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NONAXISYMMETRIC DEFORMATION OF SOLIDS OF REVOLUTION MADE OF ELASTIC ORTHOTROPIC MATERIALS WITH DIFFERENT TENSILE AND COMPRESSIVE MODULI

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A method is proposed to allow for the difference of the tensile and compressive moduli of compound orthotropic bodies of revolution subject to nonaxisymmetric loading and heating. The compliance matrix is symmetrized by introducing weighting coefficients that take into account the influence of the sign of stresses in two mutually perpendicular directions on the corresponding coefficients of this matrix

Keywords: nonaxisymmetric thermostressed state, body of revolution, orthotropic materials, different tensile and compressive moduli, nonstationary heating

Introduction. Modern mechanical engineering, aircraft construction, shipbuilding, rocketry, etc. cannot be possibly imagined without composite materials. The combination of a light, fragile, compliant matrix and very strong and rigid reinforcing fibers or particles results in light, yet strong and rigid materials, which are widely used to fabricate a great variety of articles: from crucial elements of aircraft wings, firings of reentry vehicles, and control rods of nuclear reactors to golf-clubs, hockey sticks, jumping poles, and fishing rods. A peculiar feature of composites is that their properties in different directions can be predefined, i.e., they initially possess pronounced anisotropy. Along with anisotropy, composites often have an original property: their compliance or stiffness moduli depend on whether the load is tensile or compressive [1, 7, 8]. This property follows from Table 1, which gives ratios of tensile and compressive moduli for some fibrous and particulate materials.

It follows from Table 1 that the compressive moduli are 20 to 25% less than the tensile moduli in epoxy-resin composites reinforced with unidirectional glass fibers and 15 to 20% larger than the tensile moduli in epoxy-resin composites reinforced with unidirectional boron fibers. The tensile moduli may exceed the compressive moduli by 40% and more in epoxy-resin composites reinforced with unidirectional carbon fibers. The tensile moduli of the other fibrous composites, such as carbon composite reinforced with carbon fibers, are larger than the compressive moduli by a factor of 2 to 5. Thus, we cannot clearly identify the relationship between Young's moduli for fibrous composites since their values depend on the type of loading (whether it is tensile or compressive). Particulate materials have a similar property. Further development of composite micromechanics will possibly explain this phenomenon. The modern stress analysis of composite members usually neglects their heterogeneous structure and employs the phenomenological description of the material. In other words, the behavior of a composite member with different tensile and compressive moduli is described within the framework of anisotropic elasticity theory [2].

In this paper, we set forth a method [5] for stress analysis of elastic orthotropic solids of revolution with different tensile and compressive moduli under nonaxisymmetric loading and heating.

1. Problem Formulation and Governing Equations. Let us analyze, in a cylindrical coordinate system z, r, φ, the stress state of compound solids of revolution made of elastic orthotropic materials and subjected to volume and surface forces, $\vec{K}(K_z, K_r, K_\varphi)$ and $\vec{t}_n(t_{nz}, t_{nr}, t_{n\varphi})$, and nonuniform heating. The level of loading is such that the rheological properties do not manifest themselves, though the mechanical characteristics depend on temperature. A compound body is meant a discretely

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TABLE 1

Material	Texture	$E^{\pm} = nE^{\mp}$
Glass-epoxy resin	Fibrous	$E^{+} = 1.25E^{-}$
Boron-epoxy resin		$E^{-} = 1.2E^{+}$
Carbon (graphite)-epoxy resin		$E^{+} = 1.4E^{-}$
Carbon-carbon		$E^{+} = (2–5)E^{-}$
Graphite ZTA	Particulate	$E^{-} = 1.2E^{+}$
Graphite ATJ-S		$E^{+} = 1.2E^{-}$
Graphite B-1		$E^{+} = 1.5E^{-}$
Concrete		$E^{-} = 1.3E^{+}$
Cast iron SCh 12-28		$E^{-} = 1.35E^{+}$

inhomogeneous solid of revolution whose constituents are also solids of revolution. Both the whole body and all of its components have a common axis of revolution coinciding with the z-axis. Assume that the constituents, which have different tensile and compressive moduli, are fit together without interference at a temperature T_0 so as to provide perfect mechanical and thermal contact. The stress–strain analysis of such solids of revolution reduces to the solution of the nonstationary heat conduction problem to determine the temperature T and of the thermoelastic problem to determine the displacements u_i, strains ε_{ij}, and stresses σ_{ij} $(i, j = z, r, \varphi)$ at fixed time points. As orthotropic materials we will consider elastic materials with the principal axes of thermal and mechanical anisotropy coinciding with the axes of a cylindrical coordinate system or with the axes of a Cartesian coordinate system where one of the axes coincides with the body's axis of revolution.

Since the thermophysical characteristics of materials are assumed to be independent of the sign of the stresses, the temperature fields in a heated body can be determined by the method from [6].

Assigning the displacement components u_z, u_r, and u_φ to be the basic variables, we will use the corresponding Lagrange equation to determine them [3]:

$$\int_V \left(\sigma_{ij}\delta\varepsilon_{ij} - K_i \delta u_i\right) dV - \int_{\Sigma_t} t_{ni}\delta u_i \, d\Sigma = 0 \qquad (i, j = z, r, \varphi), \tag{1}$$

where V is the volume of the body of revolution bounded by a surface Σ, and Σ_t is a portion of the surface Σ on which the surface load components $\vec{t}_n$ are specified.

For an orthotropic material with the principal axes of mechanical and thermal anisotropy coinciding with the axes of the orthogonal coordinate system, the strain and stress components are related by

$$\begin{Bmatrix} \varepsilon_{11} - \varepsilon_{11}^T \\ \varepsilon_{22} - \varepsilon_{22}^T \\ \cdots \\ \varepsilon_{23} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & \cdots & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1/2G_{23} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \cdots\cdots \\ \sigma_{23} \end{Bmatrix}, \tag{2}$$

where E_i are the elastic moduli along the principal axes of anisotropy; G_{ij} are the shear moduli in the corresponding coordinate planes; and ν_{ij} is Poisson's ratio defining the compression of an element in the direction X_j after it has been stretched in the direction X_i; $\varepsilon_{ij}^T = \alpha_{ii}^T (T - T_0)$, α_{ii}^T is the linear thermal expansion coefficient along the corresponding principal direction of anisotropy. Here X_i corresponds to either the cylindrical system z, r, φ or the Cartesian system z, x, y.

For an anisotropic material with equal tensile and compressive moduli, the compliance matrix in (2) is symmetric, which follows from the condition that a positive definite function of potential energy exists. For an inequimodulus material, the compliance matrix is asymmetric. It may be symmetrized by specifying certain relations between the tensile and compressive moduli that would satisfy the well-known relations of anisotropic elasticity theory. Such an approach has been proposed in [1]. However, these relations restrict the use of real engineering materials. The idea of another approach proposed in [8] is to sum the coefficients of the tensile and compressive compliance matrices in proportion to the corresponding compressive and tensile stresses. This can be done by introducing weighting coefficients to allow for the influence of the sign of the normal stresses in two perpendicular directions on the corresponding coefficients of the compliance matrix. Though there is yet no theory behind this approach, it allows us to simmetrize the compliance matrix and to use anisotropic elasticity theory to solve the problem posed. Depending on the sign of stresses, the coefficients of the compliance matrix in (4) have the following form:

$$\text{if} \quad \sigma_{11} < 0, \quad \text{then} \quad \frac{1}{E_1} = \frac{1}{E_1^-}, \quad \text{if} \quad \sigma_{11} > 0, \quad \text{then} \quad \frac{1}{E_1} = \frac{1}{E_1^+},$$

....

$$\text{if} \quad \begin{matrix} \sigma_{11} > 0 \\ \sigma_{22} > 0 \end{matrix}, \quad \text{then} \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} = \frac{\nu_{12}^+}{E_1^+}, \quad \text{if} \quad \begin{matrix} \sigma_{11} < 0 \\ \sigma_{22} < 0 \end{matrix}, \quad \text{then} \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} = \frac{\nu_{12}^-}{E_1^-},$$

....

$$\text{if} \quad \begin{matrix} \sigma_{11} > 0 \\ \sigma_{22} < 0 \end{matrix}, \quad \text{then} \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} = \frac{|\sigma_{11}|}{|\sigma_{11}| + |\sigma_{22}|} \frac{\nu_{12}^+}{E_1^+} + \frac{|\sigma_{22}|}{|\sigma_{11}| + |\sigma_{22}|} \frac{\nu_{21}^-}{E_2^-},$$

$$\text{if} \quad \begin{matrix} \sigma_{11} < 0 \\ \sigma_{22} > 0 \end{matrix}, \quad \text{then} \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} = \frac{|\sigma_{11}|}{|\sigma_{11}| + |\sigma_{22}|} \frac{\nu_{12}^-}{E_1^-} + \frac{|\sigma_{22}|}{|\sigma_{11}| + |\sigma_{22}|} \frac{\nu_{21}^+}{E_2^+},$$

....

$$\text{if} \quad \sigma_{12} > 0, \quad \text{then} \quad \frac{1}{G_{12}} = \frac{1}{G_{12}^+} = \frac{1}{E_{12}^{45+}} - \left(\frac{1}{E_1^+} + \frac{1}{E_2^+} - \frac{\nu_{12}^+}{E_1^+} \right),$$

$$\text{if} \quad \sigma_{12} < 0, \quad \text{then} \quad \frac{1}{G_{12}} = \frac{1}{G_{12}^-} = \frac{1}{E_{12}^{45-}} - \left(\frac{1}{E_1^-} + \frac{1}{E_2^-} - \frac{\nu_{12}^-}{E_1^-} \right). \tag{3}$$

Since the material properties depend on the stress state and vice versa, the stress–strain problem is a problem with a priori unknown mechanical characteristics. However, we can get rid of this uncertainty by using the following iterative procedure. First, the displacements and stresses are determined for originally specified characteristics (for example, average Poisson's ratios and tensile and compressive moduli); and then the corresponding material properties are determined with regard to the sign of the stresses calculated at the previous step. This process continues until the prescribed accuracy is attained.

Resolving the system of equations (2) with already symmetric compliance matrix for the stress components, we obtain formulas for stresses in terms of the strain components in the principal axes of anisotropy, i.e., in the coordinate system z, r, φ in the case of cylindrical orthotropy and in the Cartesian coordinate system z, x, y in the case of rectilinear orthotropy. If we pass from the Cartesian to cylindrical coordinate system using the well-known transformation formulas, then the stress–strain relationship becomes

$$\sigma_{ij} = A_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^T), \tag{4}$$

where the expressions for $A_{ijkl}\,(i,j,k,l=z,r,\varphi)$ depend on the type of material [3, 9].

Let us write these relations in the form of Hooke's law for a homogeneous material. To this end, we represent the coefficients A_{ijkl} as $A_{ijkl}=A^0_{ijkl}(1-\omega_{ijkl})$, where A^0_{ijkl} are some temperature-independent average values of the corresponding coefficients, and $A^0_{ijkl}\omega_{ijkl}$ are functions describing the behavior of A_{ijkl} and accounting for their temperature dependence and the difference between the compressive and tensile moduli. Then the stress–strain relationship becomes

$$\begin{Bmatrix}\sigma_{zz}\\ \sigma_{rr}\\ \sigma_{\varphi\varphi}\\ \sigma_{zr}\\ \sigma_{z\varphi}\\ \sigma_{r\varphi}\end{Bmatrix}=\begin{bmatrix}A^0_{zzzz} & A^0_{zzrr} & A^0_{zz\varphi\varphi} & 0 & 0 & 0\\ A^0_{zzrr} & A^0_{rrrr} & A^0_{rr\varphi\varphi} & 0 & 0 & 0\\ A^0_{zz\varphi\varphi} & A^0_{rr\varphi\varphi} & A^0_{\varphi\varphi\varphi\varphi} & 0 & 0 & 0\\ 0 & 0 & 0 & A^0_{zrzr} & 0 & 0\\ 0 & 0 & 0 & 0 & A^0_{z\varphi z\varphi} & 0\\ 0 & 0 & 0 & 0 & 0 & A^0_{r\varphi r\varphi}\end{bmatrix}\begin{Bmatrix}\varepsilon_{zz}\\ \varepsilon_{rr}\\ \varepsilon_{\varphi\varphi}\\ \varepsilon_{zr}\\ \varepsilon_{z\varphi}\\ \varepsilon_{r\varphi}\end{Bmatrix}-\begin{Bmatrix}\sigma^*_{zz}\\ \sigma^*_{rr}\\ \sigma^*_{\varphi\varphi}\\ \sigma^*_{zr}\\ \sigma^*_{z\varphi}\\ \sigma^*_{r\varphi}\end{Bmatrix}, \quad (5)$$

where

$$\sigma^*_{zz}=A_{zzzz}\varepsilon^T_{zz}+A_{zzrr}\varepsilon^T_{rr}+A^T_{zz\varphi\varphi}\varepsilon^T_{\varphi\varphi}+A^0_{zzzz}\omega_{zzzz}\varepsilon_{zz}+A^0_{zzrr}\omega_{zzrr}\varepsilon_{zr}+A^0_{zz\varphi\varphi}\omega_{zz\varphi\varphi}\varepsilon_{\varphi\varphi},$$

....

$$\sigma^*_{r\varphi}=2A^0_{r\varphi r\varphi}\omega_{r\varphi r\varphi}\varepsilon_{r\varphi} \quad (6)$$

for cylindrical orthotropy and

$$\sigma^*_{zz}=A^0_{zztz}\omega_{zzzz}\varepsilon_{zz}+A^0_{zzrr}\omega_{zzrr}\varepsilon_{rr}+A^0_{zz\varphi\varphi}\omega_{zz\varphi\varphi}\varepsilon_{\varphi\varphi}-2A_{zzr\varphi}(\varepsilon_{r\varphi}-\varepsilon^T_{r\varphi})+A_{zzzz}\varepsilon^T_{zz}+A_{zzrr}\varepsilon^T_{rr}+A_{zz\varphi\varphi}\varepsilon^T_{\varphi\varphi},$$

....

$$\sigma^*_{r\varphi}=2A_{r\varphi r\varphi}\omega_{r\varphi r\varphi}+2A_{r\varphi r\varphi}\varepsilon^T_{r\varphi}-A_{zzr\varphi}(\varepsilon_{zz}-\varepsilon^T_{zz})-A_{rrr\varphi}(\varepsilon_{rr}-\varepsilon^T_{rr})-A_{\varphi\varphi r\varphi}(\varepsilon_{\varphi\varphi}-\varepsilon^T_{\varphi\varphi}) \quad (7)$$

for rectilinear orthotropy.

2. Solution Method. It is very difficult to directly discretize the variational equation (1) using three-dimensional finite elements (FEs). The efficiency of the finite-element method as applied to three-dimensional problems for solids of revolution can be improved by using the semianalytic finite-element method [3, 4, 6]. It reduces the original three-dimensional problem to a series of two-dimensional problem in the meridional section of the body. We will search for the solution in the form

$$u_z(z,r,\varphi,t)=\sum_{m=0}^{\infty}\bar{\bar{u}}^{(m)}_z(z,r,t)\cos m\varphi+\sum_{m=1}^{\infty}\bar{\bar{u}}^{(m)}_z(z,r,t)\sin m\varphi \quad (z,r),$$

$$u_\varphi(z,r,\varphi,t)=\sum_{m=1}^{\infty}\bar{\bar{u}}^{(r)}_\varphi(z,r,t)\sin m\varphi+\sum_{m=0}^{\infty}\bar{\bar{u}}^{(m)}_\varphi(z,r,t)\cos m\varphi, \quad (8)$$

where the coefficients are determined from the variational equation (1) using FEs in the meridional section. The variational equation itself should be transformed as follows.

Substituting relations (5) into this equation and assuming that the secondary stresses σ^*_{ij} are not varied, we get

$$\delta E=\delta\left\{\int_V\left[\frac{1}{2}\left(A^0_{zzzz}\varepsilon^2_{zz}+A^0_{rrrr}\varepsilon^2_{rr}+A^0_{\varphi\varphi\varphi\varphi}\varepsilon^2_{\varphi\varphi}\right)+2\left(A^0_{zrzr}\varepsilon^2_{zr}+A^0_{z\varphi z\varphi}\varepsilon^2_{z\varphi}+A^0_{r\varphi r\varphi}\varepsilon^2_{r\varphi}\right)\right.\right.$$

$$+A^0_{zzrr}\varepsilon_{zz}\varepsilon_{rr}+A^0_{zz\varphi\varphi}\varepsilon_{zz}\varepsilon_{\varphi\varphi}+A^0_{rr\varphi\varphi}\varepsilon_{rr}\varepsilon_{\varphi\varphi}-\sigma^*_{zz}\varepsilon_{zz}-\sigma^*_{rr}\varepsilon_{rr}-\sigma^*_{\varphi\varphi}\varepsilon_{\varphi\varphi}-2\sigma^*_{zr}\varepsilon_{zr}-2\sigma^*_{z\varphi}\varepsilon_{z\varphi}-2\sigma^*_{r\varphi}\varepsilon_{r\varphi}-K_z u_z$$

$$\left.\left.-K_r u_r-K_\varphi u_\varphi\right] rdzdrd\varphi-\int\limits_{\Sigma_t}(t_{nz}u_z+t_{nr}u_r+t_{n\varphi}u_\varphi)rdsd\varphi\right\}=0. \quad (9)$$

For the solution of Eq. (9) to have the form of the trigonometric series (8), we need to expand the projections of the surface (t_{ni}) and volume (K_i) forces and the functions σ^*_{ij} into similar series. Substituting these series, the series (8), and the expressions for the strain components derived from (8) using the Cauchy relations into the variational equation (9), we obtain a set of variational equations for each harmonic to determine the amplitude values $\overline{u}_i^{(m)}$ and $\overline{\overline{u}}_i^{(m)}$ of the unknown displacements.

To find the stationary values of the functionals obtained, we will use triangular finite elements with linear approximation of the coefficients of the series (8) within them. As a result, we obtain a system of $3N$ linear algebraic equations for each harmonic to determine the coefficients $\overline{u}_\alpha^{(m)}$ and $\overline{\overline{u}}_\alpha^{(m)}$ $(\alpha=z,r,\varphi)$ at the nodes $(i,\ j,\ k)$ of the triangular FEs q:

$$\sum_{q=1}^{M}(B^{zi(q)}_{zp}u_{zp}+B^{zi(q)}_{rp}u_{rp}+B^{zi(q)}_{\varphi p}u_{\varphi p})=D_{zi},$$

$$\sum_{q=1}^{M}(B^{ri(q)}_{zp}u_{zp}+B^{ri(q)}_{rp}u_{rp}+B^{ri(q)}_{\varphi p}u_{\varphi p})=D_{ri},$$

$$\sum_{q=1}^{M}(B^{\varphi i(q)}_{zp}u_{zp}+B^{\varphi i(q)}_{rp}u_{rp}+B^{\varphi i(q)}_{\varphi p}u_{\varphi p})=D_{\varphi i} \qquad (p=i,j,k),\ <i=1,\ 2,\ \ldots,N>. \quad (10)$$

There will be as many such systems as there are terms retained in the solution. (8).

The elements of the matrix of system (10) are calculated in terms of the coefficients of the physical equations (5) and the nodal coordinates of FEs in the meridional plane; and the right-hand side of the system, in terms of the amplitude values of the secondary stresses σ^*_{ij} and volume and surface loads at the corresponding points.

For a triangular FE with nodes i, j, and k, they have the form

$$B^{zi}_{zj}=\int\limits_{F_\Delta}(A^0_{11}b_{2i}b_{2j}+A^0_{44}b_{3i}b_{3j}+m^2A^0_{55}\Delta_{1i}\Delta_{1j})rdzdr,$$

$$B^{zi}_{rj}=\int\limits_{F_\Delta}(A^0_{44}b_{3i}b_{2j}+A^0_{12}b_{2i}b_{3j}+A^0_{13}b_{2i}\Delta_{1j})rdzdr,$$

$$B^{zi}_{\varphi j}=m\int\limits_{F_\Delta}(-A^0_{55}b_{2j}\Delta_{1i}+A^0_{13}b_{2i}\Delta_{1j})rdzdr,$$

$$B^{ri}_{zj}=\int\limits_{F_\Delta}(A^0_{44}b_{2i}b_{3j}+A^0_{12}b_{3i}b_{2j}+A^0_{13}b_{2j}\Delta_{1i})rdzdr,$$

$$B^{ri}_{rj}=\int\limits_{F_\Delta}\left\{(A^0_{22}b_{3i}+A^0_{23}\Delta_{1i})b_{3j}+A^0_{44}b_{2i}b_{2j}+[A^0_{23}b_{3i}+(A^0_{33}+m^2A_{66})\Delta_{1j}]\Delta_{1j}\right\}rdzdr, \quad (11)$$

$$B^{ri}_{\varphi j}=m\int\limits_{F_\Delta}[(A^0_{33}\Delta_{1i}+A^0_{23}b_{3i})\Delta_{1j}+A^0_{66}\Delta_{1i}\Delta_{2j}]rdzdr,$$

$$B^{\varphi i}_{zj}=m\int\limits_{F_\Delta}(-A^0_{55}b_{2i}\Delta_{1j}+A^0_{13}b_{2j}\Delta_{1i})rdzdr,$$

$$B_{rj}^{\varphi i} = m \int_{F_\Delta} [(A_{33}^0 \Delta_{1i} + A_{66}^0 \Delta_{2i})\Delta_{1j} + A_{23}^0 b_{3j} \Delta_{1i}] rdzdr,$$

$$B_{\varphi j}^{\varphi i} = \int_{F_\Delta} (A_{55}^0 b_{2i} b_{2j} + A_{66}^0 \Delta_{2i} \Delta_{2j} + m^2 A_{33}^0 \Delta_{1i} \Delta_{1j}) rdzdr,$$

$$D_{zi} = \sum_{q=1}^{M} \int_{F_\Delta} [\sigma_{zz}^* b_{2i} + \sigma_{zr}^* b_{3i} - (m\sigma_{z\varphi}^* - K_z z)\Delta_{1i}] rdzdr + \sum_{l=1}^{L} \frac{l_{ij}}{12} \operatorname{sign} F_\Delta [t_{nz_i}(3r_i + r_j) + t_{nz_j}(r_i + r_j)],$$

$$D_{ri} = \sum_{q=1}^{M} \int_{F_\Delta} [\sigma_{rr}^* b_{3i} + \sigma_{zr}^* b_{2i} + (\sigma_{\varphi\varphi}^* - m\sigma_{r\varphi}^* + K_r r)\Delta_{1i}] rdzdr + \sum_{l=1}^{L} \frac{l_{ij}}{12} \operatorname{sign} F_\Delta [t_{nr_i}(3r_i + r_j) + t_{nr_j}(r_i + r_j)], \quad (12)$$

$$D_{\varphi i} = \sum_{q=1}^{M} \int_{F_\Delta} [\sigma_{z\varphi}^* b_{ri} - \sigma_{r\varphi}^* \Delta_{ri} + (m\sigma_{\varphi}^* + K_\varphi r)\Delta_{1i}] rdzdr + \sum_{l=1}^{L} \frac{l_{ij}}{12} \operatorname{sign} F_\Delta [t_{n\varphi_i}(3r_i + r_j) + t_{n\varphi_j}(r_i + r_j)],$$

$$b_{1j} = \frac{z_k r_i - z_i r_k}{2F_\Delta}, \quad b_{2j} = \frac{r_k - r_i}{2F_\Delta}, \quad b_{3j} = \frac{z_i - z_k}{2F_\Delta},$$

$$\Delta_{1j} = b_{3j} + b_{2j} \frac{z}{r} + b_{1j} \frac{1}{r}, \quad \Delta_{2j} = b_{2j} \frac{z}{r} + b_{1j} \frac{1}{r},$$

$$F_\Delta = \frac{1}{2}[z_i(r_j - r_k) + z_j(r_k - r_i) + z_k(r_i - r_j)], \quad l_{ij} = \sqrt{(z_i - z_j)^2 + (r_i - r_j)^2}. \quad (13)$$

The first summation in (12) is over all FEs, and the second summation is over FEs one of whose sides runs along the boundary of the meridional section and one of whose nodes has the number i. The remaining coefficients in (10) can be obtained from (11) with the subscript j replaced by i or k.

Here $\overline{K}_i^{(m)}$, $\bar{t}_{ni}^{(m)}$, and $\overline{\sigma}_i^{*(m)}$ denote the coefficients in series expansion of the corresponding quantities; F is the area of half the meridional section of the body; and S is its boundary.

After solving systems (10) to determine the amplitude values of displacements, we calculate the displacement, strain, and stress components in each approximation. The number of necessary approximations is determined from the condition that two successive solutions differ by less than a predefined parameter.

3. Examples. To test the method described above, we calculated the stresses and strains in a uniformly heated thin-walled hollow cylinder (shell) subjected to internal pressure, axial compression, and torsion. The stress state of such a cylinder can also be determined by summing elementary solutions that account for the difference of the tensile and compressive moduli of the material without conditions (3) for each load. The results practically coincide for both stresses and strains when some of the tensile and compressive moduli differ by more than 50%.

Also we analyzed the nonstationary temperature field and stress–strain state of a convectively heated two-layer cylinder. It was assumed that its layers are made of materials differently resisting tension and compression.

At time $t = 0$, the cylinder, having temperature $T_0 = 20$ °C, begins to be heated on the cylindrical surface by an ambient medium with temperature varying as $\theta = (320 + 300\cos\varphi)$ °C and at the ends $z = \pm 0.3$ m by a medium of constant temperature $\theta = 300$ °C. The heat-transfer factor α is assumed constant.

The material of the inner layer ($0.035 \le r \le 0.04$ m) is cast iron with the following temperature-independent mechanical characteristics:

$$E_z^+ = E_r^+ = E_\varphi^+ = 9.327 \cdot 10^4 \text{ MPa}, \quad \nu_{zr}^+ = \nu_{z\varphi}^+ = \nu_{r\varphi}^+ = 0.22,$$

$$G_{zr}^+ = G_{z\varphi}^+ = G_{r\varphi}^+ = 3.823 \cdot 10^4 \text{ MPa}, \quad E_z^- = E_r^- = E_\varphi^- = 12.436 \cdot 10^4 \text{ MPa},$$

$$\nu_{zr}^- = \nu_{z\varphi}^- = \nu_{r\varphi}^- = 0.27, \quad G_{zr}^- = G_{z\varphi}^- = G_{r\varphi}^- = 4.896 \cdot 10^4 \text{ MPa}.$$

TABLE 2

r, cm	$\varphi = 0$				$\varphi = \pi / 2$			
	T, °C	σ_{zz}	σ_{zz}^{+}	σ_{zz}^{-}	T, °C	σ_{zz}	σ_{zz}^{+}	σ_{zz}^{-}
3.55	74	81.2	39.8	88.2	49	41.4	20.1	44.5
3.75	75	84.8	42.2	92.3	49	41.1	19.8	44.1
3.95	76	87.8	43.9	95.5	50	40.5	19.3	43.4
4.05	96	6.2	2.3	10.4	59	2.8	0.9	4.5
4.25	178	–1.9	–4.5	–6.5	100	–2.0	–2.5	–4.1
4.45	271	–21.9	–12.2	–25.5	146	–12.1	–6.4	–13.8
4.65	373	–43.0	–20.6	–46.5	197	–23.0	–10.7	–24.6
4.85	483	–65.5	–29.7	–69.0	252	–34.5	–15.3	–36.1
4.95	539	–77.0	–34.3	–80.6	280	–40.5	–17.7	–42.0

TABLE 3

r, cm	$\varphi = 0$				$\varphi = \pi / 2$			
	T, °C	σ_{zz}	σ_{zz}^{+}	σ_{zz}^{-}	T, °C	σ_{zz}	σ_{zz}^{+}	σ_{zz}^{-}
3.55	302	39.8	17.0	43.0	206	19.3	8.9	19.7
3.75	302	47.3	23.8	52.6	206	19.2	8.8	19.6
3.95	303	54.2	30.0	61.3	206	19.2	8.6	19.2
4.05	317	30.7	1.5	6.6	211	1.4	0.4	1.9
4.25	375	–2.2	–2.9	–4.5	230	–0.9	–1.2	–2.1
4.45	434	–13.9	–7.4	–15.7	250	–5.4	–3.0	–6.4
4.65	493	–25.1	–11.9	–26.9	272	–10.0	–4.8	–10.9
4.85	550	–36.0	–16.4	–37.9	294	–14.7	–6.6	–15.6
4.95	579	–41.4	–18.6	–43.3	305	–17.0	–7.5	–17.9

TABLE 4

r, cm	$\varphi = 0$			$\varphi = \pi / 2$		
	$\sigma_{\varphi\varphi}$	$\sigma^{+}_{\varphi\varphi}$	$\sigma^{-}_{\varphi\varphi}$	$\sigma_{\varphi\varphi}$	$\sigma^{+}_{\varphi\varphi}$	$\sigma^{-}_{\varphi\varphi}$
3.55	62.4	28.7	67.7	34.3	16.3	37.3
3.75	62.0	29.3	67.8	32.3	15.2	35.1
3.95	61.3	29.4	67.4	30.3	14.0	32.7
4.05	4.2	–1.2	6.2	2.0	0.5	2.9
4.25	–6.1	–5.2	–9.9	–3.7	–2.8	–5.4
4.45	–23.8	–12.2	–27.4	–12.9	–6.4	–14.5
4.65	–42.5	–19.7	–46.4	–22.7	–10.3	–24.3
4.85	–62.1	–27.4	–66.2	–32.9	–14.3	–34.5
4.95	–72.1	–31.6	–76.1	–38.0	–16.4	–39.6

TABLE 5

r, cm	$\varphi = 0$			$\varphi = \pi / 2$		
	$\sigma_{\varphi\varphi}$	$\sigma^{+}_{\varphi\varphi}$	$\sigma^{-}_{\varphi\varphi}$	$\sigma_{\varphi\varphi}$	$\sigma^{+}_{\varphi\varphi}$	$\sigma^{-}_{\varphi\varphi}$
3.55	25.1	6.4	24.8	15.5	7.2	16.6
3.75	30.7	12.4	32.4	14.4	6.7	15.5
3.95	35.7	17.6	39.0	13.4	6.2	14.5
4.05	2.3	0.6	3.3	0.9	0.2	1.2
4.25	–5.2	–3.7	–7.4	–2.0	–1.3	–2.7
4.45	–15.4	–7.8	–17.7	–6.1	–2.9	–6.7
4.65	–25.1	–11.7	–27.6	–10.2	–4.8	–10.8
4.85	–34.4	–15.5	–37.0	–14.3	–6.2	–14.8
4.95	–38.9	–17.3	–41.6	–16.3	–7.0	–16.8

The thermal characteristics are the following: $c\rho = 4.19$ MJ/(m^3·K), $\lambda_{zz} = \lambda_{rr} = \lambda_{\varphi\varphi} = 0.05349$ kW/(m·K), $\alpha^T_{zz} = \alpha^T_{rr} = \alpha^T_{\varphi\varphi} = 10^{-5}$ 1/K. The material of the outer layer ($0.04 \le r \le 0.05$ m) is concrete with the following mechanical and thermal characteristics:

$$E^+_z = E^+_r = E^+_\varphi = 0.7\cdot 10^4 \text{ MPa}, \quad \nu^+_{zr} = \nu^+_{z\varphi} = \nu^+_{r\varphi} = 0.17,$$

$$G^+_{zr} = G^+_{z\varphi} = G^+_{r\varphi} = 0.3\cdot 10^4 \text{ MPa}, \quad E^-_z = E^-_r = E^-_\varphi = 1.75\cdot 10^4 \text{ MPa},$$

$$\nu^-_{zr} = \nu^-_{z\varphi} = \nu^-_{r\varphi} = 0.17, \quad G^-_{zr} = G^-_{z\varphi} = G^-_{r\varphi} = 0.748\cdot 10^4 \text{ MPa},$$

$$\alpha_{zz} = \alpha^T_{rr} = \alpha^T_{\varphi\varphi} = 10^{-5} \text{ 1/K}, \quad c\rho = 1.7582 \text{ MJ/(m}^3\cdot\text{K)}, \quad \lambda_{zz} = \lambda_{rr} = \lambda_{\varphi\varphi} = 0.00093 \text{ kW/(m}\cdot\text{K)}.$$

Some results obtained in the middle section of the cylinder are given in Tables 2–5.

These tables show the radial distribution of the temperature and stresses (in MPa) σ_{zz} (Table 2) and stresses $\sigma_{\varphi\varphi}$ (Table 4) at the 60th second and after 5 minutes of heating (Tables 3 and 5, respectively) for $\varphi = 0$ and $\varphi = \pi/2$. These stress components have been chosen because they are maximum. An analysis of the results shows that nonaxisymmetric loading induces tangential stresses $\sigma_{z\varphi}$ and $\sigma_{r\varphi}$ that reach from 10 to 15% of the normal stresses.

For reference, the tables include the values of the normal stresses $\sigma^+_{zz}, \sigma^+_{\varphi\varphi}$ and $\sigma^-_{zz}, \sigma^-_{\varphi\varphi}$ obtained experimentally in tension (+) or compression (–) tests. It can be seen that if the difference of the tensile and compressive moduli is disregarded, then the results differ almost twofold.

Thus, the method proposed to allow for the difference of the tensile and compressive moduli of an orthotropic material in the stress–strain analysis of compound solids of revolution subject to nonstationary heat improves considerably the solution of the corresponding boundary-value problem.

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