International Applied Mechanics, Vol. 41, No. 11, 2005

STRESS DISTRIBUTION IN PHYSICALLY AND GEOMETRICALLY NONLINEAR THIN CYLINDRICAL SHELLS WITH TWO HOLES

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UDC 539.374

The elastoplastic state of thin cylindrical shells weakened by two circular holes is analyzed. The centers of the holes are on the directrix of the shell. The shells are made of an isotropic homogeneous material and subjected to internal pressure of given intensity. The distribution of stresses along the hole boundaries and over the zone where they concentrate (when the distance between the holes is small) is analyzed using approximate and numerical methods to solve doubly nonlinear boundary-value problems. The data obtained are compared with the solutions of the physically nonlinear (plastic strains taken into account) and geometrically nonlinear (finite deflections taken into account) problems and with the numerical solution of the linearly elastic problem. The stress–strain state near the two holes is analyzed depending on the distance between them and the nonlinearities accounted for

Keywords: nonlinear problems, cylindrical shells, two circular holes, shell directrix, internal pressure, plastic strains, finite deflections, mutual influence

Introduction. Stress, strain, and strength analyses of isotropic and anisotropic structural members (plates and shells) of complex geometry, including multiple connectivity, under static and dynamic loads are still of importance. Of heightened interest are nonaxisymmetric problems for thin and nonthin shells and plates of various outlines and shapes under high surface and boundary loads. Solving such static and dynamic problems, in both linear and nonlinear formulations, involves severe mathematical and computational difficulties.

Efficient methods developed to solve linearly elastic boundary-value problems for multiply connected shells (weakened by two or more curvilinear holes or notches) are outlined in [1, 2, 6, 7, 11–13].

Concrete numerical results have been obtained for metal and composite shells of spherical, cylindrical, conical, and other shapes subjected to static surface loads, axial forces, and a system of boundary forces and moments. Note that the publications [2, 6, 7, 12] present results for elastic isotropic and orthotropic cylindrical shells with two equal or unequal circular holes (their centers are on a common generatrix or directrix) and with a finite number of circular holes, and for periodic cases.

It should also be emphasized that two-dimensional static problems for shells (plates) of various shapes and purposes with a curvilinear (circular or elliptic) hole or a notch of various geometries (simply and doubly connected domains) have mostly been solved considering only the elastic stage of deformation or only geometrical (finite deflections) or physical (plastic or creep strains) nonlinearities [6, 7, 9, 11, 13]. The principles and methods of theoretical and experimental analysis of stress concentration in load-bearing structural members were developed with an eye toward solving two-dimensional boundary-value problems in various formulations [3–8, 10, 11]. Isolated results were obtained by solving nonlinear problems for shells with both finite (large) deflections and plastic strains [6, 7, 13].

As regards nonlinear problems for multiply connected thin shells, there are only isolated theoretical results. For thin-walled isotropic spherical and cylindrical shells with two circular holes, such results are presented in [6, 14–17].

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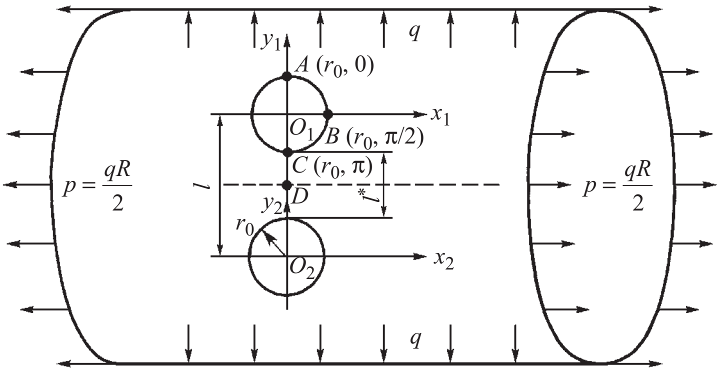

Fig. 1

The paper [14] gives a general nonlinear formulation to the stress (or strain or displacement) distribution problem for arbitrary thin shells weakened by two and more curvilinear holes (notches) and described by nonconjugate curvilinear coordinates. Also, this paper briefly outlines a theory and an approximate method for solving two-dimensional boundary-value problems for isotropic homogeneous shells characterized by two nonlinearities (finite deflections and plastic strains). Concrete numerical results have been obtained for spherical [14, 15] and cylindrical [16, 17] shells having two equal circular holes and subjected to a uniform surface load (internal pressure) of given intensity. Also, the case of an axially stretched cylindrical shell with two circular holes aligned along a common generatrix was analyzed numerically. The results obtained in [14–17] made it possible to study the mutual influence of the holes and the influence of nonlinearities on the elastoplastic state of flexible shells in stress concentration regions.

In support of the studies [14, 16, 17], here we will discuss concrete numerical results from an inelastic stress–strain analysis of flexible cylindrical shells subjected to internal pressure of increasing intensity. The shells have two circular holes with the centers lying on a common directrix and the boundaries interfering with each other. Also, we will analyze the distribution of stress-concentration factors along the hole boundaries, compare the associated numerical results, and evaluate the effect of one or two nonlinearities on these factors.

1. Governing Equations. Solution Technique for Nonlinear Problems. Let us analyze the stress–strain state of thin-walled cylindrical shells with two circular holes (Fig. 1). Their centers are on a common directrix. The shells are subjected to surface pressure of given intensity (q = const). The materials of the shells are isotropic, homogeneous (metals or their alloys), and such that high levels of the load produce small plastic strains in the zones of maximum stress concentration and normal displacements comparable with the shell thickness [6, 14].

The holes are assumed [6, 16] to be closed with special plugs that transmit only shearing forces to the hole boundary, which are equivalent to the surface load on each hole. Thus, adopting certain boundary conditions, geometrical parameters of the shells, and mechanical characteristics and deformation curves (σ versus ε and σ_i versus e_i) of their materials, we arrive at elastoplastic problems for flexible shells of complex geometry with two holes spaced at different distances.

The paper [14] suggests analyzing the elastoplastic state of multiply connected thin shells with finite deflections on the basis of the theory of flexible shells (geometrically nonlinear theory of second order), the theory of flow with isotropic hardening, the Mises yield criterion, and the associated flow rule.

For shells with the mid-surface described by nonconjugate curvilinear coordinates $(\alpha_1, \alpha_2, \alpha_3)$, the nonlinear geometrical equations can be written in the following matrix form ($\alpha_3 = \text{const}$):

$$\{e\} = \{\varepsilon\} + \alpha_3 \{\kappa\} = \{e^{\text{l}}\} + \{e^{\text{n}}\}. \tag{1.1}$$

The linear and nonlinear terms are defined by

$$\{e^{\text{l}}\} = [A_\varepsilon]\{U\} + \alpha_3 \left([A_\kappa]\{\varphi\} + [A_\kappa^*]\{U\}\right),$$

$$\{e^{\text{n}}\} = 0.5[B_\varphi]\{\varphi\}, \qquad [A_\kappa^*] = [A_\varphi]\{U\},$$

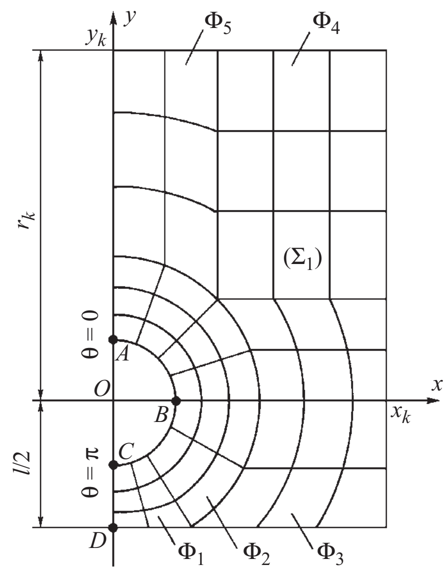

Fig. 2

where $\{U\}=\{u_1, u_2, u_3\}$, $\{\varphi\}=\{\varphi_1, \varphi_2\}$, and $\{e\}=\{e_1, e_2, e_3\}$ are the vectors of displacements, angles of rotation, and strains, respectively; and $[A_\varepsilon]$, $[A_\kappa]$, and $[B_\varphi]$ are the linear differentiation matrices.

With complex loading and the adopted flow theory, the nonlinear physical equations are given by

$$\{\sigma\}=[D]\left(\{e\}-\{e^{\mathrm{pl}}\}\right)=[D]\{e\}+\{\sigma^{\mathrm{pl}}\}=[D]\{e^{\mathrm{l}}\}+\{\sigma^{\mathrm{n}}\}, \tag{1.2}$$

where the total strain is the sum of elastic and inelastic (plastic) components ($\{e\}=\{e^{\mathrm{el}}\}+\{e^{\mathrm{pl}}\}$); $\{\sigma^{\mathrm{pl}}\}=-[D]\{e^{\mathrm{pl}}\}$, $\{\sigma^{\mathrm{n}}\}=[D]\{e^{\mathrm{n}}\}+\{\sigma^{\mathrm{pl}}\}$; and $\{\sigma\}=\{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T$ is the stress vector.

The solution technique for the nonlinear problems in question is based on the procedure of incremental loading. The governing equations are derived from the nonlinear equations (1.1) and (1.2), using [6, 14] the virtual-displacement principle, the modified Newton–Kantorovich method, the method of additional stresses, and the finite-element method (FEM).

Thus, the complex nonlinear problem is reduced to a sequence of linearly elastic problems. They are solved numerically, after the finite-element discretization of the shell's mid-surface. Within each of the finite elements, the components of the vector of displacement increments $\{\Delta U\}$ are represented by polynomials of two variables [14].

Linearizing and discretizing the corresponding variational equation and using the stationarity condition for the linearized total energy of a complex-geometry shell, we arrive at a system of governing algebraic equations. For the nth step of loading, this system is expressed as

$$[K]=\{\Delta q\}=\{\Delta F\},$$

$$[K]=[K_1]+[K_\alpha]+[K_\sigma], \qquad \{\Delta F\}=\{\Delta P\}-\{\Delta N\}+\{\Delta\Phi\}, \tag{1.3}$$

where $[K_1]$ is the incremental stiffness matrix for linearly elastic shells; $[K_\alpha]$ and $[K_\sigma]$ are the influence matrices for the initial angles of rotation and stresses; $\{\Delta q\}$ is the column vector of increments of nodal degrees of freedom (nodal variables); $\{\Delta P\}$ is the load vector; $\{\Delta N\}$ is the vector of nonlinearities; and $\{\Delta\Phi\}$ is the residual vector for the equilibrium equations at the end of the $(n-1)$th step of loading.

For further calculations, Eqs. (1.3) should be supplemented with appropriate boundary conditions on the hole edges and far from them (outer edge), and the geometrical and force symmetries of the problems should be taken into account. The software application system, which implements our method [14], allows us to solve intricate doubly nonlinear boundary-value problems for thin shells. The problem-solving process proceeds until $||\Delta q||_n^i / ||q||_n^i \le \delta$, where $10^{-2} \le \delta \le 10^{-3}$ is a prescribed error tolerance.

TABLE 1

$\tilde{l}$	point; θ	ξ	$\sigma_{\theta\theta}^{0}$			
			LP	PNP	GNP	PGNP
2.5	A; 0	0.5	2621	1693	2142	1515
		–0.5	–3487	–1945	–812	–816
	B; π / 2	0.5	1626	1271	3063	1785
		–0.5	6084	2326	3478	1844
	C; π	0.5	7296	4300	2794	1683
		–0.5	–5499	–2987	532	1310
2.7	A; 0	0.5	2623	1782	2176	1536
		–0.5	–3717	–1974	–856	–857
	B; π / 2	0.5	1638	1479	3104	1774
		–0.5	5945	2209	3466	1800
	C; π	0.5	5230	2913	2561	1646
		–0.5	–5061	–2698	91	507
3.0	A; 0	0.5	2601	1770	2165	1527
		–0.5	–3701	–1984	–854	–857
	B; π / 2	0.5	1579	1420	3066	1770
		–0.5	5755	2151	3433	1805
	C; π	0.5	3440	2127	2388	1611
		–0.5	–4520	–2369	–383	–147
4.0	A; 0	0.5	2607	1570	2160	1521
		–0.5	–3646	–1972	–806	–774
	B; π / 2	0.5	1625	1399	2994	1762
		–0.5	5517	2090	3341	1798
	C; π	0.5	2334	1736	2179	1565
		–0.5	–3667	–1987	–861	–830

TABLE 2

$\tilde{l}$	σ_{ij}^0	ξ	LP	PNP	GNP	PGNP
2.5	σ_{rr}^0	0.5	120	764	–637	–364
		–0.5	259	210	1022	1283
	$\sigma_{\theta\theta}^0$	0.5	7588	4352	2473	1483
		–0.5	–3425	–1950	753	1507
2.7	σ_{rr}^0	0.5	–256	–633	–626	–402
		–0.5	517	1300	1104	1301
	$\sigma_{\theta\theta}^0$	0.5	–5923	–3185	2136	1368
		–0.5	–2062	–784	611	896
3.0	σ_{rr}^0	0.5	–320	493	–343	–295
		–0.5	1080	1898	1089	1212
	$\sigma_{\theta\theta}^0$	0.5	–4089	2417	1794	1344
		–0.5	–804	602	502	494
4.0	σ_{rr}^0	0.5	–30	–822	429	376
		–0.5	1944	1940	992	1041
	$\sigma_{\theta\theta}^0$	0.5	851	883	851	981
		–0.5	602	746	314	291

2. Stress Concentration in Cylindrical Shells with Two Holes Aligned Along the Directrix. We will discuss numerical results from an elastoplastic stress–strain analysis of flexible thin-walled shells with two equal circular holes (Fig. 1) subjected to a static load—uniform surface pressure $q = q_0 \cdot 10^5$ Pa. The shells are made of AMg-6 alloy and have the following geometrical and mechanical characteristics:

$$\rho = r_0 / \sqrt{Rh} = 2, \qquad \tilde{l} = l / r_0 = 2.5, 2.7, 3.0, 4.0 \qquad (l^* / r_0 = 0.5, 0.7, 1.0, 2.0),$$

$$E = 65 \text{ GPa}, \quad \nu = 0.3\text{–}0.5, \quad \sigma_n = 130 \text{ MPa}, \quad \varepsilon_n = 0.002, \quad \sigma_T = 165 \text{ MPa}, \tag{2.1}$$

where r_0 and R are the radii of the holes and shell; h is its thickness; and l and l^* are the distances between the centers and edges of the holes. The shell is referred to a Cartesian coordinate system (x, y) with the origin at the center of one of the holes (Fig. 2); (r, θ) is a polar coordinate system.

It is assumed that only the shearing forces $Q_k = qr_0 / 2$ act at the boundaries of the unreinforced holes [6]. The elastic state at a sufficient distance from the holes is momentless and defined by the following conditions:

$$T_k = qR \qquad \text{at} \qquad x = x_k,$$

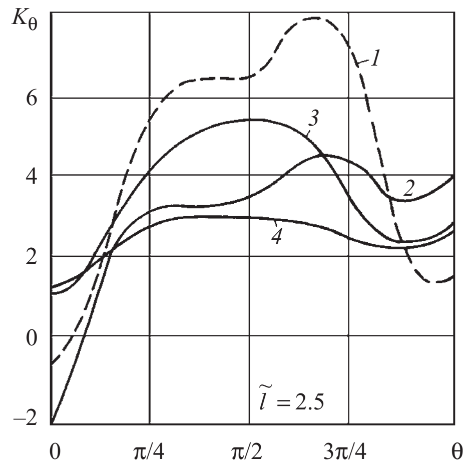

Fig. 3

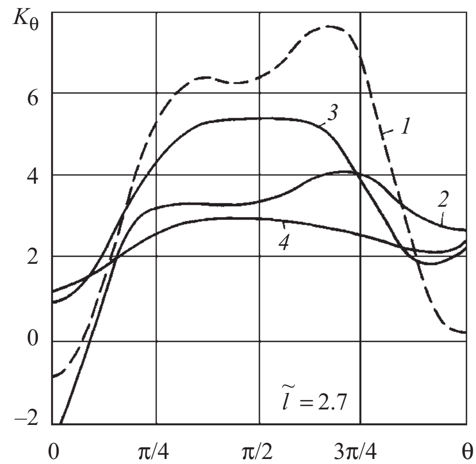

Fig. 4

$$T_k = qR/2 \qquad \text{at} \qquad y = y_k,$$

$$\text{symmetry conditions} \qquad \text{at} \qquad x=0 \text{ and } y=-l/2 \text{ (Fig. 2).} \tag{2.2}$$

Due to the force and geometry symmetry of the problem, only a quarter of the shell (Σ_1) needs to be considered. We partition it into five fragments for finite-element discretization and then solve numerically the linearly elastic ($q_0 = 1$) and nonlinear problems for shells with the parameters (2.1) and conditions (2.2) under an internal pressure of intensity $q = 1.5 \cdot 10^5$ Pa. To this end, the loading process is divided into 20 steps for $\tilde{l} = 2.5$; 15 steps for $\tilde{l} = 2.7$, 3.0; and 10 steps for $\tilde{l} = 4.0$.

3. Numerical Results and Their Analysis. The results of the solution are the values of the components of the displacement vector $\{U\}$ and the strain and stress tensors ($e_{ij}, \sigma_{ij};\ i, j = r, \theta$) calculated, using Eqs. (1.1) and (1.2), at different nodes of the hole boundary ($r = r_0$), in the area of interest ($r \le r_0 \le 4r_0$, $0 \le \theta \le \pi \le \pi/2$), and at three points across the thickness of the shell. Some of the results are presented in the figures and tables below.

Table 1 shows the distribution of the maximum hoop stresses ($\sigma_{\theta\theta} = \sigma^0_{\theta\theta} \cdot 10^5$ Pa) along the hole boundary ($r = r_0$, $0 \le \theta \le \pi$) at the points *A*, *B*, and *C* on the outer and inner surfaces ($\xi = z/h = \pm 0.5$) of the shells for $\tilde{l} = 2.5$–4.0. Table 2 collects the values of the radial and hoop stresses at the point *D* (the center of the bridge between holes, $x = 0$, $y = -l/2$). These results (Tables 1 and 2) have been obtained by solving four boundary-value problems: linear problem (LP), physically nonlinear problem (PNP), geometrically nonlinear problem (GNP), and doubly (physically and geometrically) nonlinear problem (PGNP).

Figures 3–6 show the stress-concentration factors ($K_\theta = \sigma_{\theta\theta} h / qR$) on the hole boundary ($0 \le \theta \le \pi$) for different sizes of the bridge. Curves 1, 2, 3, and 4 in these figures correspond to LP, PNP, GNP, and PGNP, respectively, in the tables.

An analysis of the results presented in Tables 1 and 2 and Figs. 3–6 suggests that (as in [16, 17]) the effect of the second hole on the stress distribution along a part of the boundary ($0 \le \theta \le \pi/2$) of the first hole increases insignificantly with decreasing distance between the holes. For example, the hoop stresses at the point *B* ($r = r_0$, $\theta = \pi/2$) increase by 10% in the LP, by 11% in the PNP, by 4% in the GNP, and by 3% in the PGNP.

The mutual influence of the holes is maximum at the point *C* ($r = r_0$, $\theta = \pi$). The hoop stresses in this section on the outer surface ($\xi = 0.5$) for $\tilde{l} = 2.5$ are greater than those for $\tilde{l} = 4.0$ by 213, 148, 28, and 8% (in the LP, PNP, GNP, and PGNP, respectively). When the distance between the hole centers is large ($\tilde{l} = 4.0$), the point *B* (r_0, $\pi/2$) on the inner surface ($\xi = -0.5$) is the most stressed. When the distance between the hole centers is small ($\tilde{l} = 2.5$), the most critical is the point *C* (r_0, π) on the outer surface ($\xi = 0.5$) according to the LP and PNP and the point *B* (r_0, $\pi/2$) according to the GNP and PGNP.

The maximum stresses at the point *D* ($r = l/2$, $\theta = \pi$) are less than the maximum stresses on the hole boundaries when $\tilde{l} = 4.0$ and are greater than these stresses, according to the LP and PNP, when $\tilde{l} = 2.5$. Thus, for a short bridge between the holes, the most critical is the point *D* (bridge center) according to the LP and PNP and the point *B* according to the GNP and PGNP.

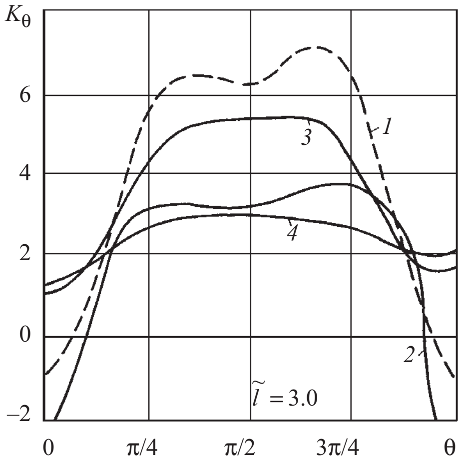

Fig. 5

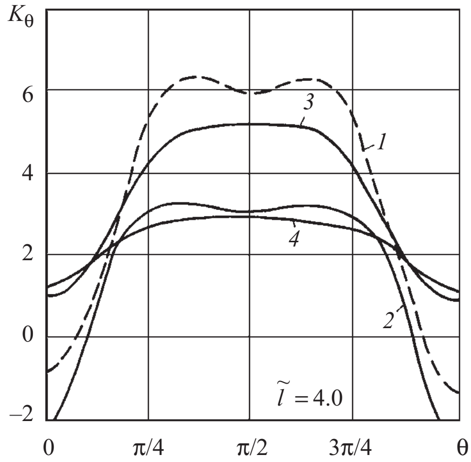

Fig. 6

The results indicate that the nonlinearities greatly reduce the maximum stresses for all distances between the holes. For example, when $\tilde{l} = 2.5$, this reduction is 43% with inelastic strains, 54% with finite deflections, and 76% with both nonlinearities taken into account.

It also follows from the results that when the bridge is larger than two hole radii ($\tilde{l} = 4.0$), the mutual influence of the holes has not to be taken into account in a stress–strain analysis of cylindrical shells (weakened by two circular holes located on a common directrix) with plastic strains and finite deflections.

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