other numerical methods) and printing out plots of the meridional section of the cavity at predefined time points.

3. Numerical Results for Ellipsoidal Cavities. Some of the results obtained are shapes of the cavity at fixed times.

Figure 1 shows several meridional sections of an initially ellipsoidal cavity with semiaxes $a=b=3$ and $c=1$ at small time intervals ($t=2, 4, 6, 10$). The major axis of the ellipsoid is perpendicular to the direction of gravity. Flow occurs when $Fr=4$ and $We=10^3$. The heavy line is half the outline of the meridional section of the initially ellipsoidal cavity, and the other lines represent only half the meridian section of the cavity, since it is symmetric. Though the initial ellipsoid is quite oblong, its section gradually becomes circular. Note also that at the initial stage of expansion, the effect of gravity on the cavity is hardly noticeable. The curve for t = 6 is still close to elliptical (though its aspect ratio is already $c/a=1.4$), whereas the curve for t = 10 is already very close to circular (though it is still an ellipse, its aspect ratio is just $c/a=1.2$).

Figure 2 illustrates meridional sections of an initially ellipsoidal cavity with semiaxes $a=b=1.5$ and $c=1$ at longer time intervals t = 10, 50, 70, 180. The parameters of the flow are the same: $Fr=4$ and $We=1000$. This ellipsoid is less oblong; therefore, the meridional sections are nearly circular at early time points. It is seen from Fig. 2 that after the time $t\approx 50$, the deformation mode of the cavity looks like that observed under gravity. A peculiar feature of the mode is a dome-shaped depression formed at the bottom of the cavity. It is well observed at $t=70$ and $t=180$ (Fig. 2). This depression indicates the beginning of formation of an axisymmetric cumulative jet. At the same time, the upper surface of the cavity moves downward, toward the jet. It is this jet that makes the cavity toroidal after reaching its upper surface, which was discovered in the experiments [4, 5], though these experiments address collapsing cavities. It should also be noted that the cross sections of cavitation bubbles occurring in heavy flows deform similarly (have a depression at the bottom).

REFERENCES

1. V. N. Buivol, *Thin Cavities in Disturbed Flows* [in Russian], Naukova Dumka, Kiev (1980).
2. V. N. Buivol and Yu. R. Shevchyuk, "Deformation equations for a floating-up bubble," *Dokl. AN USSR*, *Ser. A*, No. 5, 34–37 (1987).
3. V. V. Voronin and T. N. Machekhina, *The Present State of the Art in Research on Cavitation Flows* (*Reviews*) [in Russian], **651**, ONTI TsAGI, Moscow (1985).
4. R. T. Knapp, J. W. Daily, and F. G. Hammitt, *Cavitation*, McGraw Hill, New York (1970).
5. J. P. Best, "The formation of toroidal bubbles upon the collapse of transient cavities," *J. Fluid Mech.*, **251**, 79–107 (1993).

6. J. P. Best and A. Kucera, "A numerical investigation of nonspherical rebounding bubbles," *J. Fluid Mech.*, **245**, 137–154 (1992).
7. J. R. Blake, P. B. Robinson, A. Shima, and Y. Tomita, "Interaction of two cavitation bubbles with a rigid boundary," *J. Fluid Mech.*, **255**, 707–721 (1993).
8. A. F. Dashchenko and Yu. M. Dudzinskii, "Natural oscillations of a jet membrane under hydrostatic pressure," *Int. Appl. Mech.*, **40**, No. 12, 1385–1390 (2004).
9. V. T. Grinchenko, "Wave motion localization effects in elastic waveguides," *Int. Appl. Mech.*, **41**, No. 9, 988–994 (2005).
10. A. N. Guz and A. P. Zhuk, "Motion of solid particles in a liquid under the action of an acoustic field: the mechanism of radiation pressure," *Int. Appl. Mech.*, **40**, No. 3, 246–265 (2004).
11. W. A. H. J. Herman, "On the instability of the translating gas bubble under the influence of a pressure step," *Fhilips Res. Repts Suppl.*, No. 3, 2–26 (1973).
12. V. D. Kubenko, "Impact of blunted bodies on a liquid or elastic medium," *Int. Appl. Mech.*, **40**, No. 11, 1185–1225 (2004).
13. M. S. Plesset and A. Prosperitti, "Bubble dynamics and cavitation," *Ann. Rev. Fluid Mech.*, No. 9, 145–185 (1977).