# SPLINE-APPROXIMATION METHOD APPLIED TO SOLVE NATURAL-VIBRATION PROBLEMS FOR RECTANGULAR PLATES OF VARYING THICKNESS

**A. Ya. Grigorenko and T. L. Efimova**

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**The natural vibrations of anisotropic rectangular plates of varying thickness with complex boundary conditions are studied using the spline-collocation and discrete-orthogonalization methods. The basic principles of the approach are outlined. The natural vibrations of orthotropic plates with parabolically varying thickness are calculated. The results (natural frequencies and modes) obtained with different boundary conditions are analyzed**

**Keywords:** anisotropic rectangular plates, spline-collocation method, discrete-orthogonalization method, natural frequencies and modes

**Introduction.** Plates of varying thickness are widely used in structures of various designations. To design them, it is necessary to determine the natural frequencies and modes with high accuracy, which are needed to describe the response of plates to the operating conditions. For plates with constant thickness and hinged opposite edges, the solution can be constructed in a closed form [5, 13]. With boundary conditions of other types, however, it is impossible to obtain a similar solution for natural vibrations of elastic plates. The natural vibrations of orthotropic plates with such boundary conditions were studied quite actively, which was reflected in a number of publications. The solutions for forced and natural vibrations of orthotropic plates were obtained in [21] in the form of double trigonometric series. Lagrangian multipliers were used in [20] to solve a similar problem with allowance for shear strains in several first modes. The superposition method was used in [14] to table natural frequencies for a certain range of stiffness ratios. In [23], the superposition method and affine transformation were used to determine the natural frequencies of orthotropic plates partially clamped and partially simply supported. The Kantorovich method was used in [7] to study the natural vibrations of clamped plates. The natural vibrations of complex anisotropic plates were studied in [2, 9] using variational methods and the R-function method. The natural vibrations of rectangular plates of varying thickness were addressed by many authors. For example, the paper [12] is concerned with the general natural-vibration problem for plates of varying thickness. The transverse vibrations of plates with exponentially varying thickness are studied in [10] and inhomogeneous rectangular plates with parabolically varying thickness in [22]. The natural vibrations of simply supported plates with linearly varying thickness were investigated in [6, 9, 11, 19].

Thus, we may conclude that there is a variety of approximate approaches to natural-vibration problems for anisotropic rectangular plates with boundary conditions that do not allow closed-form solutions. Recently, computational mathematics, mathematical physics, and mechanics have widely employed spline functions to solve such problems. This is due to the following advantages of the spline-approximation method over the other ones: stability of splines against local perturbations, i.e., the behavior of a spline in the neighborhood of a point does not affect the overall behavior of the spline (as polynomial approximation does, for example); better convergence of spline-interpolation compared with polynomial interpolation; and simple and convenient computer implementation of spline algorithms. The use of spline functions in variational, projective, and other discrete–continuous methods allows us to obtain appreciable results as contrasted to the use of classical polynomials, to simplify substantially their numerical implementation, and to obtain a highly accurate solution.

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Kiev. Translated from Prikladnaya Mekhanika, Vol. 41, No. 10, pp. 90–99, October 2005. Original article submitted December 13, 2004.

 

In the present paper, we extend the spline-collocation method proposed in [1, 15] to the natural vibrations of rectangular orthotropic plates of varying thickness with complex boundary conditions. This method was also used in [16–18] to analyze the stress–strain state of elastic bodies.

**1. Basic Relations (Constitutive Equations).** Let us solve the natural-vibration problem for a rectangular orthotropic plate of varying thickness $h(x, y)$ in a rectangular coordinate system (the coordinate plane $xOy$ is the mid-surface of the plate).

Within the framework of Kirchhoff–Love theory, the vibration equations can be written [4, 5] as

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=\rho h\frac{\partial^2 w}{\partial t^2}, \quad \frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=Q_x, \quad \frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}=Q_y, \tag{1}$$

where $x$ and $y$ are the Cartesian coordinates ($0 \le x \le a$ and $0 \le y \le b$); $t$ is time; $w$ is the deflection of the plate; and $\rho$ is the density of its material.

The moments $M_x$, $M_y$, and $M_{xy}$ and the shear forces $Q_x$ and $Q_y$ satisfy the relations

$$M_x=-\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right), \quad Q_x=-\left[D_{11}\frac{\partial^3 w}{\partial x^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x\partial y^2}\right],$$

$$M_y=-\left(D_{12}\frac{\partial^2 w}{\partial x^2}+D_{22}\frac{\partial^2 w}{\partial y^2}\right), \quad Q_y=-\left[D_{22}\frac{\partial^3 w}{\partial y^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x^2\partial y}\right], \quad M_{xy}=-2D_{66}\frac{\partial^2 w}{\partial x\partial y}, \tag{2}$$

where the stiffness characteristics $D_{ij}$ of the plate are defined by

$$D_{ij}=\frac{B_{ij}h^3(x,y)}{12} \qquad \left(B_{11}=\frac{E_1}{1-\nu_1\nu_2},\ B_{12}=\frac{\nu_2 E_1}{1-\nu_1\nu_2}=\frac{\nu_1 E_2}{1-\nu_1\nu_2},\ B_{22}=\frac{E_2}{1-\nu_1\nu_2},\ B_{66}=G_{12}\right).$$

Here $E_1, E_2, G_{12}, \nu_1$, and $\nu_2$ are the elastic and shear moduli and Poisson's ratios.

The system of equations (1)–(2) yields an equivalent differential equation for the deflection $w$:

$$D_{11}\frac{\partial^4 w}{\partial x^4}+D_{22}\frac{\partial^4 w}{\partial y^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+2\frac{\partial D_{11}}{\partial x}\frac{\partial^3 w}{\partial x^3}+2\frac{\partial D_{22}}{\partial y}\frac{\partial^3 w}{\partial y^3}+2\frac{\partial}{\partial y}(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x^2\partial y}$$

$$+2\frac{\partial}{\partial x}(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x\partial y^2}+\left(\frac{\partial^2 D_{11}}{\partial x^2}+\frac{\partial^2 D_{12}}{\partial y^2}\right)\frac{\partial^2 w}{\partial x^2}+\left(\frac{\partial^2 D_{12}}{\partial x^2}+\frac{\partial^2 D_{22}}{\partial y^2}\right)\frac{\partial^2 w}{\partial y^2}+4\frac{\partial^2 D_{66}}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\rho h\frac{\partial^2 w}{\partial t^2}=0. \tag{3}$$

It is assumed that all points of the plate vibrate harmonically with a frequency $\omega$, i.e., $w(x,y,z)=\tilde{w}(x,y)e^{i\omega t}$ (the symbol "~" is omitted hereafter).

Let boundary conditions expressed in terms of the deflection be specified at the edges $x = 0$, $x = a$, $y = 0$, and $y = b$. For $y$ = const, we will consider the following boundary conditions:

$$w=0, \quad \frac{\partial w}{\partial y}=0 \quad \text{at} \quad y=0, \quad y=b \quad \text{(clamped)}; \tag{4}$$

$$w=0, \quad \frac{\partial^2 w}{\partial y^2}=0 \quad \text{at} \quad y=0, \quad y=b \quad \text{(hinged)}; \tag{5}$$

$$w=0, \quad \frac{\partial^2 w}{\partial y^2}=0 \quad \text{at} \quad y=0 \quad \text{(hinged)} \quad \text{and} \quad w=0, \quad \frac{\partial w}{\partial y}=0 \quad \text{at} \quad y=b \quad \text{(clamped)}. \tag{6}$$

Similar conditions can be specified at the boundaries $x$ = const.

**2. Solution Method.** We will search for the solution of Eq. (3) in the form

$$w = \sum_{i=0}^{N} w_i(x)\psi_i(y), \tag{7}$$

where $w_i(x)\ (i = \overline{1, N})$ are unknown functions, and $\psi_i(y)$ are functions constructed using quintic B-splines ($N \geq 6$).

The functions $\psi_i(y)$ are selected so as to satisfy the boundary conditions for $y$ = const using linear combinations of B-splines:

$$\psi_0(y) = \alpha_{11}B_5^{-2}(y) + \alpha_{12}B_5^{-1}(y) + B_5^0(y),$$

$$\psi_1(y) = \alpha_{21}B_5^{-1}(y) + \alpha_{22}B_5^0(y) + B_5^1(y),$$

$$\psi_2(y) = \alpha_{31}B_5^{-2}(y) + \alpha_{32}B_5^0(y) + B_5^2(y),$$

$$\psi_i(y) = B_5^i(y), \qquad i = 3, 4, \ldots, N-3,$$

$$\psi_{N-2}(y) = \beta_{31}B_5^{N+2}(y) + \beta_{32}B_5^N(y) + B_5^{N+2}(y),$$

$$\psi_{N-1}(y) = \beta_{21}B_5^{N+1}(y) + \beta_{22}B_5^N(y) + B_5^{N-1}(y),$$

$$\psi_N(y) = \beta_{11}B_5^{N+2}(y) + \beta_{12}B_5^{N+1}(y) + B_5^N(y), \tag{8}$$

where $B_5^i(y)$ ($i = -2, \ldots, N+2$ is spline number) are splines constructed on a uniform mesh $\Delta$ with a spacing $h_y$: $y_{-5} < y_{-4} < \ldots < y_N < \ldots < y_{N+5}$, $y_0 = 0$, and $y_N = b$:

$$B_5^i(y) = \frac{1}{120}\begin{cases} 0, & -\infty < y < y_{i-3}, \\ z^3, & y_{i-3} \leq y < y_{i-2}, \\ -5z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1, & y_{i-2} \leq y < y_{i-1}, \\ 10z^5 - 20z^4 - 20z^3 + 20z^2 + 50z + 26, & y_{i-1} \leq y < y_i, \\ -10z^5 + 30z^4 - 60z^2 + 66, & y_i \leq y < y_{i+1}, \\ 5z^5 - 20z^4 + 20z^3 + 20z^2 - 50z + 26, & y_{i+1} \leq y < y_{i+2}, \\ (1-z)^5, & y_{i+2} \leq y < y_{i+3}, \\ 0, & y_{i+3} \leq y < \infty, \end{cases}$$

where $z = (y - y_k)/h_y$ on the interval $[y_k, y_{k+1}]$, $k = i-3, \ldots, i+2$, $i = -3, \ldots, N+2$, $h_y = y_{k+1} - y_k$ = const, $\alpha_{ij}$ and $\beta_{ij}$ ($i = 1, 2, 3$, $j = 1, 2$) are constant coefficients that depend on the boundary conditions at $y = b$ and $y = a$.

Let

$$A_\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{bmatrix}, \qquad A_\beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix}.$$

Then we have

$$A_\alpha = A_\beta = \begin{bmatrix} \frac{165}{4} & -\frac{33}{8} \\ 1 & -\frac{26}{33} \\ 1 & -\frac{1}{33} \end{bmatrix}$$

when the edges $y=0$ and $y=b$ are clamped,

$$A_\alpha = A_\beta = \begin{bmatrix} 12 & -3 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}$$

when they are hinged, and

$$A_\alpha = \begin{bmatrix} \frac{165}{4} & -\frac{33}{8} \\ 1 & -\frac{26}{33} \\ 1 & -\frac{1}{33} \end{bmatrix}, \qquad A_\beta = \begin{bmatrix} 12 & -3 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}$$

under the conditions (6). We now write Eq. (3) as

$$\frac{\partial^4 w}{\partial x^4} = a_1 \frac{\partial^3 w}{\partial x^3} + a_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + a_3 \frac{\partial^3 w}{\partial x^2 \partial y} + a_4 \frac{\partial^2 w}{\partial x^2} + a_5 \frac{\partial^3 w}{\partial x \partial y^2} + a_6 \frac{\partial^2 w}{\partial x \partial y} + a_7 \frac{\partial^4 w}{\partial y^4} + a_8 \frac{\partial^3 w}{\partial y^3} + a_9 \frac{\partial^2 w}{\partial y^2} + a_{10} w, \tag{9}$$

where $a_i = a_i(x, y)$, $i = 1, 2, ..., 9$, $a_{10} = a_{10}(x, y, \omega)$.

We substitute (7) into Eq. (9) and require that it be satisfied at given collocation points $\xi_k \in [0, b]$, $k = \overline{0, N}$. Let us consider the case where the number of mesh nodes is even, i.e., $N = 2n+1$ $(n \geq 3)$, and $\xi_{2i} \in [y_{2i}, y_{2i+1}]$, $\xi_{2i+1} \in [y_{2i}, y_{2i+1}]$ $(i = 0, 1, \dots, n)$. The interval $[y_{2i}, y_{2i+1}]$ has two collocation points, and the adjacent intervals $[y_{2i+1}, y_{2i+2}]$ do not have such points. Within each of the intervals $[y_{2i}, y_{2i+1}]$, collocation points are selected as follows: $\xi_{2i} = y_{2i} + z_1 h_y$, $\xi_{2i+1} = y_{2i} + z_2 h_y$ $(i = 0, 1, 2, \dots, n)$, where $z_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$ and $z_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$ are the roots of a quadratic Legendre polynomial on the interval [0, 1]. Such collocation points are optimal and substantially increase the accuracy of approximation. As a result, we obtain a system of $N+1$ linear differential equations for $w_i$. If

$$\Psi_j = [\psi_i^{(j)}(\xi_k)], \quad k, i = 0, \dots, N, \quad j = 0, \dots, 4,$$

$$\bar{w} = \{w_0, w_1, \dots, w_N\}^T, \quad \bar{a}_r^T = \{a_r(x, \xi_0), a_r(x, \xi_1), \dots, a_r(x, \xi_N)\}, \quad r = 1, \dots, 9,$$

$$\bar{a}_{10}^T = \{a_{10}(x, \xi_0, \omega), a_{10}(x, \xi_1, \omega), \dots, a_{10}(x, \xi_N, \omega)\},$$

and $\bar{c} * A$ denotes the matrix $[c_i a_{ij}]$, where $\bar{c} = \{c_0, c_1, \dots, c_N\}^T$ and $A = [a_{ij}]$ ($i, j$ = 0, ..., $N$), then the system of differential equations becomes

$$\bar{w}^{\mathrm{IV}} = \Psi_0^{-1}(\bar{a}_7 * \Psi_4 + \bar{a}_8 * \Psi_3 + \bar{a}_9 * \Psi_2 + \bar{a}_{10} * \Psi)\bar{w}$$

$$+ \Psi_0^{-1}(\bar{a}_5 * \Psi_2 + \bar{a}_6 * \Psi_1)\bar{w}' + \Psi_0^{-1}(\bar{a}_2 * \Psi_3 + \bar{a}_3 * \Psi_1 + \bar{a}_4 * \Psi_0)\bar{w}'' + \Psi_0^{-1}(\bar{a}_1 * \Psi_0)\bar{w}'''. \tag{10}$$

This system can be normalized:

$$\frac{d\bar{Y}}{dx} = A(x, \omega)\bar{Y} \qquad (0 \leq x \leq a), \tag{11}$$

where $\bar{Y} = \{w_1, w_2, \dots, w_{N+1}, w_1', w_2', \dots, w_{N+1}', w_1'', w_2'', \dots, w_{N+1}'', w_1''', w_2''', \dots, w_{N+1}'''\}^T$, $w_K^{(I)} = w^{(I)}(x, \xi_K)$, $K = 1, \dots, N+1$, $I = 1, 2, 3$, and $A(x, \omega)$ is a square matrix of order $(N+1) \times (N+1)$.

The boundary conditions for this system can be expressed as

$$B_1 \bar{Y}(0) = \bar{0}, \qquad B_2 \bar{Y}(a) = \bar{0}. \tag{12}$$

TABLE 1

| $\alpha$ | $\bar{\omega}_i$ | N | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 10 | 12 | 14 | 16 | 18 | 20 | 22 |
| 0 | $\bar{\omega}_1$ | 61.139 | 61.132 | 61.129 | 61.127 | 61.127 | 61.127 | 61.127 |
| | $\bar{\omega}_2$ | 107.188 | 107.066 | 107.016 | 106.994 | 106.982 | 106.976 | 106.972 |
| | $\bar{\omega}_3$ | 142.550 | 142.537 | 142.532 | 142.530 | 142.529 | 142.528 | 142.528 |
| 0.3 | $\bar{\omega}_1$ | 62.108 | 62.102 | 62.099 | 62.099 | 62.098 | 62.098 | 62.098 |
| | $\bar{\omega}_2$ | 97.737 | 97.637 | 97.598 | 97.580 | 97.570 | 97.566 | 97.562 |
| | $\bar{\omega}_3$ | 145.289 | 145.276 | 145.271 | 145.269 | 145.268 | 145.268 | 145.267 |

To solve the eigenvalue problem for the system of ordinary differential equations (11) with the boundary conditions (12), we will combine discrete orthogonalization with incremental search [2].

**3. Numerical Solution.** We will use the proposed approach to study the spectrum of natural vibrations of a square plate with varying thickness and different boundary conditions. The thickness of the plate varies by the formula

$$h(x)=[\alpha(6x^2-6x+1)+1]h_0. \quad (13)$$

The material of the plate is orthotropic (fiberglass fabric) with Young's moduli $E_1=4.76\cdot 10^4$ MPa and $E_2=2.07\cdot 10^4$ MPa, shear modulus $G_{12}=0.531\cdot 10^4$ MPa, and Poisson's ratios $\nu_1=0.149$ and $\nu_1=0.0647$.

The dimensionless frequencies $\bar{\omega}=\omega a^2(\rho h/D_0)^{1/2}$ ($D_0=(h_0^3/12)\cdot 10^4$ MPa) of the clamped plate determined by the spline-collocation method with different number of collocation points ($N$ = 10, 12, 14, 16, 18, 20, 22) differ a little (Table 1).

Table 2 collect dimensionless frequencies $\bar{\omega}_i$ $(i=1,2,3)$ (ordered by value) of the orthotropic square plate for $\alpha\le 0$ and $\alpha>0$, respectively, and the following boundary conditions:

(i) all edges are clamped (boundary conditions of type A)

$$w=0,\quad \frac{\partial w}{\partial y}=0 \quad \text{at} \quad y=0,\quad y=a,$$

$$w=0,\quad \frac{\partial w}{\partial x}=0 \quad \text{at} \quad x=0,\quad x=a,$$

(ii) three edges are clamped and the fourth one is hinged (boundary conditions of type B)

$$w=0,\quad \frac{\partial w}{\partial y}=0 \quad \text{at} \quad y=a,$$

$$w=0,\quad \frac{\partial^2 w}{\partial y^2}=0 \quad \text{at} \quad y=0,$$

$$w=0,\quad \frac{\partial w}{\partial x}=0 \quad \text{at} \quad x=0,\quad x=a$$

TABLE 2

| Boundary conditions | $\overline{\omega}_i$ | α | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | –0.5 | –0.4 | –0.3 | –0.2 | –0.1 | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| A | $\overline{\omega}_1$ | 58.375 | 59.012 | 59.605 | 60.159 | 60.674 | 61.139 | 61.542 | 61.871 | 62.108 | 62.237 | 62.238 |
| | $\overline{\omega}_2$ | 121.526 | 119.010 | 116.240 | 113.311 | 110.281 | 107.188 | 104.058 | 100.904 | 97.737 | 94.557 | 91.364 |
| | $\overline{\omega}_3$ | 124.339 | 129.656 | 134.030 | 137.585 | 140.405 | 142.550 | 144.062 | 144.968 | 145.289 | 145.035 | 144.212 |
| B | $\overline{\omega}_1$ | 50.227 | 51.605 | 52.905 | 54.129 | 55.270 | 56.320 | 57.266 | 58.095 | 58.791 | 59.339 | 59.724 |
| | $\overline{\omega}_2$ | 102.334 | 100.723 | 98.953 | 97.083 | 95.147 | 93.166 | 91.149 | 89.104 | 87.026 | 84.911 | 82.749 |
| | $\overline{\omega}_3$ | 121.396 | 126.863 | 131.384 | 135.083 | 138.045 | 140.330 | 141.977 | 143.017 | 143.469 | 138.723 | 131.720 |
| C | $\overline{\omega}_1$ | 52.733 | 52.211 | 51.624 | 50.995 | 50.337 | 49.659 | 48.965 | 48.257 | 47.535 | 46.801 | 46.051 |
| | $\overline{\omega}_2$ | 109.151 | 112.403 | 110.490 | 107.334 | 104.080 | 100.783 | 97.477 | 94.187 | 90.928 | 87.713 | 84.546 |
| | $\overline{\omega}_3$ | 116.222 | 113.496 | 114.497 | 116.613 | 117.777 | 118.403 | 118.535 | 118.208 | 117.477 | 116.269 | 114.688 |
| D | $\overline{\omega}_1$ | 43.939 | 44.009 | 43.998 | 43.923 | 43.790 | 43.607 | 43.376 | 43.098 | 42.774 | 42.402 | 41.982 |
| | $\overline{\omega}_2$ | 97.299 | 95.178 | 92.905 | 90.549 | 88.155 | 85.755 | 83.369 | 81.013 | 78.694 | 76.420 | 74.190 |
| | $\overline{\omega}_3$ | 105.958 | 109.322 | 111.879 | 113.750 | 115.019 | 115.748 | 115.981 | 115.752 | 115.085 | 113.998 | 112.506 |
| E | $\overline{\omega}_1$ | 48.701 | 47.412 | 46.059 | 44.676 | 43.289 | 41.918 | 40.583 | 39.297 | 38.077 | 36.933 | 35.878 |
| | $\overline{\omega}_2$ | 95.664 | 97.177 | 97.994 | 98.235 | 97.985 | 96.671 | 93.185 | 89.743 | 86.364 | 83.065 | 79.858 |
| | $\overline{\omega}_3$ | 113.313 | 110.306 | 107.049 | 103.676 | 100.170 | 97.306 | 96.249 | 94.852 | 93.149 | 91.160 | 88.934 |
| G | $\overline{\omega}_1$ | 45.045 | 46.929 | 48.698 | 50.354 | 51.893 | 53.306 | 54.586 | 55.718 | 56.642 | 57.495 | 58.109 |
| | $\overline{\omega}_2$ | 85.738 | 85.046 | 84.248 | 83.378 | 82.453 | 81.479 | 80.460 | 79.389 | 78.258 | 77.055 | 75.765 |
| | $\overline{\omega}_3$ | 119.425 | 124.989 | 129.605 | 133.396 | 136.448 | 138.819 | 140.291 | 134.815 | 129.283 | 123.717 | 118.135 |

and boundary conditions of type C

$$w = 0, \quad \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = 0, \quad y = a,$$

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0,$$

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = a;$$

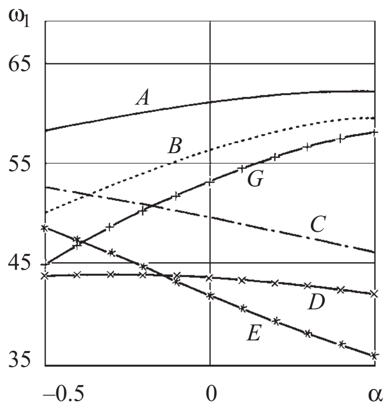


Fig. 1

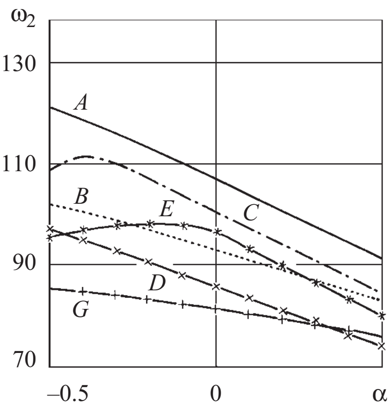


Fig. 2

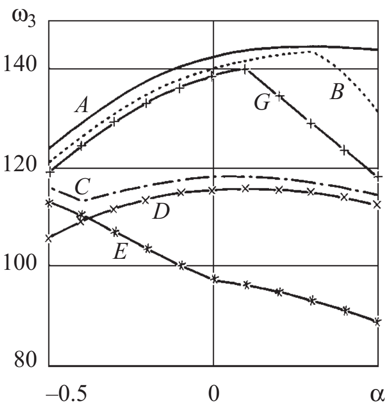


Fig. 3

(iii) two edges are clamped and two are hinged (boundary conditions of type D)

$$w = 0, \quad \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = 0,$$

$$w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = a,$$

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0,$$

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = a,$$

boundary conditions of type E

$$w = 0, \quad \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = 0, \quad y = a,$$

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, \quad x = a,$$

and boundary conditions of type G

$$w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = 0, \quad y = a,$$

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0, \quad x = a.$$

Figures 1–3 show the dimensionless frequency $\overline{\omega}_i$ of the square orthotropic plate with different boundary conditions as a function of the parameter $\alpha$ for $N = 10$. These results enable us to compare the first three frequencies of a plate with thickness varying in different ways. The frequency of the clamped plate is maximum among all the frequencies computed for different boundary conditions and different values of $\alpha$. As the parameter $\alpha$ in (13) increases, the frequency behaves monotonically in the first mode and nonmonotonically in higher modes for some types of boundary conditions. The maximum and minimum frequencies $\overline{\omega}_2$ and $\overline{\omega}_3$, shown in Figs. 2 and 3, correspond to the reorganization of vibration modes.

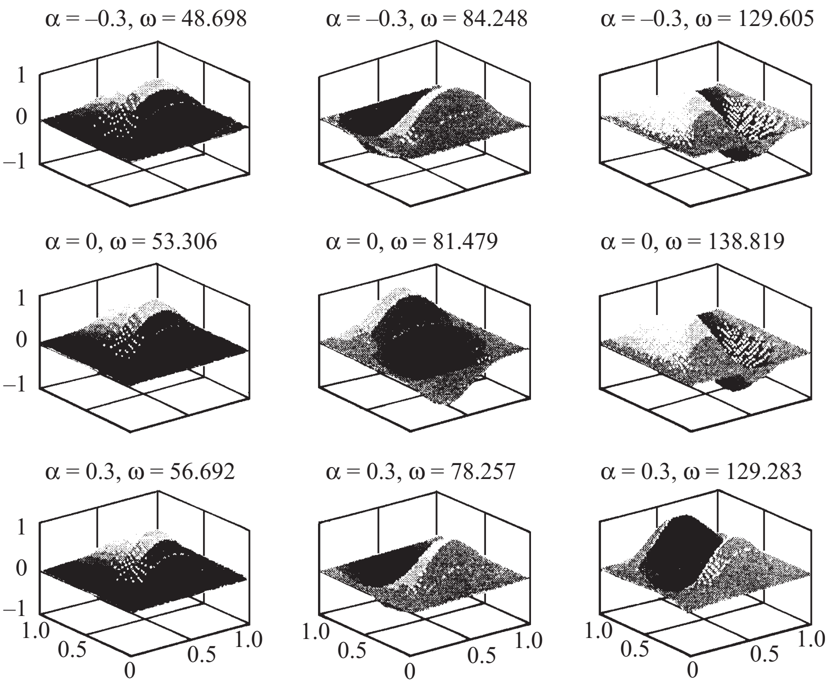


Fig. 4

Figure 4 shows natural modes of the plate with type G boundary conditions. Note that the third modes for different values of α differ significantly.

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