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ON RELIABLE STABILIZATION OF LINEAR PERIODIC SYSTEMS

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An algorithm is proposed to synthesize a reliable controller with a given stability margin for linear and periodic systems optimized with respect to a quadratic performance criterion. A reliable controller synthesized by the algorithm guarantees the stability margin and is close to the linear-quadratic requlator. The importance of ensuring the stability margin is demonstrated. The proposed algorithm is based on methods of linear matrix inequalities and can be implemented using standard MATLAB routines. As an example, a reliable controller that stabilizes the program motion of a hopping machine is synthesized

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Introduction. The stabilization problem for linear periodic systems has a wide variety of applications [9, 13] among which, in particular, is the synthesis of a stabilization system for walking and hopping robots [17–21, 24–27]. This kind of problems is solved using different approaches (generalized to periodic systems) intended to analyze stationary systems. Noteworthy are synthesis algorithms based on a stabilizing solution of the Riccati equation [2, 6, 7, 12, 16, 28], algorithms based on linear matrix inequalities (LMIs) [1, 3, 8, 22], etc. Note that researchers usually try to generalize a synthesis algorithm for stationary systems to periodic systems. In this connection, we will solve the stabilization problem for periodic systems using an approach developed to synthesize robust controllers with a guaranteed stability margin for stationary systems [29]. As indicated in [31], the first formulations of robust H_2-optimization problems were to ensure a stability margin of synthesized controllers. The difficulties faced then gave rise to problems of synthesis of robust controllers in terms of H_∞-norm optimization. Such approaches, however, give less attention to other important characteristics that are more naturally described in terms of H_2-norms. This is why robust controllers were synthesized using various combinations of H_2- and H_∞-norms. Alternatively, it was proposed, e.g. in [29], to minimize a quadratic performance criterion ensuring a certain stability margin. The approach from [3, 23] is modified to apply to the synthesis of a reliable controller with a guaranteed stability margin. The reliable-stabilization problem [4, 5] is formulated as follows: given a plant P, find controllers C_1 and C_2 such that the closed-loop system is stable when either both controllers C_1 and C_2 or only one of them operates. In Sect. 3, this problem for a stationary system is formalized so that it can easily be generalized to a periodic system (Sect. 4).

The proposed approach for stationary systems is detailed in the first part of the paper (Sects. 1–3) and is generalized to periodic systems in the second part (Sects. 4–6). The importance of ensuring a given stability margin for a synthesized reliable controller will be demonstrated by examples.

1. Synthesis of a Controller with Feedback Gain Constraints and a Given Stability Margin. Following [1], we will outline an algorithm, based on linear matrix inequalities, for the synthesis of a stabilizing controller, including a procedure of restricting (minimizing in a sense) the norm of the feedback gain matrix. To demonstrate the algorithm, let us consider a linear stationary discrete-time system:

$$x(t+1) = Ax(t) + Bu(t), \qquad t = 0, 1, 2, \ldots, \tag{1.1}$$

where x and u are the phase and control vectors, respectively; and A and B are time-independent matrices.

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It is necessary to find the feedback gain matrix K (controller $u = Kx$) that would stabilize the system (1.1) in the sense that the eigenvalues of the matrix $A + BK$ are inside a unit circle. According to [8], this problem is reduced to the solution (determination of matrices Q and W) of the following LMIs:

$$Q = Q', \quad \begin{bmatrix} -Q & AQ + BW \\ QA' + W'B' & -Q \end{bmatrix} < 0, \tag{1.2}$$

since

$$K = WQ^{-1}. \tag{1.3}$$

Hereafter the prime denotes transposition.

The LMI (1.2) can be considered a parametrization of the set of controllers stabilizing the system (1.1). In this connection, there may be other formulations of the problem that impose different constraints on (1.2) to endow the controller with certain characteristics. Examples of such formulations are stabilization by minimum (in one sense or another (see, e.g., [10, Sect. 7.2.3])) controls and limitation or minimization of the $||K||$-norm of the feedback gain matrix, etc. (hereafter $||\bullet||$ denotes spectral norm; for example, if $\lambda(X)$ are the eigenvalues of a matrix X, then $||X|| = \sqrt{\max\lambda(XX')}$). Considering that $Q > 0$, it is computationally convenient to minimize the norm of the matrix $KQ^{1/2}$ to bound $||K||$.

One more problem is to minimize the norm of the matrix $KQ^{1/2}$ provided that the eigenvalues of the matrix of the closed-loop system are inside a circle of radius $\rho < 1$ (ρ is a guaranteed stability margin)

Let us first address the minimization problem for the norm of the matrix $KQ^{1/2}$.

Consider the following inequalities:

$$Z - KQK' > 0, \quad Z = Z' < \lambda I. \tag{1.4}$$

Hereafter I is a unit matrix and $\lambda > 0$ is a scalar.

It follows from (1.4) that $KQK' < \lambda I$. Thus, minimizing λ, we can reduce the spectral norm of the matrix $KQ^{1/2}$. According to [10], the nonlinear matrix inequalities (1.4) can be replaced with the following LMIs:

$$\begin{bmatrix} Z & W \\ W' & Q \end{bmatrix} > 0, \quad Z < \lambda I.$$

Thus, the synthesis of a stabilizing controller with matrix $KQ^{1/2}$ of minimum norm can be reduced to a standard problem for LMIs that involves the minimization of eigenvalues (eigenvalue problem [10]).

Naturally, such a procedure does not guarantee finding the minimum value of $||K||$. However, as shown in [1], it makes it possible in some cases to synthesize controllers with $||K||$ less than that obtained by minimizing $\sum_{i=0}^{\infty} u(t)'u(t)$ in the corresponding linear quadratic regulator (LQR) problem [6, 11].

Let us consider a second problem: given $\mu > 1$, find the eigenvalues of the matrix $A + BK$ that fall inside a circle of radius $\rho \le 1/\mu$. It is well known (see, e.g., [32]) that in this case there exist two matrices $Q > 0$ and $Q_1 > 0$ such that

$$\mu^2 (A + BK)'Q(A + BK) - Q = -Q_1. \tag{1.5}$$

Note that the relation (1.5) can be written in the form of LMI similar to (1.2), namely:

$$Q = Q', \quad \begin{bmatrix} -Q & \mu(AQ + BW) \\ \mu(QA + WB)' & -Q \end{bmatrix} < 0, \quad K = WQ^{-1}. \tag{1.6}$$

Thus, the problem of synthesizing a controller with guaranteed stability margin ($\rho \le 1/\mu$) and matrix $KQ^{1/2}$ of minimum norm reduces to the problem of minimizing the scalar λ when the LMIs (1.5) and (1.6) hold, i.e., a standard problem for LMIs (eigenvalue problem [10]).

2. Optimization of a Controller with a Given Stability Margin. We will modify the approach from [1, 22, 23], i.e., we will outline a procedure for the synthesis of a controller with given stability margin that is close to the regulator synthesized to minimize a quadratic performance criterion. The procedure includes the following steps:

(i) an optimal controller (minimizing a given quadratic criterion) is synthesized. In the general case, however, this controller does not ensure the stability margin of the closed-loop system. Hence the second step;

(ii) based on the results of Sect. 1, the coefficients of the controller synthesized at the first step are minimally (in a sense) adjusted to ensure the stability margin.

Consider the first step (LQR problem). Let there be given a quadratic performance criterion

$$J=\sum_{t=0}^{\infty}(x'(t)Px(t)+u(t)'Ru(t)), \qquad P\geq 0, \quad R\geq 0, \tag{2.1}$$

which is to be minimized to optimize the system (1.1), i.e., it is necessary to find a regulator (matrix K_0),

$$u(t)=K_0x(t), \tag{2.2}$$

that would minimize the criterion (2.1) on the class of stable closed-loop systems ((1.1) + (2.2)). An algorithm for the synthesis of such a regulator (selection of the matrix K_0) is well-known [11]. It is implemented by the dlqr.m MATLAB routine. Therefore, we will omit the details of solving the LQR problem governed by (1.1) and (2.1).

We will restrict ourselves to computing the value of the functional (2.1). Let $x_0=x(0)$ be initial conditions for the system (1.1) with feedback (2.2). Then (2.1) can be written as

$$J=\sum_{t=0}^{\infty}x_0'(\bar{A}')^t(P+K_0'RK_0)(\bar{A})^t x_0=x_0'Yx_0, \qquad \bar{A}=A+BK_0. \tag{2.3}$$

The matrix Y appearing in (2.3) satisfies the Lyapunov equation

$$Y-\bar{A}'Y\bar{A}=P+K_0'RK_0. \tag{2.4}$$

Considering the initial conditions (x_0) as a random vector with a zero expectation and covariance matrix $V=\langle x_0x_0'\rangle$ (where $\langle\,\rangle$ is the averaging operator), we write the following expression for the functional (2.1):

$$J=\mathrm{trace}\,(YV), \tag{2.5}$$

where Y is defined by (2.4). We will further assume that $V=I$.

Consider the second step. Let us generalize the procedure (outlined in Sect. 1) of synthesis of a controller K that guarantees the stability margin ($\rho\leq 1/\mu$) of the closed-loop system. To this end, we supplement the problem formulated in Sect. 1 with a minimality condition for

$$||(K-K_0)Q^{1/2}||, \tag{2.6}$$

which specifies how close the synthesized controller K is to the regulator K_0 (minimizing the criterion (2.1)), and replace the inequality (1.4) with

$$Z-(WQ^{-1}-K_0)Q(WQ^{-1}-K_0)'>0, \qquad Z=Z'<\lambda I. \tag{2.7}$$

The linear version of the inequality (2.4) is given by

$$\begin{bmatrix} Z & T \\ T' & Q \end{bmatrix}>0, \qquad Z<\lambda I, \qquad T=W-K_0Q. \tag{2.8}$$

The LMIs (1.6) and (2.8) determine a procedure of optimizing a controller with stability margin. This procedure is reduced to the minimization of the scalar λ in the LMIs (1.6) and (2.8), i.e., to a standard problem for LMIs.

Thus, we have outlined a procedure for synthesizing a controller that provides a given stability margin ($\rho \le 1/\mu$) of a closed-loop system and that is minimally different from the linear quadratic requlator K_0 (minimizing the criterion (2.1)).

3. Reliable Stabilization of a System with a Given Stability Margin. As indicated in the introduction, the reliable-stabilization problem is formulated in [4] as follows: given a plant P, find controllers C_1 and C_2 such that the closed-loop system is stable when either both controllers or only one of them operates. For example, if the matrix B in (1.1) has two columns (two controls), then the controllers C_1 and C_2 are defined by the first and second column of the matrix B, respectively. In this case, the reliable-stabilization problem can be formalized as follows: find a matrix K that would stabilize the system (1.1) when the matrix B has either both columns nonzero (both controllers operate) or one zero column (one of the controllers operates). Next, we will consider the reliable-stabilization problem for the system (1.1) in the following formulation.

Given a controlled plant (system (1.1)) and matrices $B_1, B_2, \ldots, B_L$ resulted from zeroing some columns of the matrix B, synthesize a controller (2.1) that would stabilize, in addition to the system (1.1), L more systems derived from (1.1) by substituting a matrix B_i $(i = 1, \ldots, L)$ for the matrix B.

Add two conditions:

(i) the reliable controller must be a little different from the optimal one (which minimizes the criterion (2.1));

(ii) all the systems defined by the matrices $B, B_1, \ldots, B_L$ must have a stability margin $\rho \le 1/\mu$, where μ is given.

Next, assuming that the reliable-stabilization problem with these conditions has a solution, we will develop a computational algorithm.

Let the system (1.1) be optimized according to the performance criterion (2.1), resulting in the matrix K_0 appearing in (2.2). Moreover, let there be given L matrices $B_1, \ldots, B_L$ obtained from the matrix B by zeroing some of its columns. Find a feedback gain matrix K that would

(i) ensure stability of the system (1.1) and all the systems derived from (1.1) by replacing the matrix B with matrices $B_1, \ldots, B_L$;

(ii) guarantee the stability margin $\rho \le 1/\mu$ to all these systems; and

(iii) be close to the matrix K_0 in the sense of (2.6).

We will formulate a solution algorithm for this problem in terms of LMIs. It is obvious that for all the systems defined by the matrices $B, B_1, \ldots, B_L$ to be stabilizable, it is necessary that the relation (1.2) hold. Also, for these systems to have the stability margin $\rho \le 1/\mu$, it is necessary that the relations (1.6) hold for each of them. For example, when $L = 2$ (in this case we have B, B_1, and B_2), for reliable stabilization of the system (1.1) with the stability margin $\rho \le 1/\mu$, the following LMIs must hold:

$$Q = Q', \quad \begin{bmatrix} -Q & \mu(AQ + BW) \\ \mu(AQ + BW)' & -Q \end{bmatrix} < 0,$$

$$\begin{bmatrix} -Q & \mu(AQ + B_1 W) \\ \mu(AQ + B_1 W)' & -Q \end{bmatrix} < 0, \quad \begin{bmatrix} -Q & \mu(AQ + B_2 W) \\ -\mu(AQ + B_2 W)' & -Q \end{bmatrix} < 0. \tag{3.1}$$

Note that the matrices Q and W satisfying (3.1) parametrize the set of controllers ($K = WQ^{-1}$) that reliably stabilize the system (1.1) with the stability margin $\rho \le 1/\mu$.

Adding (3.1) to the relations (2.8) and determining the matrices Q and W corresponding to the minimum value of λ, we obtain a controller $K = WQ^{-1}$ closest to K_0, i.e., the solution of the reliable-stabilization problem for the system (1.1) with a stability margin $\rho \le 1/\mu$.

Let us demonstrate the algorithm by an example from [5, 30].

Example [5, 30]. The matrices A and B appearing in (1.1) are defined by

$$A = \begin{bmatrix} 0.2113 & 0.0087 & 0.4524 \\ 0.0824 & 0.8096 & 0.8075 \\ 0.7599 & 0.8474 & 0.4832 \end{bmatrix}, \quad B = \begin{bmatrix} 0.6135 & 0.6538 \\ 0.2749 & 0.4899 \\ 0.8807 & 0.7741 \end{bmatrix}.$$

Let there be given matrices B_1 and B_2 derived by zeroing the first and second column B, respectively, i.e.,

TABLE 1

$1/\mu$	ρ	$\frac{\|\|K-K_0\|\|}{\|\|K_0\|\|}$	$\frac{J-J_m}{J_m}$
1.0	0.994	0.1168	0.0107
0.8	0.7706	0.4219	0.1933
0.6	0.5876	0.6446	0.5902

$$B_1 = \begin{bmatrix} 0 & 0.6538 \\ 0 & 0.4899 \\ 0 & 0.7741 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.6135 & 0 \\ 0.2749 & 0 \\ 0.8807 & 0 \end{bmatrix}.$$

Assume that the weight matrices in (2.1) are unit, i.e., $P = I$ and $R = I$. To obtain the matrix K_0, we use the dlqr.m MATLAB routine. The resulting matrix K_0 provides the following eigenvalues of the matrix of the closed-loop system $A + BK_0$:

$$\lambda(A + BK_0) = -0.4826,\ 0.4385,\ 0.2085.$$

The optimal value J_m of the functional (2.1) found according to (2.5) ($V = I$) is $J_m = 5.2447$. However, the matrix K_0 cannot be considered a solution of the reliable-stabilization problem, since it does not ensure stability of all the systems:

$$A + BK_0, \quad A + B_1K_0, \quad A + B_2K_0. \tag{3.2}$$

The reason is that the absolute value of the maximum eigenvalue of the matrices (3.2) is equal to 1.1, i.e., not all eigenvalues of these matrices fall inside a unit circle.

Using the algorithm outlined above, we will synthesize a reliable controller that would ensure stability of all the systems (3.2) with the stability margin $\rho \le 1/\mu$ and be close to K_0. Table 1 summarizes the results obtained in this example using the relations (2.8) and (3.1) for different values of $1/\mu$.

In Table 1, ρ denotes the absolute value of the maximum eigenvalue of the matrices (3.2), and J is the value of the functional (2.1) corresponding to the system $A + BK$. The closeness of $1/\mu$ and ρ can be judged from Table 1. It can also be seen that the performance of control in the main system ($A + BK$) deteriorates as the stability margin in all the systems (column $(J - J_m)/J_m$) increases. The table also demonstrates that $||K - K_0||$ increases with increase in the guaranteed stability margin.

It should be noted that this problem does not have solution for all values of μ. This is the way we can obtain a solution for $\mu = 1/\sqrt{0.29}$, but not for $\mu = 1/\sqrt{0.28}$.

4. Reliable Stabilization of a Periodic System with a Given Stability Margin. We will generalize the algorithm described in Sect. 3 to the case of a periodic (with a period τ) system:

$$x(t+1) = A(t)x(t) + B(t)u(t), \qquad t = 0, 1, \ldots, \tag{4.1}$$

where $x(t)$ is the phase vector, $u(t)$ is the control vector; $A(t)$ and $B(t)$ are periodic (with a period τ) matrices, i.e., $A(t+\tau) = A(t)$, $B(t+\tau) = A(t)\ \forall t$. Assume that in addition to sets of matrices $A(0), A(1), \ldots, A(\tau-1)$, $B(0), B(1), \ldots, B(\tau-1)$ determining the matrices $A(t)$ and $B(t)$ (their values over one period), L sets are given:

$$B_i(0),\ B_i(1),\ \ldots,\ B_i(\tau-1), \qquad i = 1, \ldots, L, \tag{4.2}$$

which are derived by zeroing some columns in the matrices $B(0), B(1), \ldots, B(\tau-1)$. With the periodicity condition, the sets (4.2) determine (in addition to (4.1)) L periodic systems:

$$x(t+1) = A(t)x(t) + B_i(t)u(t), \qquad t = 0, 1, \ldots, \tag{4.3}$$

where the periodic matrices $B_i(t)$ are defined by the sets (4.2).

Let us introduce some definitions.

(i) A controller

$$u(t) = K(t)x(t), \tag{4.4}$$

where the periodic (with a period τ) matrix $K(t)$ stabilizes the system (4.1) if the eigenvalues of the matrix (the monodromy matrix of the system (4.1) closed by the controller (4.4))

$$\Phi = (A(\tau-1)+B(\tau-1)K(\tau-1))\ldots(A(0)+B(0)K(0)) \tag{4.5}$$

fall inside a unit circle.

(ii) The controller (4.4) stabilizes the system (4.1) with a stability margin $\rho < 1$ if the eigenvalues of the matrix (4.5) lie inside a circle of radius ρ.

(iii) The controller (4.4) reliably stabilizes the system (4.1) if it also stabilizes the systems (4.2).

(iv). The controller (4.4) reliably stabilizes the system (4.1) with the stability margin ρ if it stabilizes the systems (4.1) and (4.3) with the stability margin ρ.

Let us look into the synthesis of such controllers. First, we will address the stabilization problem for the system (4.1). According to [8], the set of controllers (4.4) that stabilizes the system (4.1) is defined by the following periodic LMIs (PLMIs) in which the argument t is omitted:

$$\begin{bmatrix} -\sigma Q & AQ+BW' \\ (AQ+BW)' & -Q \end{bmatrix} < 0, \qquad t = 0,\ldots,\tau-1. \tag{4.6}$$

Hereafter σ is the shift operator ($\sigma Q(t) = Q(t+1)$).

If the matrices W and Q exist and satisfy the conditions

$$W(t+\tau) = W(\tau),$$

$$Q(t+\tau) = Q(t) = Q'(t) > 0 \ \ \forall t, \tag{4.7}$$

then the feedback gain matrix K appearing in (4.4) and stabilizing the system (4.1) is given by

$$K = W'Q^{-1}. \tag{4.8}$$

Let us now address the stabilization problem for the system (4.1) with the stability margin $\rho < 1$. Consider the following PLMIs (an analog of the LMIs (1.6)):

$$\begin{bmatrix} -\sigma Q & \mu\,(AQ+BW') \\ \mu\,(AQ+BW')' & -Q \end{bmatrix} < 0, \qquad t = 0,\ldots,\tau, \qquad \mu > 1. \tag{4.9}$$

If the PLMIs (4.9) have solutions that satisfy (4.7), then the controller (4.8) stabilizes the system (4.1) with the stability margin $\rho \le 1/\mu^{\tau}$. Indeed, it is arguable that the PLMIs (4.9) correspond to the periodic system

$$x(t+1) = \mu\left(A(t)+B(t)u(t)\right). \tag{4.10}$$

The monodromy matrix of the system (4.10) stabilized by the controller (4.8) is given by

$$\Phi_{\mu} = \mu\left(A(\tau-1)+B(\tau-1)K(\tau-1)\right)\cdot\mu\left(A(\tau-2)+B(\tau-2)K(\tau-2)\right)\cdot\ldots\cdot\mu\left(A(0)+B(0)K(0)\right)$$

$$= \mu^{\tau}\left(A(\tau-1)+B(\tau-1)K(\tau-1)\right)\cdot\left(A(\tau-2)+B(\tau-2)K(\tau-2)\right)\cdot\ldots\cdot\left(A(0)+B(0)K(0)\right). \tag{4.11}$$

Since the conditions (4.9) and (4.7) place the eigenvalues of the matrix Φ_{μ} defined by (4.11) inside a unit circle, it is arguable that the matrix (4.5) has eigenvalues lying inside a circle of radius $\rho \le 1/\mu^{\tau}$, i.e., the synthesis of the controller that stabilizes the system (4.1) with the stability margin $\rho \le 1/\mu^{\tau}$ is reduced to finding the solution of the PLMIs (4.9) that satisfies the conditions (4.7).

Let us finally address the reliable-stabilization problem of the system (4.1) with a stability margin ρ. For the controller (4.8) to reliably stabilize the system (4.1) with the stability margin $\rho \le 1/\mu^{\tau}$, it is necessary that the PLMIs (4.9) hold both for the system (4.1) and for all the systems (4.3), i.e., it is necessary to find matrices W and Q that satisfy the conditions (4.7) and the following PLMIs:

$$\begin{bmatrix} -\sigma Q & \mu\,(AQ+B_i W') \\ \mu\,(AQ+B_i W')' & -Q \end{bmatrix} < 0, \qquad \begin{matrix} t=0,1,\ldots,\tau-1, \\ i=1,\ldots,L. \end{matrix} \tag{4.12}$$

Thus, if the matrices W and Q satisfying (4.7) and (4.12) have been found, then the controller (4.8) would reliably stabilize the system (4.1) with the stability margin $\rho \le 1/\mu^{\tau}$. In other words, the solutions of (4.12) satisfying (4.7) parametrize the set of controllers (4.8) that reliably stabilize the system (4.1) with the stability margin $\rho \le 1/\mu^{\tau}$. The next step of the synthesis procedure is to choose out of this set a controller optimal in one sense or another (see Sect. 5).

5. Optimization of a Reliable Periodic Controller. Let us generalize the synthesis algorithms outlined in Sects. 2 and 3 to periodic systems. We will start with the synthesis of a periodic analog of the controller (2.2). Let the system (4.1) be optimized by minimizing the quadratic performance criterion

$$J=\sum_{t=0}^{\infty}\left(x'(t)P(t)x(t)+u'(t)R(t)u(t)\right), \tag{5.1}$$

where $P(t)=P'(t)\ge 0$ and $R(t)=R'(t)\ge 0$ are periodic matrices with a period τ ($P(t+\tau)=P(t)$ and $R(t+\tau)=R(t)$), i.e., it is required to find a controller (periodic matrix $K_0(t)$ with period τ)

$$u(t)=K_0(t)x(t) \tag{5.2}$$

that minimizes the criterion (5.1) on the class of stability of the closed-loop systems (4.1) and (5.2).

According to [6, 11], the matrix $K_0(t)$ is defined by

$$K_0(t)=-\left[B'(t)S(t+1)B(t)+R(t)\right]^{-1}B(t)S(t+1)A(t), \tag{5.3}$$

where the set $S(t)$ of periodic (with period τ) symmetric matrices satisfies the recurrent relation

$$S(t)=A(t)\left[S(t+1)-S(t+1)B(t)\left(R(t)+B'(t)S(t+1)B(t)\right)^{-1}S(t+1)\right]A(t)+P(t). \tag{5.4}$$

The set (5.4) with the periodicity condition $S(t+\tau)=S(t)$ will be called, as in [7], periodic discrete Riccati equation (PDRE). Algorithms of solving PDREs are described in [2, 5, 7, 12, 15, 16, 28].

Let us outline the algorithm from [2]. The solution of the PDRE (set of matrices $S(1),\ldots,S(\tau)$) can be found from the following problem: maximize

$$\operatorname{trace}\left(S(1)+S(2)+\ldots+S(\tau)\right) \tag{5.5}$$

given the following LMIs:

$$S(1)\ge 0,\quad S(2)\ge 0,\ \ldots,\quad S(\tau)\ge 0,$$

$$\begin{bmatrix} A'(1)S(2)A(1)-S(1)+P(1) & A'(1)S(2)B(1) \\ B'(1)S(2)A(1) & R(1)+B'(1)S(2)B(1) \end{bmatrix}\ge 0,$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{bmatrix} A'(\tau)S(1)A(\tau)-S(\tau)+P(\tau) & A'(\tau)S(1)B(\tau) \\ B'(\tau)S(1)A(\tau) & R(\tau)+B'(\tau)S(1)B(\tau) \end{bmatrix}\ge 0. \tag{5.6}$$

This problem (maximization of (5.5) with constraints (5.6)) can be solved with the help of the mincx.m MATLAB routine [14].

Thus, after solving the problem (5.5), (5.6), the relations (5.3) define the set of periodic matrices $K_0(t)$ that optimally stabilize the system (4.1) according to the criterion (5.1).

Let us synthesize a reliable controller for the system (4.1). We choose, out of the set of controllers defined by (4.7)–(4.9), a controller close to the controller $K_0(t)$ defined by (5.3). Considering that $Q > 0$, we will formalize the requirement that the controllers (4.8) and $K_0(t)$ be close, as in [3]. Let us introduce a symmetric matrix $Z(t) > 0$ and a scalar λ:

$$Z(t) - (W'(t)Q^{-1}(t) - K_0(t))Q(t)(W'(t)Q^{-1}(t) - K_0(t))' > 0,$$

$$Z(t) = Z'(t) < \lambda I, \qquad t = 0, 1, \ldots, \tau - 1. \tag{5.7}$$

Minimizing the scalar λ in (5.7) provided that the PLMIs (4.12) hold, we can make the matrix (4.8) tend to the matrix $K_0(t)$. Let us rearrange the nonlinear inequalities (5.7) into LMIs for $Z(t)$, $W(t)$, and $Q(t)$ (the argument t will be omitted; if the inequality will have matrices with different values of t, then this will be identified by the shift operator σ)

$$\begin{bmatrix} Z & V \\ V & Q \end{bmatrix} > 0, \quad \operatorname{diag}\{Z, \sigma Z, \ldots, \sigma^{\tau-1} Z\} < \lambda I, \quad V = W' - K_0 Q. \tag{5.8}$$

Based on the aforesaid, we can now formulate the following algorithm for the synthesis of a reliable controller with a given stability margin for the periodic system (4.1), this controller being closest (in the sense of (5.7)) to the optimal (in the sense of (5.1)) one. Namely, after determination of the matrix $K_0(t)$ from the solution of the problem (5.5), (5.6), such a controller can be found by minimizing the scalar λ in (5.8) if the PLMIs (4.12) hold.

To avoid possible computational difficulties, it might be reasonable to introduce, as in [1], a matrix $-I$ into the right-hand side of (4.12):

$$\begin{bmatrix} -\sigma Q & \mu(AQ + B_i W') \\ \mu(AQ + B_i W')' & -Q \end{bmatrix} < -I, \tag{5.9}$$

$$t = 1, \ldots, \tau, \qquad i = 1, \ldots, L.$$

In contrast to (4.12), the PLMIs (5.9) have the matrix $-I$ on the right-hand side. This modification is due to the following: the PLMIs (4.12) are homogeneous in Q and W, i.e., if Q_* and W_* satisfy these LMIs, then the matrices εQ_* and εW_* satisfy them too for any $\varepsilon > 0$. It is important that the matrix $K = W_*' Q_*^{-1}$ does not depend on ε. This fact (the homogeneity of (4.12)) may make it difficult to solve the problem, since in this case the minimization of λ does not necessarily mean minimization of $VQ^{-1}V'$. Indeed, for arbitrary $\lambda > 0$ and arbitrary pair of matrices Q_* and W_* satisfying (4.12), there always exists a sufficiently small value of $\varepsilon > 0$ such that the PLMIs (4.12) hold if $W = \varepsilon W_*$ and $Q = \varepsilon Q_*$. This is impossible because of the matrix $-I$ on the right-hand side of the PLMN (5.9), which provides the minimization of the matrix $VQ^{-1}V'$ for minimum λ.

6. Example of a Reliable Controller with a Given Stability Margin for a Periodic System. Let us illustrate the algorithm outlined in Sect. 5 by an example of a stabilization system for a hopping machine [19]. To this end, we will borrow the initial data of Example 1 from [19]. According to these data, the system (4.1) has period 2, i.e., $\tau = 2$, and the matrices $A(0)$, $A(1)$, $B(0)$, and $B(1)$ are defined by

$$A(0) = \begin{bmatrix} 1.4838 & 0.0002 & 0.0002 & -0.0547 & 0.2022 \\ 0 & 1.0000 & 0.1747 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 5.9480 & 0.0026 & 0.0026 & -0.7115 & 1.4862 \end{bmatrix},$$

$$A(1)=\begin{bmatrix} 1.0003 & 0.0006 & 0.0006 & 0.0905 & 0.3256 \\ 0 & 1.0000 & 0.3253 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0.0018 & 0.0035 & 0.0035 & 0.5891 & 1.0018 \end{bmatrix},$$

$$B(0)=\begin{bmatrix} -0.4066 & 0.3267 & 0.0167 & -0.0023 \\ 0 & 0 & 0 & 0.0147 \\ 0 & 0 & 0 & 0.1687 \\ 2.2261 & 2.5068 & 0 & 0.1551 \\ -5.3757 & 3.8351 & 0.2055 & -0.0516 \end{bmatrix}, \quad B(1)=\begin{bmatrix} -0.3299 \\ 0 \\ 0 \\ 0 \\ -1.9132 \end{bmatrix}.$$

The matrices $P(t)$ and $R(t)$ appearing in (5.1) are given by

$$P(0)=P(1)=\mathrm{diag}\{10^4, \ 10, \ 10^{-1}, \ 10^2, \ 10^2\},$$

$$R(0)=\mathrm{diag}\{4\cdot 10^2, \ 4\cdot 10^2, \ 0, \ 7, \ 10^{-2}\}, \quad R(1)=2\cdot 10^3.$$

The controller synthesized from these data (the matrix $K_0(t)$ in (5.2)) is as follows:

$$K_0(0)=\begin{bmatrix} 0.6038 & 0.0938 & 0.1182 & -0.1490 & 0.1206 \\ -0.4282 & 0.0966 & 0.1223 & -0.0996 & -0.1011 \\ -17.7789 & -0.1049 & -0.1233 & 0.8524 & -3.2036 \\ 0.9377 & -3.3184 & -3.9446 & -2.4637 & 0.1348 \end{bmatrix},$$

$$K_0(1)=[1.3405 \quad 0.0011 \quad 0.0007 \quad 0.1729 \quad 0.5390]. \tag{6.1}$$

These data are given in [19], where it is pointed out that the eigenvalues of the matrix (4.5), i.e., of the matrix $(A(1)+B(1)K_0(1))\ (A(0)+B(0)K_0(0))$, are the following: $0.5714\pm 0.2684i$, –0.00699, –0.0195, and 0.0018. All these eigenvalues lie inside a unit circle, and, hence, the closed-loop system is asymptotically stable.

Let us synthesize a reliable controller with a given stability margin for this system. To this end, we specify the set (4.2). Let $L=1$ and the matrices $B_1(0)$ and $B_1(1)$ be given by

$$B_1(0)=\begin{bmatrix} -0.4066 & 0 & 0.0167 & -0.0023 \\ 0 & 0 & 0 & 0.0147 \\ 0 & 0 & 0 & 0.1687 \\ 2.2261 & 0 & 0 & 0.1551 \\ -5.3757 & 0 & 0.2055 & -0.0516 \end{bmatrix}, \quad B_1(1)=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.2}$$

Note that the controller defined by (6.1) does not stabilize the system (4.3) in which the matrices $B_i(t)$ are defined by (6.2). Indeed, the maximum eigenvalue of the matrix $(A(1)+B_1(1)K_0(1))\ (A(0)+B_1(0)K_0(0))$ is equal to 1.4336, i.e., lies outside a unit circle. Using the algorithm for different values of $1/\mu^2$, we have synthesized a reliable controller. The results are summarized in Table 2 (ρ is the maximum (in absolute value) eigenvalue of the matrices $(A(1)+B(1)K(1))(A(0)+B(0)K(0))$ and $(A(1)+B_1(1)K(1))(A(0)+B_1(0)K(0))$).

The feedback gain matrix $K(t)$ is obtained from the solution of the problem (5.8), (5.9) for a given value of μ.

As follows from Tables 1 and 2, when $\mu=1$ (no restrictions for the stability margin), the system has a small stability margin (the value of ρ is very close to 1). Thus, the results indicate the expediency of synthesizing a reliable controller with a guaranteed stability margin.

Conclusions. An algorithm has been proposed for synthesizing a reliable controller with a given stability margin for linear stationary and periodic systems optimized by minimizing a quadratic performance criterion. The essence of the algorithm

TABLE 2

$1/\mu^2$	ρ	$\frac{\|\|K(0)-K_0(0)\|\|}{\|\|K_0(0)\|\|}$	$\frac{\|\|K(1)-K_0(1)\|\|}{\|\|K_0(1)\|\|}$
1.0	0.9722	0.0188	0.1387
0.8	0.7637	0.410	0.3091
0.6	0.5441	0.1220	0.5732

is to synthesize a reliable controller that guarantees a given stability margin and is close to the linear quadratic regulator. The importance of ensuring a given stability margin has been demonstrated. The algorithm is based on methods associated with linear matrix inequalities and can be implemented by standard MATLAB routines. As an example, a reliable controller has been synthesized to stabilize the program motion of a hopping machine.

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