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APPLICATION OF DISCRETE FOURIER SERIES IN THE STRESS ANALYSIS OF CYLINDRICAL SHELLS OF VARIABLE THICKNESS WITH ARBITRARY END CONDITIONS

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An approach is proposed to solve boundary-value stress–strain problems for cylindrical shells with thickness varying in two coordinate directions. The approach employs discrete Fourier series to separate circumferential variables. This makes it possible to reduce the problem to a one-dimensional one, which can be solved by the stable discrete-orthogonalization method. Examples are given

Keywords: cylindrical shell, variable thickness, discrete Fourier series, discrete orthogonalization

Modern structural elements and devices include circular cylindrical shells with constant or variable thickness and different end conditions under nonuniform or local loads [3, 4, 7, 12–15]. Methods of stress–strain analysis are available for cylindrical shells of constant thickness. For shells of variable thickness, however, the available methods apply only under various simplified assumptions. For example, some methods for cylindrical shells of variable thickness employ the simplified equations of the Mushtari–Donnell–Vlasov theory of shells [1, 2, 6, 8, 10, 11] or certain end conditions that allow separating variables [5].

The present paper addresses stress–strain problems for cylindrical shells of thickness varying in two coordinate directions under surface load and with arbitrary end conditions. These problems are solved using the exact equations of thin shells [2, 8] and discrete Fourier series [9]. By expanding all functions into Fourier series in the circumferential coordinate on continuous and discrete point sets, a two-dimensional boundary-value problem is reduced to a system of ordinary differential equations with certain boundary conditions, which is solved by the stable discrete-orthogonalization method [3].

1. Let the median surface of the shell be referred to an orthogonal coordinate system, with s and θ being the arc length and central angle, respectively. The basic relations include:

the expressions for strains

$$\varepsilon_s = \frac{\partial u}{\partial s}, \quad \varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r}, \quad l_{s\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial s},$$

$$\kappa_s = -\frac{\partial^2 w}{\partial s^2}, \quad \kappa_\theta = \frac{1}{r^2}\left(\frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2}\right), \quad \kappa_{s\theta} = \frac{1}{r}\frac{\partial v}{\partial s} - \frac{1}{r}\frac{\partial^2 w}{\partial s \partial \theta}, \tag{1}$$

the equilibrium equations

$$r\frac{\partial N_s}{\partial s} + \frac{\partial S}{\partial \theta} + rq_s = 0, \quad r\frac{\partial S}{\partial s} + \frac{\partial N_\theta}{\partial \theta} + \frac{\partial H}{\partial s} + Q_\theta + rq_\theta = 0,$$

$$r\frac{\partial Q_s}{\partial s} + \frac{\partial Q_\theta}{\partial \theta} - N_\theta + rq_\gamma = 0, \quad r\frac{\partial M_s}{\partial s} + \frac{\partial H}{\partial \theta} - rQ_s = 0, \quad r\frac{\partial H}{\partial s} + \frac{\partial M_\theta}{\partial \theta} - rQ_\theta = 0, \tag{2}$$

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and the elastic relations

$$N_s = D_M\left(\varepsilon_s + \nu\varepsilon_\theta\right), \quad N_\theta = D_N\left(\varepsilon_\theta + \nu\varepsilon_s\right), \quad S = \frac{1-\nu}{2} D_N \varepsilon_{s\theta}, \quad M_s = D_M\left(\kappa_s + \nu\kappa_\theta\right),$$

$$M_\theta = D_M\left(\kappa_\theta + \nu\kappa_s\right), \quad H = (1-\nu) D_M \kappa_{s\theta} \quad (0 \le s \le L,\ 0 \le \theta \le \pi), \tag{3}$$

where

$$D_N = \frac{E \cdot h(s,\theta)}{1-\nu^2}, \quad D_M = \frac{E \cdot h^3(s,\theta)}{12(1-\nu^2)} \tag{4}$$

are tangential and flexural stiffnesses; u, v, and w are the displacements along the generatrix, directrix, and normal to the median surface; ε_s, ε_θ, $\varepsilon_{s\theta}$ and κ_s, κ_θ, $\kappa_{s\theta}$ are tangential and flexural strains; N_s, N_θ, S, Q_s, Q_θ, M_s, M_θ, and H are forces and moments; $h = h(s, \theta)$ is the thickness of the shell; r is the radius of curvature of its cross section; E is Young's modulus; ν is Poisson's ratio; and q_s, q_θ, and q_γ are the load components.

Let $\hat{Q}_s, N_s, \hat{S}, M_s, w, u, v$, and ϑ_s be resolving functions, where

$$\hat{Q}_s = Q_s + \frac{1}{r}\frac{\partial H}{\partial\theta}, \quad \hat{S} = S + \frac{2}{r} H, \quad \vartheta_s = -\frac{\partial w}{\partial s}. \tag{5}$$

Assuming that $q_s = q_\theta = 0$ and performing some transformations in (1)–(4), we obtain a system of resolving equations:

$$\frac{\partial\hat{Q}_s}{\partial s} = \frac{\nu}{r} N_s - \frac{\nu}{r^2}\frac{\partial^2 M_s}{\partial\theta^2} + \frac{1-\nu^2}{r^2} D_N \frac{\partial v}{\partial\theta} - \frac{1-\nu^2}{r^4}\frac{\partial^2}{\partial\theta^2}\left(D_M \frac{\partial v}{\partial\theta}\right) + \frac{1-\nu^2}{r^2} D_N w + \frac{1-\nu^2}{r^4}\frac{\partial^2}{\partial\theta^2}\left(D_M \frac{\partial^2 w}{\partial\theta^2}\right) - q_r,$$

$$\frac{\partial N_s}{\partial s} = -\frac{1}{r}\frac{\partial\hat{S}}{\partial\theta} + \frac{4}{r^3}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0 D_N}\hat{S}\right) + \frac{2(1-\nu)}{r^3}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0}\frac{\partial\vartheta_s}{\partial\theta}\right) - \frac{2(1-\nu)}{r^4}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0}\frac{\partial u}{\partial\theta}\right),$$

$$\frac{\partial\hat{S}}{\partial s} = -\frac{\nu}{r}\frac{\partial N_s}{\partial\theta} - \frac{\nu}{r^2}\frac{\partial M_s}{\partial\theta} - \frac{1-\nu^2}{r^2}\frac{\partial}{\partial\theta}\left(D_N \frac{\partial v}{\partial\theta}\right) - \frac{1-\nu^2}{r^4}\frac{\partial}{\partial\theta}\left(D_M \frac{\partial v}{\partial\theta}\right) - \frac{1-\nu^2}{r^2}\frac{\partial}{\partial\theta}\left(D_N w\right) + \frac{1-\nu^2}{r^4}\frac{\partial}{\partial\theta}\left(D_M \frac{\partial^2 w}{\partial\theta^2}\right),$$

$$\frac{\partial M_s}{\partial s} = \hat{Q}_s - \frac{4}{r^2}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0 D_N}\hat{S}\right) - \frac{2(1-\nu)}{r^2}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0}\frac{\partial\vartheta_s}{\partial\theta}\right) + \frac{2(1-\nu)}{r^3}\frac{\partial}{\partial\theta}\left(\frac{D_M}{p_0}\frac{\partial u}{\partial\theta}\right),$$

$$\frac{\partial w}{\partial s} = -\vartheta_s, \quad \frac{\partial u}{\partial s} = \frac{1}{D_N} N_s - \frac{\nu}{r}\frac{\partial v}{\partial\theta} - \frac{\nu}{r} w, \tag{6}$$

$$\frac{\partial v}{\partial s} = \frac{2}{1-\nu}\frac{1}{p_0 D_N}\hat{S} - \frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{4}{r^3}\frac{D_M}{p_0 D_N}\frac{\partial u}{\partial\theta} - \frac{4}{r^2}\frac{D_M}{p_0 D_N}\frac{\partial\vartheta_s}{\partial\theta},$$

$$\frac{\partial\vartheta_s}{\partial s} = \frac{1}{D_M} M_s + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial\theta^2} - \frac{\nu}{r^2}\frac{\partial v}{\partial\theta}, \quad p_0 = 1 + \frac{4}{r^2}\frac{D_M}{D_N} = 1 + \frac{4h^2}{12r^2}. \tag{7}$$

Boundary conditions for the resolving functions are specified at the shell ends $s = 0$ and $s = L$.

Since the stiffnesses D_N and D_M depend on the variable θ, it is impossible to separate variables using Fourier series in the circumferential coordinate. Therefore, we introduce the following additional functions that include terms that prevent the separation of circumferential variables:

$$\varphi_1^j = D_M\left\{\frac{\partial v}{\partial\theta}, \frac{\partial^2 w}{\partial\theta^2}\right\} \quad (j = 1, 2), \quad \varphi_2^j = D_N\left\{w, \frac{\partial v}{\partial\theta}\right\} \quad (j = 1, 2),$$

$$\varphi_3^j = \frac{D_M}{p_0 D_N}\left\{\hat{S}, \frac{\partial u}{\partial\theta}, \frac{\partial\vartheta_s}{\partial\theta}\right\} \quad (j=1,2,3), \qquad \varphi_4^j = \frac{D_M}{p_0}\left\{\frac{\partial u}{\partial\theta}, \frac{\partial\vartheta_s}{\partial\theta}\right\} \quad (j=1,2),$$

$$\varphi_5 = \frac{1}{D_N}N_s, \quad \varphi_6 = \frac{1}{p_0 D_N}\hat{S}, \quad \varphi_7 = \frac{1}{D_M}M_s. \tag{8}$$

In view of (8), the resolving system (6) becomes

$$\frac{\partial\hat{Q}_s}{\partial s} = \frac{\nu}{r}N_s - \frac{\nu}{r^2}\frac{\partial^2 M_s}{\partial\theta^2} + \frac{1-\nu^2}{r^2}\varphi_2^2 - \frac{1-\nu^2}{r^4}\frac{\partial^2\varphi_1^1}{\partial\theta^2} + \frac{1-\nu^2}{r^2}\varphi_2^1 + \frac{1-\nu^2}{r^4}\frac{\partial^2\varphi_1^2}{\partial\theta^2} - r_r,$$

$$\frac{\partial N_s}{\partial s} = -\frac{1}{r}\frac{\partial\hat{S}}{\partial\theta} + \frac{4}{r^3}\frac{\partial\varphi_3^1}{\partial\theta} + \frac{2(1-\nu)}{r^3}\frac{\partial\varphi_4^2}{\partial\theta} - \frac{2(1-\nu)}{r^4}\frac{\partial\varphi_4^1}{\partial\theta},$$

$$\frac{\partial\hat{S}}{\partial s} = -\frac{\nu}{r}\frac{\partial N_s}{\partial\theta} - \frac{\nu}{r^2}\frac{\partial M_s}{\partial\theta} - \frac{1-\nu^2}{r^2}\frac{\partial\varphi_2^2}{\partial\theta} - \frac{1-\nu^2}{r^4}\frac{\partial\varphi_1^1}{\partial\theta} - \frac{1-\nu^2}{r^2}\frac{\partial\varphi_2^1}{\partial\theta} + \frac{1-\nu^2}{r^4}\frac{\partial\varphi_1^2}{\partial\theta},$$

$$\frac{\partial M_s}{\partial s} = \hat{Q}_s - \frac{4}{r^2}\frac{\partial\varphi_3^1}{\partial\theta} - \frac{2(1-\nu)}{r^2}\frac{\partial\varphi_4^2}{\partial\theta} + \frac{2(1-\nu)}{r^3}\frac{\partial\varphi_4^1}{\partial\theta}, \qquad \frac{\partial w}{\partial s} = -\vartheta_s,$$

$$\frac{\partial u}{\partial s} = \varphi_5 - \frac{\nu}{r}\frac{\partial v}{\partial\theta} - \frac{\nu}{r}w, \qquad \frac{\partial v}{\partial s} = \frac{2}{1-\nu}\varphi_6 - \frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{4}{r^3}\varphi_3^2 - \frac{4}{r^2}\varphi_3^3, \qquad \frac{\partial\vartheta_s}{\partial s} = \varphi_7 + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial\theta^2} - \frac{\nu}{r^2}\frac{\partial v}{\partial\theta}. \tag{9}$$

Let us expand all the functions appearing in (9) into Fourier series in θ:

$$X(s,\theta) = \sum_{n=0}^{N} X_n(s)\cos\lambda_n\theta, \qquad Y(s,\theta) = \sum_{n=1}^{N} Y_n(s)\sin\lambda_n\theta, \qquad \lambda_n = \frac{\pi n}{2\pi} = \frac{n}{2},$$

$$X = \{\hat{Q}_s, N_s, M_s, w, u, \vartheta_s, \varphi_1^j, \varphi_2^j, \varphi_5, \varphi_7, q_\gamma\}, \qquad Y = \{\hat{S}, v, \varphi_3^j, \varphi_4^j, \varphi_6\}. \tag{10}$$

Substituting (10) into Eqs. (9) and the boundary conditions and separating the variables, we arrive at a boundary-value problem for a system of ordinary differential equations with corresponding boundary conditions for the amplitude values of the functions appearing in (9):

$$\frac{d\hat{Q}_{s,n}}{ds} = \frac{\nu}{r}N_{s,n} + \frac{\nu\lambda_n^2}{r^2}M_{s,n} + \frac{1-\nu^2}{r^2}\varphi_{2,n}^2 - \frac{(1-\nu^2)\lambda_n^2}{r^4}\varphi_{1,2}^1 + \frac{(1-\nu^2)\lambda_n}{r^2}\varphi_{2,n}^1 - \frac{(1-\nu^2)\lambda_n}{r^4}\varphi_{1,n}^2,$$

$$\frac{dN_{s,n}}{ds} = -\frac{\lambda_n}{r}\hat{S}_n + \frac{4\lambda_n}{r^3}\varphi_{3,n}^1 + \frac{2(1-\nu)\lambda_n}{r^3}\varphi_{4,n}^2 - \frac{2(1-\nu)\lambda_n}{r^4}\varphi_{4,n}^1,$$

$$\frac{d\hat{S}_n}{ds} = \frac{\nu\lambda_n}{r}N_{s,n} + \frac{\nu\lambda_n}{r^2}M_{s,n} + \frac{(1-\nu^2)\lambda_n}{r^2}\varphi_{2,n}^2 + \frac{(1-\nu^2)\lambda_n}{r^4}\varphi_{1,n}^1 + \frac{(1-\nu^2)\lambda_n}{r^2}\varphi_{2,n}^1 - \frac{(1-\nu^2)\lambda_n}{r^4}\varphi_{1,n}^2,$$

$$\frac{dM_{s,n}}{ds} = \hat{Q}_{s,n} - \frac{4\lambda_n}{r^2}\varphi_{3,n}^1 - \frac{2(1-\nu)\lambda_n}{r^2}\varphi_{4,n}^2 + \frac{2(1-\nu)\lambda_n}{r^3}\varphi_{4,n}^1,$$

$$\frac{dw_n}{ds} = -\vartheta_{s,n}, \qquad \frac{du_n}{ds} = \varphi_{5,n} - \frac{\nu\lambda_n}{r}v_n - \frac{\nu}{r}w_n,$$

$$\frac{dv_n}{ds} = \frac{2}{1-\nu}\varphi_{6,n} + \frac{\lambda_n}{r}u_n + \frac{4}{r^3}\varphi_{3,n}^2 - \frac{4}{r^2}\varphi_{3,n}^3, \qquad \frac{d\vartheta_{s,n}}{ds} = \varphi_{7,n} - \frac{\nu\lambda_n^2}{r^2}w_n - \frac{\nu\lambda_n}{r^2}v_n \quad (n=\overline{0,N}). \tag{11}$$

The boundary conditions are

$$B_1\bar{Z}(0)=\bar{b}_1, \quad B_2\bar{Z}(L)=\bar{b}_2, \tag{12}$$

where $\bar{Z}(s)=\{\hat{Q}_{s,n}, N_{s,n}, \hat{S}_n, M_{s,N}, w_n, u_n, v_n, \vartheta_{s,n}\}^T$ is the column vector of resolving functions, B_1 and B_2 are rectangular matrices, and $\bar{b}_1$ and $\bar{b}_2$ are vectors.

We will solve the boundary-value problem (11), (12) by the discrete-orthogonalization method on the interval $0 \le s \le L$. In view of (8), for each n in Eqs. (11) we have

$$\varphi_{1,n}^j = \varphi_{1,n}^j(s; w; v) \quad (j = 1, 2), \qquad \varphi_{2,n}^j = \varphi_{2,n}^j(s; w; v) \quad (j = 1, 2),$$

$$\varphi_{3,n}^j = \varphi_{3,n}^j(s; \hat{S}; u; \vartheta_s) \quad (j = 1, 2, 3), \qquad \varphi_{4,n}^j = \varphi_{4,n}^j(s; u; \vartheta_s) \quad (j = 1, 2),$$

$$\varphi_{5,n}^j = \varphi_{5,n}^j(s; N_s), \qquad \varphi_{6,n}^j = \varphi_{6,n}^j(s; \hat{S}), \qquad \varphi_{7,n}^j = \varphi_{7,n}^j(s; M_s) \quad (n = \overline{0, N}). \tag{13}$$

The functions φ_{in}^j appearing in the expressions for the coefficients of the Fourier series (10) are not expressed explicitly in terms of the Fourier coefficients of the resolving functions and are calculated during the integration of system (11) using discrete Fourier series at each step s = const. Relations (13) show the dependence of these coefficients on the amplitude values of certain resolving functions and the interrelation of all the equations in (11).

2. Equations (11) are integrated simultaneously for all harmonics using the discrete-orthogonalization method. To determine, during the integration, the amplitude values of the functions (13) from the current values of the resolving functions for a fixed value $s = s_k$ $(k = \overline{0, K})$, we calculate the values of the functions (13) at a number of points θ_r $(r = \overline{0, R})$ of the interval $0 \le \theta \le \pi$ and expand the discrete function into a Fourier series, which will be called a discrete Fourier series. As the number of points θ_r at which the values of the additional functions are calculated increases, the discrete Fourier series tends to the exact Fourier series. At the beginning of the integration, the boundary conditions are satisfied. The amplitude values of the functions (13) are substituted into system (11) and the integration over s is continued, using the Runge-Kutta method with orthogonalization at isolated points of the interval $0 \le s \le L$ [3]. Thus, using the current amplitude values of the resolving functions, we calculate the following quantities at the points θ_r $(1 \le r \le R)$ for the fixed value $s = s_k$ $(0 \le s \le L)$:

$$D_M^r, D_N^r, \frac{D_M^r}{p_0^r D_N^r}, \frac{D_M^r}{p_0^r}, \frac{1}{D_N^r}, \frac{1}{p_0^r D_N^r}, \frac{1}{D_M^r}.$$

Using the numerical values of these quantities and the expressions of the functions (8), we obtain

$$\varphi_1^1(s_k, \theta_r) = D_M(s_k, \theta_r) \sum_{n=0}^{N} v_n(s_k) \lambda_n \cos \lambda_n \theta_r,$$

$$\varphi_1^2(s_k, \theta_r) = -D_M(s_k, \theta_r) \sum_{n=1}^{N} w_n(s_k) \lambda_n^2 \sin \lambda_n \theta_r \quad (1 \le r \le R). \tag{14}$$

The values of the functions $\varphi_2^j, \varphi_3^j, \varphi_4^j, \varphi_5, \varphi_6$, and φ_7 at the points θ_r $(1 \le r \le R)$ are calculated similarly.

Using the obtained values of the additional functions, we construct discrete Fourier series, similar to (10), whose coefficients are the deficient amplitude values of the functions (13) at $s = s_k$ for system (11). Using the standard procedure of determining the Fourier coefficients for a tabulated function, we find the numerical values of the Fourier coefficients of the additional functions [9]. These coefficients are substituted into system (11) and the integration over s is continued, passing from the point s_k to the point s_{k+1}.

To solve the boundary-value problem (11), (12), we apply the stable discrete-orthogonalization method. At the beginning and end of integration, the boundary conditions are satisfied.

As the number of points R at which the additional functions are calculated increases, the discrete Fourier series tends to the continuous Fourier series. In practice, just few first terms of the trigonometric series are used since the Fourier coefficients

TABLE 1

Method	R	N	$WE/10^3 q_0$				$N_\theta/10q_0$			
			$\theta/\pi = 0$		$\theta/\pi = 1$		$\theta/\pi = 0$		$\theta/\pi = 1$	
			$s/L = 0.1$	$s/L = 0.5$	$s/L = 0.1$	$s/L = 0.5$	$s/L = 0.1$	$s/L = 0.5$	$s/L = 0.1$	$s/L = 0.5$
$H = 0.25$										
1	12	3	0.654	2.117	1.887	6.108	0.794	2.569	0.794	2.571
	16	5	0.770	2.492	2.098	6.788	0.963	3.115	0.891	2.884
	24	8	0.745	2.411	2.146	6.945	0.917	2.968	0.917	2.969
	36	12	0.749	2.424	2.157	6.982	0.926	2.997	0.926	2.998
2	—	—	0.748	2.421	2.157	6.981	0.926	2.996	0.927	3.000
$H = 0.5$										
1	12	3	0.326	1.055	0.943	3.053	0.791	2.561	0.794	2.570
	16	5	0.385	1.245	1.049	3.393	0.962	3.112	0.891	2.884
	24	8	0.374	1.211	1.073	3.472	0.917	2.966	0.917	2.969
	36	12	0.372	1.206	1.079	3.491	0.925	2.995	0.926	2.997
2	—	—	0.372	1.205	1.078	3.490	0.922	2.984	0.926	2.998

and, hence, the influence of higher harmonics decrease rapidly. Obviously, the accuracy of Fourier-series approximation essentially depends on the rate of decrease in the Fourier coefficients, which in turn is related to the differential properties of a function continued to the whole interval $(-\infty, \infty)$. There are approximate approaches that allow us to compare the coefficient of a discrete Fourier series on a finite point set with the exact values of this and other coefficients of a Fourier series for the same function defined analytically. For example, it was shown in [9] that if the number of terms in a discrete Fourier series is $R = 12$, then it is necessary to retain only two to three terms. If $R = 24$, then acceptable accuracy may be expected to be achieved with seven to eight terms.

3. Let us discuss the solutions of some problems obtained using our approach.

To illustrate the convergence of an approximate solution to an exact one, let us consider a cylindrical shell of thickness varying in the circumferential direction as

$$h = H(1+\beta\cos\theta) \qquad (0 \le \theta \le 2\pi). \tag{15}$$

The shell is subject to the load $q_\gamma = q_0 \sin(\pi s / L)$ and its ends are hinged, i.e.,

$$N_s = M_s = w = v = 0 \quad \text{at} \quad s = 0, s = L. \tag{16}$$

The end conditions (16) allow us to solve the problem in two ways:

(i) using the approach based on discrete Fourier series and

(ii) separating longitudinal variables and solving a one-dimensional problem by the discrete-orthogonalization method.

The latter approach is considered to be exact.

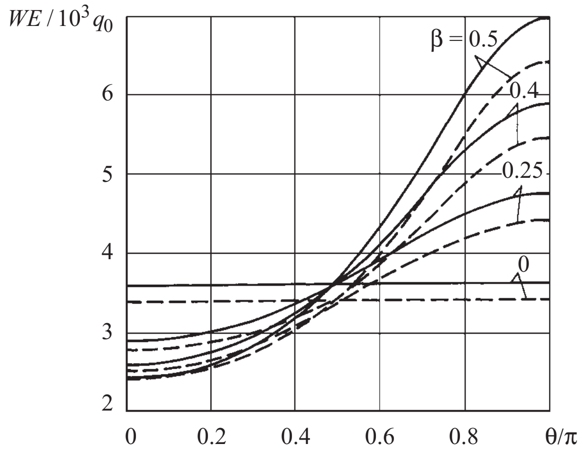

Fig. 1

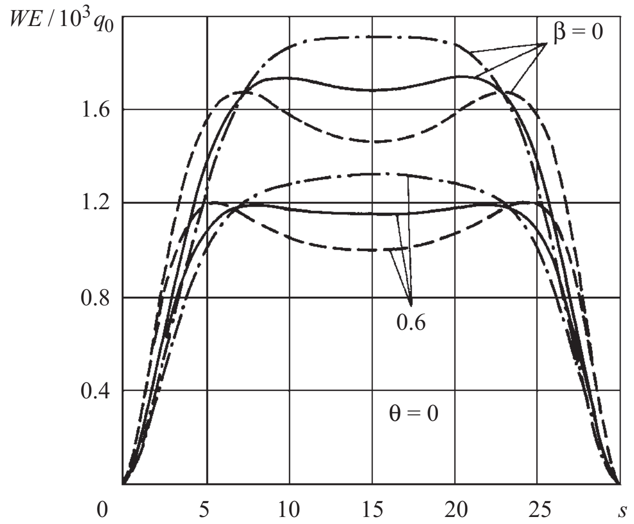

Fig. 2

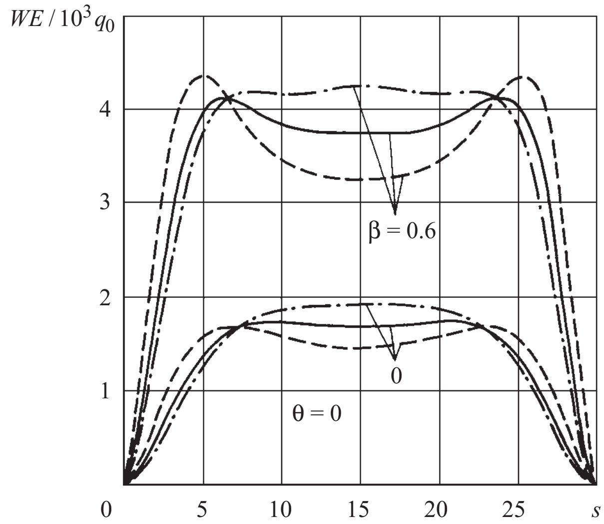

Fig. 3

The problem has been solved for the following initial data: $R = 30$; $L = 30$; $H = 0.25, 0.50$; $\beta = 0.5$; and $\nu = 0.3$. The results of solution (deflection w and force N_θ) for some values of s and θ are summarized in Table 1, where R is the number of terms of the discrete Fourier series and N is the number of terms retained in the series. It is seen that as R and N increase, w and N_θ tend to their exact values. When $R = 36$ and $N = 12$, the results coincide up to the third or fourth significant digit, i.e., up to several hundredths of a percent, which indicates that our approach is quite accurate.

Figure 1 shows the solution for a cylindrical shell of variable thickness (15) for two cases of end conditions: clamping (dashed line) and hinging (solid line). The clamping conditions are expressed as

$$u = v = w = \vartheta_s = 0 \quad \text{at} \quad s = 0, s = L. \tag{17}$$

The initial data: $R = 30$; $L = 30$; $H = 0.25$; $q = q_0 \sin(\pi s / L)$; $\beta = 0, 0.25, 0.40, 0.50$; $s = L/2$; and $\nu = 0.3$.

Figure 1 shows the influence of the end conditions on the deflection in the cross section $s = L/2$ when the thickness varies and the mass remains constant.

Figures 2 and 3 show the solution for a cylindrical shell of thickness varying in two coordinate directions by the formula

$$h(s,\theta) = \frac{H}{1-\alpha/3}\left[1-\alpha\left(\frac{2s}{L}-1\right)^2(1+\beta\cos\theta)\right] \qquad (0 \le s \le L,\ 0 \le \theta \le 2\pi). \tag{18}$$

TABLE 2

β	θ/π	$WE/10^3 q_0$				
		α = 0	α = 0.2	α = 0.4	α = –0.2	α = –0.4
0	0–1.0	3.310	3.091	2.872	3.529	3.748
0.2	0	2.848	2.660	2.470	3.037	3.225
	0.2	2.925	2.732	2.537	3.119	3.312
	0.4	3.153	2.944	2.735	3.362	3.570
	0.6	3.501	3.269	3.038	3.733	3.964
	0.8	3.859	3.604	3.350	4.115	4.370
	1.0	4.020	3.755	3.490	4.286	4.552
0.4	0	2.531	2.364	2.195	2.699	2.866
	0.2	2.650	2.474	2.298	2.826	3.000
	0.4	3.029	2.828	2.627	3.229	3.429
	0.6	3.730	3.484	3.237	3.978	4.224
	0.8	4.681	4.373	4.065	4.991	5.300
	1.0	5.214	4.871	4.529	5.559	5.903
0.6	0	2.307	2.153	1.999	2.460	2.613
	0.2	2.459	2.296	2.132	2.622	2.785
	0.4	2.940	2.745	2.550	3.135	3.328
	0.6	4.011	3.745	3.480	4.276	4.541
	0.8	6.021	5.624	5.228	6.419	6.817
	1.0	7.551	7.054	6.557	8.050	8.548

TABLE 3

β	θ/π	$N_\theta/10q_0$				
		α = 0	α = 0.2	α = 0.4	α = –0.2	α = –0.4
0	0–1.0	2.993	2.993	2.992	3.026	3.024
0.2	0	3.040	3.039	3.038	3.080	3.076
	0.2	3.031	3.031	3.029	3.072	3.068
	0.4	3.009	3.008	3.007	3.046	3.043
	0.6	2.981	2.980	2.980	3.007	3.006
	0.8	2.960	2.959	2.960	2.968	2.969
	1.0	2.952	2.952	2.952	2.951	2.953
0.4	0	3.089	3.088	3.087	3.112	3.107
	0.2	3.074	3.073	3.072	3.101	3.096
	0.4	3.028	3.027	3.026	3.064	3.060
	0.6	2.969	2.969	2.968	2.992	2.991
	0.8	2.927	2.927	2.928	2.909	2.911
	1.0	2.912	2.912	2.913	2.874	2.877
0.6	0	3.129	3.128	3.127	3.134	3.133
	0.2	3.121	3.120	3.119	3.121	3.118
	0.4	3.050	3.050	3.049	3.081	3.076
	0.6	2.955	2.955	2.954	2.987	2.985
	0.8	2.890	2.891	2.892	2.863	2.864
	1.0	2.852	2.852	2.853	2.822	2.824

TABLE 4

β	θ/π	$WE/10^3q_0$				
		$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = -0.2$	$\alpha = -0.4$
0	0–1.0	1.684	1.572	1.458	1.795	1.905
0.2	0	1.453	1.358	1.261	1.548	1.641
	0.2	1.493	1.395	1.295	1.590	1.686
	0.4	1.607	1.500	1.393	1.712	1.816
	0.6	1.776	1.658	1.537	1.894	2.010
	0.8	1.945	1.814	1.681	2.075	2.203
	1.0	2.019	1.883	1.745	2.154	2.288
0.4	0	1.283	1.199	1.114	1.366	1.448
	0.2	1.345	1.257	1.168	1.432	1.518
	0.4	1.542	1.441	1.339	1.642	1.742
	0.6	1.890	1.764	1.637	2.015	2.139
	0.8	2.334	2.177	2.018	2.491	2.647
	1.0	2.580	2.405	2.230	2.754	2.928
0.6	0	1.160	1.083	1.005	1.236	1.313
	0.2	1.234	1.153	1.071	1.315	1.395
	0.4	1.493	1.396	1.299	1.590	1.686
	0.6	2.038	1.904	1.768	2.171	2.304
	0.8	2.990	2.790	2.590	3.190	3.390
	1.0	3.748	3.498	3.251	3.998	4.250

TABLE 5

β	θ/π	$N_\theta/10^3q_0$				
		$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = -0.2$	$\alpha = -0.4$
0	0–1.0	3.027	3.025	3.018	3.026	3.024
0.2	0	3.084	3.085	3.082	3.080	3.076
	0.2	3.075	3.076	3.072	3.072	3.068
	0.4	3.048	3.047	3.043	3.046	3.043
	0.6	3.006	3.004	2.997	3.007	3.006
	0.8	2.966	2.962	2.954	2.968	2.969
	1.0	2.949	2.944	2.936	2.951	2.953
0.4	0	3.116	3.118	3.118	3.112	3.107
	0.2	3.105	3.107	3.107	3.101	3.096
	0.4	3.067	3.068	3.066	3.064	3.060
	0.6	2.992	2.990	2.985	2.992	2.991
	0.8	2.906	2.901	2.894	2.909	2.911
	1.0	2.870	2.865	2.858	2.874	2.877
0.6	0	3.134	3.133	3.131	3.134	3.133
	0.2	3.123	3.124	3.124	3.121	3.118
	0.4	3.085	3.088	3.090	3.081	3.076
	0.6	2.988	2.988	2.986	2.987	2.985
	0.8	2.860	2.858	2.855	2.863	2.864
	1.0	2.820	2.818	2.818	2.822	2.824

For any values of α and β, the mass of the shell remains constant and equals that of a shell of constant thickness H = const ($\alpha = \beta = 0$).

The problem has been solved for the following initial data: $R = 30$, $L = 30$, $H = 0.5$, $q_\gamma = q_0$ = const, and $\nu = 0.3$. The shell ends are rigidly clamped. The values of β are indicated in the figures, which show the distribution of the deflections w along the length of the shell on the interval $0 \le s \le L$ for $\theta = 0$ (Fig. 2) and $\theta = \pi$ (Fig. 3) and for $\alpha = 0$ (solid line), $\alpha = 0.4$ (dashed line), and $\alpha = -0.4$ (dash-and-dot line). It can be seen how the thickness influences the deformation of the shell and how to obtain the distribution and level of deflections by selecting the values of α and β at constant mass of the shell.

The circumferential deflections in the middle section $s = L/2$ of a shell with constant mass is given in Table 2 for $H = 0.25$ and Table 4 for $H = 0.5$ depending on α and β. Table 2 demonstrates how the deflection of the shell depends on its thickness. For example, for $\alpha = 0$ the deflections in thicker ($\theta = 0$) and thinner ($\theta = \pi$) parts of the shell of varying thickness ($\beta = 0.2$) are less than in the shell of constant thickness ($\beta = 0$) by factors of 1.16 and 1.21, respectively. Also, such factors are equal to 1.31 and 1.57 for $\theta = 0$ and $\theta = \pi$, respectively, when $\beta = 0.4$ and to 1.43 and 2.28 for $\theta = 0$ and $\theta = \pi$, respectively, when $\beta = 0.6$. With $\beta = 0.2$, the deflection in the middle section decreases (increases) for $\theta = 0$ and increases (decreases) for $\theta = \pi$ with increase (decrease) in α. The same pattern is observed for $\beta = 0.4$ and $\beta = 0.6$, i.e., when β is fixed and α varies, the mid-section deflection may either increase or decrease, the mass of the shell remaining constant.

The values of the force N_θ are given in Table 3 for $H = 0.25$ and in Table 5 for $H = 0.5$. As is seen from Tables 3 and 5, N_θ depends insignificantly on α and β. From Tables 4 and 5 it follows that the qualitative pattern is similar to those represented by Tables 2 and 3, but the deflection decreases by half and the force N_θ differs insignificantly from the values in Table 3.

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