International Applied Mechanics, Vol. 41, No. 11, 2005

ON THE STRESS STATE OF A PIEZOCERAMIC BODY WITH A FLAT CRACK UNDER SYMMETRIC LOADS

V. S. Kirilyuk

UDC 539.3

A static-equilibrium problem is solved for an electroelastic transversely isotropic medium with a flat crack of arbitrary shape located in the plane of isotropy. The medium is subjected to symmetric mechanical and electric loads. A relationship is established between the stress intensity factor (SIF) and electric-displacement intensity factor (EDIF) for an infinite piezoceramic body and the SIF for a purely elastic material with a crack of the same shape. This allows us to find the SIF and EDIF for an electroelastic material directly from the corresponding elastic problem, not solving electroelastic problems. As an example, the SIF and EDIF are determined for an elliptical crack in a piezoceramic body assuming linear behavior of the stresses and the normal electric displacement on the crack surface

Keywords: piezoelectricity, flat crack, elliptical crack, stress intensity factor, electric-displacement intensity factor

Introduction. The wide use of piezoelectric ceramic materials, which are highly brittle, in various transducers (based on the coupling of mechanical and electric fields) necessitates a careful study into the concentration of mechanical and electric fields in electroelastic bodies with imperfections such as cavities, inclusions, and cracks. However, the solution of three-dimensional problems of electroelasticity involves severe mathematical difficulties since the original system of equations describing the electrostressed state of a body consists of complicated coupled differential equations [1, 4]. This is why plane problems of electroelasticity have recently been studied in more detail. Noteworthy are the papers [2, 11, 14, 17, 18] that address the two-dimensional electroelastic state around a single cavity, inclusion, and crack and the interaction of concentrators of electric and mechanical fields. Three-dimensional problems of electroelasticity for an infinite medium with cavities, inclusions, and cracks are solved in [5–7, 9, 10, 13, 15, 16]. The papers [5, 15, 16] propose approaches to finding the general solutions of coupled equations of electroelasticity for a transversely isotropic body. The exact solutions of electroelastic problems for spheroidal and hyperboloidal cavities and inclusions have been found in [6, 13]. The electrostressed state and stress intensity factors (SIFs) and electric-displacement intensity factors (EDIFs) for an infinite medium with penny-shaped and elliptic cracks are studied in [1, 9, 10] and [7, 15, 16], respectively.

1. Problem Formulation and Governing Equations. Consider a transversely isotropic electroelastic body with a flat crack in the plane of isotropy. Symmetric mechanical and electric forces act on the surfaces of the crack. An electroelastic body with the axis of transtropy coinciding with the *Oz*-axis is described by the following complete system of equations [1]:

the equilibrium equations (no body forces)

$$\sigma_{ij,j} = 0, \tag{1}$$

the electrostatic equations

$$D_{i,i} = 0, \quad E_i = -\Psi_{,i}, \tag{2}$$

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Kiev. Translated from Prikladnaya Mekhanika, Vol. 41, No. 11, pp. 67–77, November 2005. Original article submitted December 10, 2004.

the equations of state

$$\sigma_{xx}=c_{11}^E\varepsilon_x+c_{12}^E\varepsilon_y+c_{13}^E\varepsilon_z-e_{31}E_z,\quad \sigma_{yy}=c_{12}^E\varepsilon_x+c_{11}^E\varepsilon_y+c_{13}^E\varepsilon_z-e_{31}E_z,\quad \sigma_{zz}=c_{13}^E(\varepsilon_x+\varepsilon_y)+c_{33}^E\varepsilon_z-e_{33}E_z,$$

$$\sigma_{yz}=2c_{44}^E\varepsilon_{yz}-e_{15}E_y,\quad \sigma_{xz}=2c_{44}^E\varepsilon_{xz}-e_{15}E_x,\quad \sigma_{xy}=(c_{11}^E-c_{12}^E)\varepsilon_{xy},$$

$$D_x=\varepsilon_{11}^S E_x+e_{15}\varepsilon_{xz},\quad D_y=\varepsilon_{11}^S E_y+e_{15}\varepsilon_{yz},\quad D_z=\varepsilon_{33}^S E_z+e_{31}(\varepsilon_x+\varepsilon_y)+e_{33}\varepsilon_z, \tag{3}$$

and the Cauchy relations

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),$$

where $c_{11}^E, c_{12}^E, c_{13}^E, c_{33}^E$, and c_{44}^E are the independent elastic moduli (measured at constant electric field); e_{31}, e_{15}, and e_{33} are piezoelectric moduli; and ε_{11}^S and ε_{33}^S are the dielectric permittivities (measured at constant strain).

Substituting the equations of state into Eqs. (1) and (2), we obtain a system of equations for the displacements u_x, u_y, and u_z and electric potential Ψ:

$$c_{11}^E u_{x,xx}+\frac{1}{2}(c_{11}^E-c_{12}^E)u_{x,yy}+c_{44}^E u_{x,zz}+\frac{1}{2}(c_{11}^E+c_{12}^E)u_{y,xy}+(c_{13}^E+c_{44}^E)u_{z,xz}+(e_{31}+e_{15})\Psi_{,xz}=0,$$

$$c_{11}^E u_{y,yy}+\frac{1}{2}(c_{11}^E-c_{12}^E)u_{y,xx}+c_{44}^E u_{y,zz}+\frac{1}{2}(c_{11}^E+c_{12}^E)u_{x,xy}+(c_{13}^E+c_{44}^E)u_{z,yz}+(e_{31}+e_{15})\Psi_{,yz}=0,$$

$$(c_{13}^E+c_{44}^E)(u_{x,xz}+u_{y,yz})+c_{44}^E(u_{z,xx}+u_{z,yy})+c_{33}^E u_{z,zz}+e_{15}(\Psi_{,xx}+\Psi_{,yy})+e_{33}\Psi_{,zz}=0,$$

$$(e_{31}+e_{15})(u_{x,xz}+u_{y,yz})+e_{15}(u_{z,xx}+u_{z,yy})+e_{33}u_{z,zz}-\varepsilon_{11}^S(\Psi_{,xx}+\Psi_{,yy})-\varepsilon_{33}^S\Psi_{,zz}=0. \tag{4}$$

To solve the system of equations (4), we will use the representation of solution proposed in [5]. Note that such a solution was also used in [5, 15, 16]. The displacement components and electric potential can be expressed as follows [5]:

$$u_x=\sum_{j=1}^{3}\Phi_{j,x}+\Phi_{4,y},\quad u_y=\sum_{j=1}^{3}\Phi_{j,y}-\Phi_{4,x},\quad u_z=\sum_{j=1}^{3}k_j\Phi_{j,z},\quad \Psi=\sum_{j=1}^{3}l_j\Phi_{j,z}, \tag{5}$$

where k_i and l_i are some constants to be determined.

After substitution of the expressions (5) into Eqs. (4), it becomes clear that these equations hold if the functions Φ_j satisfy the equations

$$\Phi_{j,xx}+\Phi_{j,yy}+\nu_j\Phi_{j,zz}=0 \qquad (j=1,2,3,4), \tag{6}$$

where $\nu_4=2c_{44}^E/(c_{11}^E-c_{12}^E)$, and the remaining ν_i $(i=1,2,3)$ are the roots of the following cubic equation:

$$\nu^3(A_1B_2-C_1D_2)+\nu^2(A_1B_3+A_2B_2-C_1D_3-C_2D_2)$$

$$+\nu(A_2B_3+A_3B_2-C_2D_3-C_3D_2)+A_3B_3-C_3D_3=0. \tag{7}$$

The coefficients are defined by

$$A_1=c_{11}^E e_{15},\quad A_2=(c_{13}^E+c_{44}^E)(e_{31}+e_{15})-c_{11}^E e_{33}-c_{44}^E e_{15},\quad A_3=c_{44}^E e_{33},$$

$$B_2=-[\varepsilon_{11}^S(c_{13}^E+c_{44}^E)+e_{15}(e_{31}+e_{15})],\quad B_3=\varepsilon_{33}^S(c_{13}^E+c_{44}^E)+e_{33}(e_{31}+e_{15})],$$

$$C_1=-c_{11}^E\varepsilon_{11}^S,\quad C_2=(e_{31}+e_{15})^2+c_{11}^E\varepsilon_{33}^S+c_{44}^E\varepsilon_{11}^S,\quad C_3=-c_{44}^E\varepsilon_{33}^S,$$

$$D_2 = e_{15}(c_{13}^E + c_{44}^E) - c_{44}^E(e_{31} + e_{15}), \qquad D_3 = c_{33}^E(e_{31} + e_{15}) - e_{33}(c_{13}^E + c_{44}^E). \tag{8}$$

The constants k_j and l_j $(i = 1, 2, 3)$ in (5) are related to the roots ν_j as

$$\frac{a_j + c_{13}^E k_j + e_{31} l_j}{c_{11}^E} = \frac{c_{33}^E k_j + e_{33} l_j}{c_{13}^E + a_j} = \frac{c_{33}^E k_j - \varepsilon_{33}^S l_j}{e_{31} + d_j} = \nu_j \qquad (j = 1, 2, 3), \tag{9}$$

where

$$a_j = c_{44}^E(1 + k_j) + e_{15} l_j, \quad d_j = e_{15}(1 + k_j) - \varepsilon_{11}^S l_j \qquad (j = 1, 2, 3). \tag{10}$$

The constants k_j and l_j are determined from the formulas

$$k_j = \frac{[(\nu_j c_{11}^E - c_{44}^E)(e_{15}\nu_j - e_{33}) + \nu_j(c_{44}^E + c_{13}^E)(e_{31} + e_{15})]}{[(c_{13}^E + c_{44}^E)(e_{15}\nu_j - e_{33}) - (c_{44}^E \nu_j - c_{33}^E)(e_{31} + e_{15})]},$$

$$l_j = \frac{[(\nu_j c_{11}^E - c_{44}^E)(\nu_j c_{44}^E - c_{33}^E) + \nu_j(c_{44}^E + c_{13}^E)^2]}{[(\nu_j c_{44}^E - c_{33}^E)(e_{31} + e_{15}) - (c_{13}^E + c_{44}^E)(e_{15}\nu_j - e_{33})]} \qquad (j = 1, 2, 3). \tag{11}$$

With the notation $z_j = \nu_j^{-1/2} z$, it becomes clear that the functions Φ_j are harmonic in the coordinate systems (x, y, z_j), $j = 1, 2, 3, 4$. Then the stress and electric-displacement components are given by

$$\sigma_{xx} = \sum_{j=1}^{3} \left(c_{11}^E \Phi_{j,xx} + c_{12}^E \Phi_{j,yy} + c_{13}^E k_j \Phi_{j,zz} + e_{31} l_j \Phi_{j,zz} \right) + (c_{12}^E - c_{11}^E)\Phi_{4,xy},$$

$$\sigma_{yy} = \sum_{j=1}^{3} \left(c_{12}^E \Phi_{j,xx} + c_{11}^E \Phi_{j,yy} + c_{13}^E k_j \Phi_{j,zz} + e_{31} l_j \Phi_{j,zz} \right) + (c_{12}^E - c_{11}^E)\Phi_{4,xy},$$

$$\sigma_{zz} = \sum_{j=1}^{3} \left(c_{13}^E(\Phi_{j,xx} + \Phi_{j,yy}) + c_{33}^E k_j \Phi_{j,zz} + e_{33} l_j \Phi_{j,zz} \right),$$

$$\sigma_{xy} = \frac{1}{2}(c_{11}^E - c_{12}^E)\left(2\sum_{j=1}^{3} \Phi_{j,xy} + \Phi_{4,xx} - \Phi_{4,yy} \right),$$

$$\sigma_{xz} = \sum_{j=1}^{3} [c_{44}^E(1 + k_j) + e_{15} l_j]\Phi_{j,xz} - c_{44}\Phi_{4,yz},$$

$$\sigma_{yz} = \sum_{j=1}^{3} [c_{44}^E(1 + k_j) + e_{15} l_j]\Phi_{j,yz} - c_{44}\Phi_{4,xz},$$

$$D_x = \sum_{j=1}^{3} [e_{15}(1 + k_j) - \varepsilon_{11}^S l_j]\Phi_{j,xz} - e_{15}\Phi_{4,yz},$$

$$D_y = \sum_{j=1}^{3} [e_{15}(1 + k_j) - \varepsilon_{11}^S l_j]\Phi_{j,yz} - e_{15}\Phi_{4,xz},$$

$$D_z = \sum_{j=1}^{3} [e_{31}(\Phi_{j,xx} + \Phi_{j,yy}) + (e_{33}k_j - \varepsilon_{33}^S l_j)\Phi_{j,zz}]. \tag{12}$$

Boundary conditions symmetric about the crack plane ($z = 0$) are the following:

$$u_3^{\pm} = \Psi^{\pm} = 0, \quad \sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0 \quad (\vec{x} \in R^2 / S),$$

$$\sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0, \quad \sigma_{33}^{\pm} = -P(x, y), \quad D_3^{\pm} = -q(x, y) \quad (\vec{x} \in S), \tag{13}$$

where S is the region occupied by the crack.

2. Relationship between the SIFs in the Electroelastic and Purely Elastic Problems. For the sake of comparison, we will write boundary conditions to the elastic problem for an isotropic medium with a flat crack of the same shape as in the electroelastic problem under a symmetric load:

$$u_3^{\pm} = 0, \quad \sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0 \quad (\vec{x} \in R^2 / S),$$

$$\sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0, \quad \sigma_{33}^{\pm} = -P(x, y) \quad (\vec{x} \in S). \tag{14}$$

The displacement components for an elastic isotropic body can be expressed in terms of a harmonic function f as follows [12]:

$$u_x = (1-2\nu)f_{,x} + zf_{,xz}, \quad u_y = (1-2\nu)f_{,y} + z f_{,yz}, \quad u_z = -2(1-\nu)f_{,z} + z f_{,zz}. \tag{15}$$

Then, the boundary conditions (14) yield the following conditions to determine the harmonic function f:

$$2\mu\nabla_1^2 f^{\pm} = -P(x, y) \quad (\vec{x} \in S) \quad \text{or} \quad \nabla_1^2 f^{\pm} = -P(x, y)/(2\mu) \quad (\vec{x} \in S),$$

$$f_{,z}^{\pm} = 0 \quad (\vec{x} \in R^2 / S), \tag{16}$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and μ is the shear modulus. Note that the superscripts "+" and "–", which refer to the upper and lower surfaces of the crack, can be omitted owing to symmetry.

In this case, $K_{\mathrm{I}} = \lim\limits_{r\to 0} \sqrt{2\pi r}\,\sigma_{zz}|_{z=0}$.

Using the formulas (14), we can express the stress intensity factor K_{I} in terms of the harmonic function f:

$$K_{\mathrm{I}} = \lim_{r\to 0} \sqrt{2\pi r}\, 2\mu\nabla_1^2 f|_{z=0} \quad (\vec{x} \in R^2 / S), \tag{17}$$

provided that the function f satisfies the condition (16). Note that the formulas (16) and (17) show that K_{I} does not depend on the elastic constants since first the function f on the crack surface is determined from the load magnitude multiplied by $1/(2\mu)$, and then, in determining K_{I}, its expression in terms of f is multiplied by 2μ.

Now we suppose that the problem for a flat crack of the same shape (located in an elastic isotropic body rather than a piezoceramic medium) and for the same load $-P(x, y)$ has been solved and the harmonic function $f(x, y, z)$ has been determined. Then, in (5) we set $\Phi_4 = 0; \Phi_j(x, y, z_j) = \alpha_j f(x, y, z_j)$ $(j = 1, 2, 3)$, where α_j are some constants to be determined, and all the functions $\Phi_j(x, y, z_j)$ $(z_j = \nu_j^{-1/2} z)$ can fully be determined in terms of the harmonic function $f(x, y, z)$. Note that $z_j = 0$ $(j = 1, 2, 3)$ in the crack plane $z = 0$. Using the functions $\Phi_j(x, y, z_j)$, we solve the homogeneous system of equations (4) and derive the expressions for the stress components and the normal component of the electric-displacement vector in the crack plane:

$$\sigma_{zz}|_{z=0} = \left[\sum_{j=1}^{3} \alpha_j \left(c_{44}^E(1+k_j) + e_{15}l_j\right)\right] \nabla_1^2 f|_{z=0},$$

$$D_z|_{z=0} = \left[\sum_{j=1}^{3} \alpha_j \left(e_{15}(1+k_j) - \varepsilon_{11}^S l_j\right)\right] \nabla_1^2 f|_{z=0},$$

$$\sigma_{xz}|_{z=0} = \left[\sum_{j=1}^{3} \alpha_j \frac{(c_{44}^E(1+k_j)+e_{15} l_j)}{\sqrt{\nu_j}}\right] f_{,xz}|_{z=0},$$

$$\sigma_{yz}|_{z=0} = \left[\sum_{j=1}^{3} \alpha_j \frac{(c_{44}^E(1+k_j)+e_{15} l_j)}{\sqrt{\nu_j}}\right] f_{,yz}|_{z=0}. \tag{18}$$

It follows from (18) that if the unknown constants α_j $(j=1, 2, 3)$ are determined from the system of linear equations

$$\sum_{j=1}^{3} \alpha_j \left(c_{44}^E(1+k_j)+e_{15} l_j\right) = 2\mu, \quad \sum_{j=1}^{3} \alpha_j \left(e_{15}(1+k_j) - \varepsilon_{11}^S l_j\right) = 0,$$

$$\sum_{j=1}^{3} \alpha_j \frac{(c_{44}^E(1+k_j)+e_{15} l_j)}{\sqrt{\nu_j}} = 0, \tag{19}$$

then the resulting electroelastic solution will satisfy the boundary conditions

$$u_3^{\pm} = \Psi^{\pm} = 0, \quad \sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0 \quad (\vec{x} \in R^2 / S),$$

$$\sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0, \quad \sigma_{33}^{\pm} = -P(x, y), \quad D_3^{\pm} = 0 \quad (\vec{x} \in S). \tag{20}$$

Let us now solve a new purely elastic problem for an isotropic medium with a flat crack of the same shape, but with the boundary conditions

$$u_3^{\pm} = 0, \quad \sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0 \quad (\vec{x} \in R^2 / S),$$

$$\sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0, \quad \sigma_{33}^{\pm} = -q(x, y) \quad (\vec{x} \in S), \tag{21}$$

where the function $q(x, y)$ appears in (13).

If the solution of this problem is expressed in terms of the harmonic function $f_1(x, y, z)$, i.e.,

$$2\mu \nabla_1^2 f_1^{\pm} = -q(x, y) \quad (\vec{x} \in S), \quad f_{1,z}^{\pm} = 0 \quad (\vec{x} \in R^2 / S), \tag{22}$$

then we express the electroelastic solution in term of the functions $\Phi_4 = 0$ and $\Phi_j(x, y, z_j) = \beta_j f_1(x, y, z_j)$ $(j=1, 2, 3)$ and derive the following system of linear algebraic equations for the unknown constants β_j $(j=1, 2, 3)$:

$$\sum_{j=1}^{3} \beta_j \left(c_{44}^E(1+k_j)+e_{15} l_j\right) = 0, \quad \sum_{j=1}^{3} \beta_j \left(e_{15}(1+k_j) - \varepsilon_{11}^S l_j\right) = 2\mu, \quad \sum_{j=1}^{3} \beta_j \frac{(c_{44}^E(1+k_j)+e_{15} l_j)}{\sqrt{\nu_j}} = 0. \tag{23}$$

With such β_j $(j=1, 2, 3)$, the following boundary conditions are satisfied:

$$u_3^{\pm} = \Psi^{\pm} = 0, \quad \sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0 \quad (\vec{x} \in R^2 / S),$$

$$\sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0, \quad \sigma_{33}^{\pm} = 0, \quad D_3^{\pm} = -q(x, y) \quad (\vec{x} \in S). \tag{24}$$

We represent the solution of the original electroelastic problem with the boundary conditions (13) as a superposition of two states, i.e.,

$$\Phi_4 = 0, \quad \Phi_j(x, y, z_j) = \alpha_j f(x, y, z_j) + \beta_j f_1(x, y, z_j) \quad (j = 1, 2, 3).$$

As a result, we get

$$K_{\mathrm{I}} = \lim_{r\to 0} \sqrt{2\pi r} \left(\sum_{j=1}^{3} \alpha_j \left(c_{44}^E (1 + k_j) + e_{15} l_j \right) \right) \nabla_1^2 (f + f_1)|_{z=0} = \lim_{r\to 0} \sqrt{2\pi r}\, 2\mu \nabla_1^2 f|_{z=0},$$

$$K_{\mathrm{IV}} = K_D = \lim_{r\to 0} \sqrt{2\pi r} D_3|_{z=0} = \lim_{r\to 0} \sqrt{2\pi r}\, 2\mu \nabla_1^2 f_1|_{z=0} \quad (\vec{x} \in R^2 / S). \tag{25}$$

Note that an arbitrary finite (nonzero) number, including unity, may appear instead of 2μ on the right-hand sides of Eqs. (19) and (23), since first the function f is determined from the magnitude of the load on the crack surface divided by this number and then is multiplied by the same number in determining the SIF K_{I}. It may be seen that with similar nature (structure) of mechanical and electric loads on the crack surface there is no need to decompose the problem with the boundary conditions (13) into two problems with the conditions (19) and (24).

Thus, for a flat crack of arbitrary shape (internal or external) located in the plane of isotropy of a piezoceramic material under symmetric loads (13), the SIF K_{I} depends neither on the material properties nor on the normal electric displacement on the crack surface and completely coincides with the SIF K_{I} for a purely elastic isotropic medium (with the same crack and under the same load). Similarly, K_{IV} also depends on neither the material properties nor the mechanical loads on the crack surface and completely coincides with the SIF K_{I} in the purely elastic problem for an isotropic medium with the same crack and under a load represented by the same function as the normal electric displacement in the original problem.

If the electrostressed state (σ_{ij}^0, D_i^0) of an electroelastic medium is symmetric about the crack plane, satisfies the homogeneous equations (4), and is disturbed by the crack, then it can be represented as a superposition of the principal and disturbed states. In so doing, we arrive at a boundary-value problem similar to that above.

Note that similar results have been obtained in [1, 7, 9, 10, 15, 16] for specific shapes of a flat crack and some types of loading. For example, it was established in [1, 9, 10, 15] that the SIFs in the electroelastic and purely elastic problems for a circular crack with constant stresses on its surface coincide. The same conclusion was drawn in [9] for an arbitrary symmetric load on the surface of a circular crack and in [7, 15, 16] for a constant load and normal electric displacement on the surface of an elliptic crack.

3. Electroelastic Solution for a Piezoceramic Material with an Elliptic Crack. Let us consider, as an example, an elliptic crack (with semiaxes a_1 and a_2) in the plane of isotropy of a piezoceramic medium under linearly varying stresses and normal electric displacement on the crack surface. The boundary conditions are given by

$$u_3^{\pm} = \Psi^{\pm} = 0, \quad \sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0 \quad (\vec{x} \in R^2 / S),$$

$$\sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0, \quad \sigma_{33}^{\pm} = -\left(P_0 + p\frac{x}{a_1} + q\frac{y}{a_2} \right), \quad D_3^{\pm} = -\left(\alpha_0 + \alpha\frac{x}{a_1} + \beta\frac{y}{a_2} \right) \quad (\vec{x} \in S).$$

To solve this problem and determine the functions $\Phi_j(x, y, z_j)$ ($j = 1, 2, 3$), we will use the following harmonic functions:

$$\omega_n(x, y, z_j) = \int_{\xi_j}^{\infty} \left[\frac{x^2}{a_1^2 + s} + \frac{y^2}{a_2^2 + s} + \frac{z_j^2}{s} - 1 \right]^n \frac{ds}{\sqrt{Q(s)}},$$

where $Q(s) = s(a_1^2 + s)(a_2^2 + s)$, and the elliptic coordinate $\xi_j = \xi(x, y, z_j)$ is a function of Cartesian coordinates and determined as the maximum positive root of the following cubic equation for the variable s:

$$\frac{x^2}{a_1^2 + s} + \frac{y^2}{a_2^2 + s} + \frac{z_j^2}{s} = 1.$$

Since mechanical and electric loads are structurally similar (linear functions of Cartesian coordinates), we will not decompose the original problem into two problems, and the functions $\Phi_j(x, y, z_j)$ $(j=1, 2, 3)$ (where $\Phi_4=0$) can be taken to have the form

$$\Phi_j(x, y, z_j) = A_j^{(0)} \int_{\xi_j}^{\infty} \left[\frac{x^2}{a_1^2+s} + \frac{y^2}{a_2^2+s} + \frac{z^2}{s} - 1 \right] \frac{ds}{\sqrt{Q(s)}}$$

$$+ \left(A_j^{(1)} \frac{\partial}{\partial x} + B_j^{(1)} \frac{\partial}{\partial y} \right) \int_{\xi_j}^{\infty} \left[\frac{x^2}{a_1^2+s} + \frac{y^2}{a_2^2+s} + \frac{z^2}{s} - 1 \right]^2 \frac{ds}{\sqrt{Q(s)}}, \tag{26}$$

where $A_j^{(0)}, A_j^{(1)}$, and $B_j^{(1)}$ are unknown constants to be determined from the boundary conditions.

After tedious manipulations and asymptotic expansion of the ellipsoidal coordinates at the crack front [12], we obtain (for $a_1 > a_2$) the formulas

$$K_{\mathrm{I}} = \left(\frac{P_0\sqrt{\pi}}{E(k)} \sqrt{\frac{a_2}{a_1}} + \frac{p\sqrt{\pi}\cos\varphi}{\left(\frac{a_2}{a_1}\right)^{3/2} \left[\left(\frac{1}{(k')^2} - \frac{1}{k^2} \right) E(k) + \frac{1}{k^2} K(k) \right]} \right.$$

$$\left. + \frac{q\sqrt{\pi}\sin\varphi}{\left[\left(\frac{1}{k^2} + 1 \right) E(k) - \frac{(k')^2}{k^2} K(k) \right]} \sqrt{\frac{a_2}{a_1}} \right) \times (a_1^2 \sin^2\varphi + a_2^2 \cos^2\varphi)^{1/4},$$

$$K_D = \left(\frac{\alpha_0\sqrt{\pi}}{E(k)} \sqrt{\frac{a_2}{a_1}} + \frac{\alpha\sqrt{\pi}\cos\varphi}{\left(\frac{a_2}{a_1}\right)^{3/2} \left[\left(\frac{1}{(k')^2} - \frac{1}{k^2} \right) E(k) + \frac{1}{k^2} K(k) \right]} \right.$$

$$\left. + \frac{\beta\sqrt{\pi}\sin\varphi}{\left[\left(\frac{1}{k^2} + 1 \right) E(k) - \frac{(k')^2}{k^2} K(k) \right]} \sqrt{\frac{a_2}{a_1}} \right) \times (a_1^2 \sin^2\varphi + a_2^2 \cos^2\varphi)^{1/4}, \tag{27}$$

where $k=(1-a_2^2/a_1^2)^{1/2}$, $k'=a_2/a_1$, and $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds.

As mentioned above, K_{I} depends only on the mechanical loads, and K_D only on the normal electric displacement on the crack surface. As indicated above (Sect. 2), with such a crack and symmetric loads, the elastic and electric properties of the material do not affect the values of the corresponding intensity factors. Note that the expressions for K_{I} and K_D could be derived not by solving the electroelastic problem, but by using the established relationship between them and the SIF K_{I} for a purely elastic material, i.e., by solving only the purely elastic problem for an isotropic material.

4. Numerical Results and Their Analysis. Figures 1–3 shows K_{I} for an elliptic crack with a_1 = 1 and a_2 = 0.6 (curve *1*), a_2 = 0.4 (curve *2*), and a_2 = 0.2 (curve *3*) and linearly varying stresses on its surface. Figure 1 corresponds to a linear dependence of the crack surface stress on the variable x: $\sigma_{33}^{\pm} = -P_0(1+0.5x/a_1)$; Fig. 2 to a linear dependence on the variable y: $\sigma_{33}^{\pm} = -P_0(1+0.5y/a_2)$; and Fig. 3 to a linear dependence on both variables: $\sigma_{33}^{\pm} = -P_0(1+0.5x/a_1+0.5y/a_2)$. If the normal

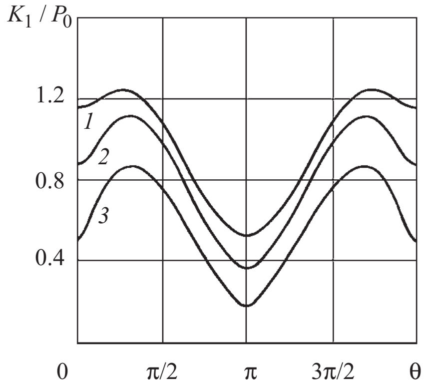

Fig. 1

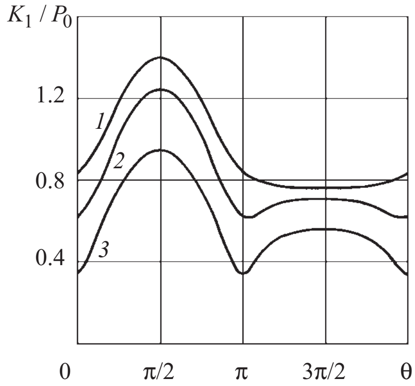

Fig. 2

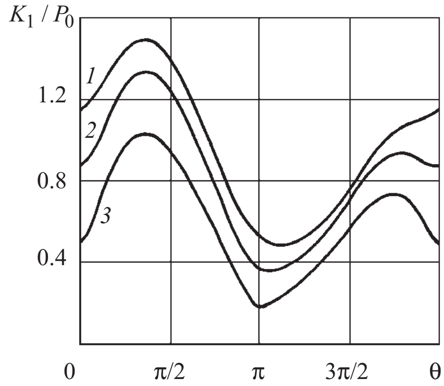

Fig. 3

electric displacement on the crack surface (as well as normal loads) varies in a similar manner, then the EDIF K_D will exhibit similar behavior.

Note that the relationship between the SIF and EDIF for a piezoelectric medium with a flat crack (in the plane of isotropy) and the SIF for an isotropic medium allows us to calculate the SIF and EDIF directly from the SIF for an isotropic medium, not solving specific problems for a piezoelectric material. For example, if the constant stresses $\sigma_{11}^0, \sigma_{22}^0$, and σ_{33}^0 and the electric potential $\Psi^0 = -E_3^0 z$ define the principal electroelastic field in a piezoceramic body with a flat crack (located in the plane of isotropy $z = 0$), then we can immediately determine D_z^0 from the following formula [1]:

$$D_z^0 = \varepsilon_{33}^S E_3^0 + d_{31}(\sigma_{11}^0 + \sigma_{22}^0) + d_{33}\sigma_{33}^0, \tag{28}$$

where ε_{33}^S is an elastic compliance, and d_{31} and d_{33} are piezoelectric constants.

Let the crack surface be stress-free and electrically impermeable (which is a natural assumption because the permittivity of, say, air is hundreds of times less than that of ceramics). Then the boundary conditions are

$$u_3^{\pm} = \Psi^{\pm} = 0, \qquad \sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0 \qquad (\vec{x} \in R^2 / S),$$

$$\sigma_{13}^{\pm} = \sigma_{23}^{\pm} = 0, \qquad \sigma_{33}^{\pm} = -\sigma_z^0, \qquad D_3^{\pm} = -\left(\varepsilon_{33}^S E_3^0 + d_{31}(\sigma_{11}^0 + \sigma_{22}^0) + d_{33}\sigma_{33}^0\right) \qquad (\vec{x} \in S).$$

Thus, with a homogeneous principal electrostressed state, the expression for the SIF completely coincides with that for a purely elastic isotropic medium, and the EDIF can be calculated from the same formulas with σ_z^0 replaced by D_z^0, according to (28). This is true for a flat crack of arbitrary shape. In the case of an elliptic crack, the expression for K_D can be derived from formulas (26) with $\alpha_0 = \varepsilon_{33}^S E_3^0 + d_{31}(\sigma_{11}^0 + \sigma_{22}^0) + d_{33}\sigma_{33}^0$ and $\alpha = \beta = 0$. If only the tensile (compressive) stresses on the crack plane are nonzero, $\sigma_x^0 \neq 0$ and $\sigma_y^0 \neq 0$, then $K_{\mathrm{I}} = 0$ and $K_D \neq 0$ since $\sigma_z^0 = 0$ and $D_z^0 = d_{31}(\sigma_x^0 + \sigma_y^0)$.

Conclusions. We have established a relationship between the SIF and EDIF for an arbitrarily shaped flat crack in the plane of isotropy of a piezoceramic body under symmetric mechanical and electric loads and the SIF for a purely elastic isotropic body. This makes it possible to calculate K_{I} and K_D immediately from K_{I} for a purely elastic body, not solving the electroelastic problem. Thus, we can now, on the one hand, determine the SIF and EDIF for many other electroelastic problems, using, for example, the results from [3, 8, 12, etc.], from the SIF for elastic bodies and, on the other hand, reduce an electroelastic problem to a more simple elastic problem. We have outlined a simple universal algorithm for solving the electroelastic problem for an arbitrarily shaped flat crack in the plane of isotropy of a medium under symmetric mechanical and electric loads. This algorithm employs the known harmonic function $f(x, y, z)$ for the purely elastic problem to determine the functions $\Phi_j(x, y, z_j)$, which not only allow determination of the stress and electric-displacement intensity factors, but also completely describe the electrostressed state of a cracked piezoceramic body.

REFERENCES

1. V. T. Grinchenko, A. F. Ulitko, and N. A. Shul'ga, *Electroelasticity*, Vol. 1 of the six-volume series *Mechanics of Coupled Fields in Structural Members* [in Russian], Naukova Dumka, Kiev (1989).
2. S. A. Kaloerov, A. I. Baeva, and Yu. A. Glushchenko, "Two-dimensional problem of electroelasticity for a multiply connected piezoelectric body with cavities and flat cracks," *Teor. Prikl. Mekh*., **32**, 64–79 (2001).
3. M. P. Savruk, *Stress Intensity Factors for Cracked Bodies*, Vol. 2 of the four-volume *Handbook of Fracture Mechanics and Strength of Materials* [in Russian], Naukova Dumka, Kiev (1988).
4. V. Z. Parton and B. A. Kudryavtsev, *Electroelasticity of Piezoceramic and Electroconductive Bodies* [in Russian], Nauka, Moscow (1988).
5. Yu. N. Podil'chuk, "Harmonic-function representation of the general solution to the static equations of electroelasticity for a transversely isotropic piezoceramic body," *Prikl. Mekh*., **34**, No. 7, 20–26 (1998).
6. Yu. N. Podil'chuk, "Electroelastic equilibrium of transversally isotropic, piezoceramic media containing cavities, inclusions, and cracks," *Int. Appl. Mech*., **34**, No. 10, 1023–1033 (1998).
7. A. H. Passos Morgado, Yu. N. Podil'chuk, and I. V. Lebedeva, "Investigation of stress state of a piezoceramic body containing an external elliptical crack," *Int. Appl. Mech*., **35**, No. 1, 33–40 (1999).
8. Y. Murakami (ed.), *Stress Intensity Factors Handbook*, Vol. 2, Ch. 9, Pergamon Press (1986).
9. W. Chen and T. Shioya, "Fundamental solution for a penny-shaped crack in a piezoelectric medium," *J. Mech. Phys. Solids*, **47**, 1459–1475 (1999).
10. S. Lin, F. Narita, and Y. Shindo, "Electroelastic analysis of a penny-shaped crack in a piezoelectric ceramic under mode I loading," *Mech. Res. Com*., **30**, 371–386 (2003).
11. S. A. Kaloerov, A. I. Baeva, and Yu. A. Glushchenko, "Two-dimensional electroelastic problem for a multiply connected piezoelectric body," *Int. Appl. Mech*., **39**, No. 1, 77–84 (2003).
12. M. K. Kassir and G. Sih, *Three-Dimensional Crack Problems*, Vol. 2 of *Mechanics of Fracture*, Nordhoff, Leyden (1975).
13. Yu. N. Podil'chuk and T. M. Proshchenko, "Flexural stress concentration near hyperboloidal neck in a transversally isotropic piezoceramic cylinder," *Int. Appl. Mech*., **39**, No. 4, 445–456 (2003).
14. Yu. N. Podil'chuk and T. M. Proshchenko, "The stress state of a piezoceramic medium with an elliptic inclusion under pure shear and pure bending," *Int. Appl. Mech*., **39**, No. 5, 573–582 (2003).
15. F. Shang, M. Kuna, and M. Abendroth, "Finite element analysis of a three-dimensional crack problem in piezoelectric structures," *Eng. Fract. Mech*., **70**, 143–160 (2003).
16. Z. K. Wang and S. H. Huang, "Stress intensification near an elliptical border," *Theor. Appl. Fract. Mech*., **22**, 229–237 (1995).
17. H. A. Sosa, "Plane problems in piezoelectric media with defects," *Int. Solids Struct*., **28**, No. 4, 491–505 (1991).
18. H. Sosa and N. Khutoryansky, "New developments concerning piezoelectric materials with defects," *Int. J. Solids Struct*., **33**, No. 23, 3399–3414 (1996).