QUADRATICALLY NONLINEAR CYLINDRICAL HYPERELASTIC WAVES: DERIVATION OF WAVE EQUATIONS FOR AXISYMMETRIC AND OTHER STATES

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A rigorous approach of nonlinear continuum mechanics is used to derive nonlinear wave equations that describe the propagation and interaction of hyperelastic cylindrical waves. Nonlinearity is introduced by means of metric coefficients, the Cauchy–Green strain tensor, and the Murnaghan potential and corresponds to the quadratic nonlinearity of all basic relationships. Quadratically nonlinear wave equations are derived for three states (configurations): (i) axisymmetric configuration dependent on the radial and axial coordinates and independent of the angular coordinate, (ii) configuration dependent on the angular coordinate, and (iii) axisymmetric configuration dependent on the radial coordinate. Four ways of introducing physical and geometrical nonlinearities to the wave equations are analyzed. Six different systems of wave equations are written

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The nonlinear wave equations describing the propagation of hyperelastic waves have been derived in [6]. This paper demonstrated a rigorous approach, passed over even in the textbooks [7, 9, 10, 16], to deriving nonlinear wave equations in cylindrical (orthogonal) coordinates. The approach is based on the concepts of modern nonlinear continuum mechanics. Nonlinearity was introduced by means of metric coefficients, the Cauchy–Green strain tensor, and the Murnaghan potential and corresponds to the quadratic nonlinearity of all basic relationships. A configuration (state) of an elastic medium dependent on the coordinates r, ϑ and independent of the coordinate z was analyzed. This configuration is often called plane-strain state. For this case, the corresponding equations of motion and analytical expressions for the stress tensor in terms of the deformation gradient were derived. Four ways of introducing physical and geometrical nonlinearities to the wave equations were analyzed. For one of the ways, the nonlinear wave equations were written explicitly.

The present paper extends the analysis performed in [6] to several partial configurations (states) omitted there.

State I. It is an axisymmetric configuration dependent on the coordinates r and z and independent of the coordinate ϑ. The Oz-axis is the axis of symmetry. This state is typical of, for example, a longitudinal torsional wave propagating along a cylinder.

State II. It is a configuration that depends only on the angular coordinate ϑ. Its axis of symmetry is Oz. This state is typical of, for example, a transverse torsional wave propagating along a cylinder.

State III. It is an axisymmetric configuration that depends only on the coordinate r. Its axis of symmetry is Oz. This state is typical of, for example, a classical cylindrical wave or Volterra translational distortions in a hollow cylinder.

As in [6], we introduce a cylindrical (orthogonal) coordinate system: $\theta^1 = r$, $\theta^2 = \vartheta$, $\theta^3 = z$. In this system, the length of a vector is defined by the formula

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$$(ds)^2 = g_{ik}\, d\theta^i\, d\theta^k = (dr)^2 + r^2 (d\vartheta)^2 + (dz)^2. \quad (1)$$

The metric tensors have the components

$$\| g_{ik} \| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \| g^{ik} \| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \quad (2)$$

The lengths of the basis vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and $(\vec{e}^{\,1}, \vec{e}^{\,2}, \vec{e}^{\,3})$ are defined by

$$|\vec{e}_1| = 1, \quad |\vec{e}_2| = r, \quad |\vec{e}_3| = 1, \quad |\vec{e}^{\,1}| = 1, \quad |\vec{e}^{\,2}| = (1/r), \quad |\vec{e}^{\,3}| = 1, \quad (3)$$

$$\delta_n^k = \vec{e}^{\,k} \cdot \vec{e}_n. \quad (4)$$

Only three Christoffel symbols of the first kind Γ_{ki}^m are nonzero:

$$\Gamma_{22}^1 = -r, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = (1/r). \quad (5)$$

Now our goal is to derive the nonlinear equations of motion for the above-mentioned configurations and to demonstrate a method for calculating the nonlinear strain and stress tensors.

State I. The displacement vector $\vec{u}\,(\theta^1, \theta^2, \theta^3)$ is defined by

$$\vec{u}\,(\theta^1, \theta^2, \theta^3) = \vec{u}\,(r, \vartheta, z) = \{ u_1 = u_r(r,z),\ u_2 = r \cdot u_\vartheta = 0,\ u_3 = u_z(r,z) \}. \quad (6)$$

The components of the nonlinear Cauchy–Green strain tensor are calculated using the covariant derivatives of the covariant and contravariant components of the displacement vector [1, 4, 7]:

$$\varepsilon_{ik} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i + \nabla_i u_k \nabla_j u^k \right), \quad \nabla_i u^k = \frac{\partial u^k}{\partial \theta^i} + u^j \Gamma_{ji}^k, \quad \nabla_i u_j = \frac{\partial u_j}{\partial \theta^i} - u_k \Gamma_{ji}^k,$$

$$\nabla_1 u_1 = u_{1,1} - u_1 \cancel{\Gamma_{11}^1} - u_2 \cancel{\Gamma_{11}^2} = u_{r,r}, \quad \nabla_1 u^1 = u^1_{,1} + u^1 \cancel{\Gamma_{11}^1} + u^2 \cancel{\Gamma_{21}^1} = u_{r,r}.$$

From here on, the crossed terms are equal to zero.

$$\nabla_1 u_2 = u_{2,1} - u_1 \cancel{\Gamma_{21}^1} - u_2 \Gamma_{21}^2 = \left(r u_\vartheta \right)_{,r} - r u_\vartheta \frac{1}{r} = r u_{\vartheta,r},$$

$$\nabla_1 u^2 = u^2_{,1} + u^1 \cancel{\Gamma_{11}^2} + u_2 \Gamma_{21}^2 = \left(\frac{u_\vartheta}{r} \right)_{,r} + \frac{u_\vartheta}{r} \cdot \frac{1}{r} = \frac{u_{\vartheta,r}}{r},$$

$$\nabla_2 u_2 = u_{2,2} - u_1 \Gamma_{22}^1 - u_2 \cancel{\Gamma_{22}^2} = \left(r u_\vartheta \right)_{,\vartheta} - u_r(-r) = r\left(u_{\vartheta,\vartheta} + u_r \right),$$

$$\nabla_2 u^2 = u^2_{,2} + u^1 \Gamma_{12}^2 + u^2 \cancel{\Gamma_{22}^2} = \left(\frac{u_\vartheta}{r} \right)_{,\vartheta} + \frac{u_r}{r} = \frac{1}{r}\left(u_{\vartheta,\vartheta} + u_r \right),$$

$$\nabla_2 u_1 = u_{1,2} - u_1 \cancel{\Gamma_{12}^1} - u_2 \Gamma_{12}^2 = u_{r,\vartheta} - r u_\vartheta \left(\frac{1}{r} \right) = u_{r,\vartheta} - u_\vartheta,$$

$$\nabla_2 u^1 = u^1_{,2} + u^1 \cancel{\Gamma_{12}^1} + u^2 \Gamma_{22}^1 = u_{r,\vartheta} + \left(\frac{u_\vartheta}{r} \right)(-r) = u_{r,\vartheta} - u_\vartheta,$$

$$\varepsilon_{11}=\varepsilon_{rr}=\nabla_1 u_1+\frac{1}{2}\left(\nabla_1 u_1 \nabla_1 u^1\right)+\frac{1}{2}\left(\nabla_1 u_3 \nabla_1 u^3\right)=u_{r,r}+\frac{1}{2}\left(u_{r,r}\right)^2+\frac{1}{2}\left(u_{z,r}\right)^2, \tag{7}$$

$$\varepsilon_{22}=r^2\varepsilon_{\vartheta\vartheta}=\nabla_2 u_2+\frac{1}{2}\left(\nabla_2 u_2 \nabla_2 u^2\right)+\frac{1}{2}\left(\nabla_2 u_1 \nabla_2 u^1\right)=\cancel{\frac{\partial u_2}{\partial\theta^2}}-u_m\Gamma^m_{22}$$

$$+\frac{1}{2}\left(\cancel{\frac{\partial u_2}{\partial\theta^2}}-u_m\Gamma^m_{22}\right)\left(\cancel{\frac{\partial u^2}{\partial\theta^2}}+u^m\Gamma^2_{m2}\right)+\left(\cancel{\frac{\partial u_1}{\partial\theta^2}}-u_m\Gamma^m_{12}\right)\left(\cancel{\frac{\partial u^1}{\partial\theta^2}}+u^m\cancel{\Gamma^1_{m2}}\right)$$

$$=-u_m\Gamma^m_{22}+\frac{1}{2}\left(-u_m\Gamma^m_{22}\right)\left(u^m\Gamma^2_{m2}\right)=u_r r+\frac{1}{2}\left(u_r\right)^2, \tag{8}$$

$$\varepsilon_{33}=\varepsilon_{zz}=\nabla_3 u_3+\frac{1}{2}\left(\nabla_3 u_3 \nabla_3 u^3\right)+\frac{1}{2}\left(\cancel{\nabla_3 u_2 \nabla_3 u^2}\right)+\frac{1}{2}\left(\nabla_3 u_1 \nabla_3 u^1\right)=\frac{\partial u_3}{\partial\theta^3}-u_m\cancel{\Gamma^m_{33}}$$

$$+\frac{1}{2}\left(\frac{\partial u_3}{\partial\theta^3}-u_m\cancel{\Gamma^m_{33}}\right)\left(\frac{\partial u^3}{\partial\theta^3}+u^m\cancel{\Gamma^m_{m3}}\right)+\frac{1}{2}\left(\frac{\partial u_1}{\partial\theta^3}-u_m\cancel{\Gamma^m_{33}}\right)\left(\frac{\partial u^1}{\partial\theta^3}+u^m\cancel{\Gamma^m_{m3}}\right)^2=u_{z,z}+\frac{1}{2}\left(u_{z,z}\right)^2+\frac{1}{2}\left(u_{r,z}\right)^2, \tag{9}$$

$$\varepsilon_{12}=r\varepsilon_{r\vartheta}=\frac{1}{2}\left(\nabla_1 u_2+\nabla_2 u_1+\nabla_1 u_1 \nabla_2 u^1+\nabla_1 u_2 \nabla_2 u^2\right)=0, \tag{10}$$

$$\varepsilon_{23}=r\varepsilon_{\vartheta z}=\frac{1}{2}\left(\nabla_2 u_3+\nabla_3 u_2+\nabla_2 u_3 \nabla_3 u^3+\nabla_2 u_2 \nabla_3 u^2+\nabla_2 u_1 \nabla_3 u^1\right)=0, \tag{11}$$

$$\varepsilon_{13}=\varepsilon_{rz}=\frac{1}{2}\left(\nabla_1 u_3+\nabla_3 u_1+\nabla_1 u_1 \nabla_3 u^1\cancel{+\nabla_1 u_2 \nabla_3 u^2}+\nabla_1 u_3 \nabla_3 u^3\right)$$

$$=\frac{1}{2}\left[\frac{\partial u_3}{\partial\theta^1}+\frac{\partial u_1}{\partial\theta^3}\cancel{-u_m\Gamma^m_{31}}\cancel{-u_m\Gamma^m_{13}}+\left(\frac{\partial u_1}{\partial\theta^1}\cancel{-u_m\Gamma^m_{11}}\right)\left(\frac{\partial u^1}{\partial\theta^3}\cancel{+u^m\Gamma^1_{m3}}\right)\right.$$

$$\left.+\left(\frac{\partial u_3}{\partial\theta^3}\cancel{-u_m\Gamma^m_{33}}\right)\left(\frac{\partial u^3}{\partial\theta^1}\cancel{+u^m\Gamma^3_{m1}}\right)\right]=\frac{1}{2}\left(u_{z,r}+u_{r,z}+u_{r,r}u_{r,z}+u_{z,r}u_{z,z}\right). \tag{12}$$

State II. The displacement vector $\vec{u}\,(\theta^1,\theta^2,\theta^3)$ is defined by

$$\vec{u}\,(\theta^1,\theta^2,\theta^3)=\vec{u}\,(r,\vartheta,z)=\left\{u^1=u_r=0,\ u^2=r\cdot u_\vartheta(r,z),\ u^3=u_z=0\right\}. \tag{13}$$

The components of the nonlinear Cauchy–Green tensor are calculated in the same way,

$$\varepsilon_{11}=\varepsilon_{rr}=\cancel{\nabla_1 u_1}+\frac{1}{2}\left(\cancel{\nabla_1 u_1 \nabla_1 u^1}\right)+\frac{1}{2}\left(\nabla_1 u_2 \nabla_1 u^2\right)+\frac{1}{2}\left(\cancel{\nabla_1 u_3 \nabla_1 u^3}\right)=\frac{1}{2}\left(u_{\vartheta,r}\right)^2, \tag{14}$$

$$\varepsilon_{22}=r^2\varepsilon_{\vartheta\vartheta}=\nabla_2 u_2+\frac{1}{2}\left(\nabla_2 u_2 \nabla_2 u^2\right)+\frac{1}{2}\left(\nabla_2 u_1 \nabla_2 u^1\right)+\frac{1}{2}\left(\nabla_2 u_3 \nabla_2 u^3\right)=\frac{1}{2}\left(u_\vartheta\right)^2,$$

$$\varepsilon_{33}=\varepsilon_{zz}=\nabla_3 u_3+\frac{1}{2}\left(\nabla_3 u_3 \nabla_3 u^3\right)+\frac{1}{2}\left(\nabla_3 u_2 \nabla_3 u^2\right)+\frac{1}{2}\left(\nabla_3 u_1 \nabla_3 u^1\right)=\frac{1}{2}\left(u_{\vartheta,z}\right)^2, \tag{15}$$

$$\varepsilon_{12}=r\varepsilon_{r\vartheta}=\frac{1}{2}\left(\nabla_1 u_2+\nabla_2 u_1+\nabla_1 u_1 \nabla_2 u^1+\nabla_1 u_2 \nabla_2 u^2+\nabla_1 u_3 \nabla_2 u^3\right)=ru_{\vartheta,r}-u_\vartheta, \tag{16}$$

$$\varepsilon_{23}=r\varepsilon_{\vartheta z}=\frac{1}{2}\left(\nabla_2 u_3+\nabla_3 u_2+\nabla_2 u_3 \nabla_3 u^3+\nabla_2 u_2 \nabla_3 u^2+\nabla_2 u_1 \nabla_3 u^1\right)=ru_{\vartheta,z}, \tag{17}$$

$$\varepsilon_{13}=\varepsilon_{rz}=\frac{1}{2}\left(\nabla_1 u_3+\nabla_3 u_1+\nabla_1 u_1\nabla_3 u^1+\nabla_1 u_2\nabla_3 u^2+\nabla_1 u_3\nabla_3 u^3\right)=\frac{1}{2}u_{\vartheta,r}u_{\vartheta,z}. \tag{18}$$

State III. The displacement vector $\vec{u}\left(\theta^1,\theta^2,\theta^3\right)$ is defined by

$$\vec{u}\left(\theta^1,\theta^2,\theta^3\right)=\vec{u}\left(r,\vartheta,z\right)=\left\{u^1=u_r(r),\ u^2=r\cdot u_\vartheta=0,\ u^3=u_z=0\right\}. \tag{19}$$

The components of the nonlinear Cauchy–Green tensor are calculated in the same way as in the previous cases,

$$\varepsilon_{11}=\varepsilon_{rr}=\frac{\partial u_1}{\partial\theta^1}-u_m\Gamma_{11}^m+\frac{1}{2}\left(\frac{\partial u_1}{\partial\theta^1}-u_m\Gamma_{11}^m\right)\left(\frac{\partial u^1}{\partial\theta^1}+u^m\Gamma_{m1}^1\right)=u_{r,r}+\frac{1}{2}\left(u_{r,r}\right)^2, \tag{20}$$

$$\varepsilon_{22}=r^2\varepsilon_{\vartheta\vartheta}=\frac{\partial u_2}{\partial\theta^2}-u_m\Gamma_{22}^m+\frac{1}{2}\left(\frac{\partial u_2}{\partial\theta^2}-u_m\Gamma_{22}^m\right)\left(\frac{\partial u^2}{\partial\theta^2}+u^m\Gamma_{m2}^2\right)$$

$$=-u_m\Gamma_{22}^m+\frac{1}{2}\left(-u_m\Gamma_{22}^m\right)\left(u^m\Gamma_{m2}^2\right)=u_r r+\frac{1}{2}\left(u_r\right)^2, \tag{21}$$

$$\varepsilon_{33}=\varepsilon_{zz}=\frac{\partial u^3}{\partial\theta^3}+u^m\Gamma_{m3}^3+\left(\frac{\partial u^3}{\partial\theta^3}+u^m\Gamma_{m3}^3\right)^2=0, \tag{22}$$

$$\varepsilon_{12}=\varepsilon_{r\vartheta}=\frac{1}{2}\left[\frac{\partial u^2}{\partial x^1}+\frac{\partial u^1}{\partial x^2}-u^m\Gamma_{m1}^2-u^m\Gamma_{m2}^1\right]=0, \tag{23}$$

$$\varepsilon_{13}=\varepsilon_{rz}=\frac{1}{2}\left[\frac{\partial u^3}{\partial x^1}+\frac{\partial u^1}{\partial x^3}-u^m\Gamma_{m1}^3-u^m\Gamma_{m3}^1\right]=0, \tag{24}$$

$$\varepsilon_{23}=\varepsilon_{\vartheta z}=\frac{1}{2}\left[\frac{\partial u^2}{\partial x^3}+\frac{\partial u^3}{\partial x^2}-u^m\Gamma_{m3}^2-u^m\Gamma_{m2}^3\right]=0. \tag{25}$$

Following the approach of [6], we return to the general nonlinear equations of motion

$$\nabla_k\left[\sigma^{ki}\left(\delta_i^n+\nabla_i u^n\right)\right]=\rho\ddot{u}^i. \tag{26}$$

In many instances, it is convenient to analyze Eqs. (26) as those for displacement components alone. Therefore, we should determine the stress components

$$\sigma^{11}=\sigma^{rr},\quad \sigma^{22}=\frac{1}{r^2}\sigma^{\vartheta\vartheta},\quad \sigma^{33}=\sigma^{zz},\quad \sigma^{12}=\frac{1}{r}\sigma^{r\vartheta},\quad \sigma^{13}=\sigma^{rz},\quad \sigma^{23}=\frac{1}{r}\sigma^{\vartheta z} \tag{27}$$

as nonlinear functions of strain components using the formula $\sigma_{ik}=\left(\partial W/\partial\varepsilon_{ik}\right)$.

Remark 1. We additionally assume that these functions are quadratically nonlinear, thus neglecting higher nonlinearities. Such an assumption was earlier adopted in nonlinear acoustics to analyze nonlinear elastic waves in Cartesian coordinates [2, 3, 5, 11–15].

The deformation of a hyperelastic medium is described by the Murnaghan potential

$$W(I_1,I_2,I_3)=\frac{1}{2}\lambda I_1^2+\mu I_2+\frac{1}{3}AI_3+BI_1I_2+\frac{1}{3}CI_1^3, \tag{28}$$

where the first three algebraic invariants are defined by

$$I_1(\varepsilon_{ik})=\varepsilon_{ik}g^{ik}=\varepsilon_{11}\cdot 1+\varepsilon_{22}\cdot\frac{1}{r^2}+\varepsilon_{33}\cdot 1, \tag{29}$$

$$I_2(\varepsilon_{ik}) = \varepsilon_{im}\varepsilon_{nk} g^{ik} g^{nm} = (\varepsilon_{11}\cdot 1)^2 + \left(\varepsilon_{22}\cdot\frac{1}{r^2}\right)^2 + (\varepsilon_{33}\cdot 1)^2 + \left(\varepsilon_{12}\cdot\frac{1}{r}\right)^2 + \left(\varepsilon_{23}\cdot\frac{1}{r}\right)^2 + (\varepsilon_{13}\cdot 1)^2, \tag{30}$$

$$I_3(\varepsilon_{ik}) = \varepsilon_{pm}\varepsilon_{in}\varepsilon_{kq} g^{im} g^{pq} g^{kn} = (\varepsilon_{11})^3 + \left(\varepsilon_{22}\frac{1}{r^2}\right)^3 + (\varepsilon_{33})^3 + (\varepsilon_{13}\cdot 1)\left(\varepsilon_{13}\varepsilon_{11} + \varepsilon_{23}\varepsilon_{12}\frac{1}{r^2} + \varepsilon_{13}\varepsilon_{33}\right)$$

$$+\left(\varepsilon_{12}\cdot\frac{1}{r^2}\right)\left(\varepsilon_{12}\varepsilon_{11} + \varepsilon_{12}\varepsilon_{22}\frac{1}{r^2} + \varepsilon_{13}\varepsilon_{23}\right) + \left(\varepsilon_{23}\cdot\frac{1}{r^2}\right)\left(\varepsilon_{12}\varepsilon_{13} + \varepsilon_{23}\varepsilon_{22}\frac{1}{r^2} + \varepsilon_{23}\varepsilon_{33}\right), \tag{31}$$

and λ and μ are the Lamé constants (second-order constants) and A, B, and C are the Murnaghan constants (third-order constants).

The components of the stress tensor necessary for further discussion are

$$\sigma^{11} = \lambda I_1 \frac{\partial I_1}{\partial\varepsilon_{11}} + \mu\frac{\partial I_2}{\partial\varepsilon_{11}} + \frac{1}{3}A\frac{\partial I_3}{\partial\varepsilon_{11}} + B\left(I_1\frac{\partial I_2}{\partial\varepsilon_{11}} + I_2\frac{\partial I_1}{\partial\varepsilon_{11}}\right) + CI_1^2\frac{\partial I_1}{\partial\varepsilon_{11}}$$

$$= \lambda I_1 + 2\mu\varepsilon_{11} + A\left[(\varepsilon_{11})^2 + \frac{1}{3r^2}(\varepsilon_{12})^2 + \frac{1}{3}(\varepsilon_{13})^2\right] + B\left(2\varepsilon_{11}I_1 + I_2\right) + CI_1^2, \tag{32}$$

$$\sigma^{22} = \frac{1}{r^2}\left[\lambda I_1\frac{\partial I_1}{\partial\varepsilon_{22}} + \mu\frac{\partial I_2}{\partial\varepsilon_{22}} + \frac{1}{3}A\frac{\partial I_3}{\partial\varepsilon_{22}} + B\left(I_1\frac{\partial I_2}{\partial\varepsilon_{22}} + I_2\frac{\partial I_1}{\partial\varepsilon_{22}}\right) + \frac{1}{3}CI_1^2\frac{\partial I_1}{\partial\varepsilon_{kk}}\right]$$

$$= \frac{1}{r^2}\left[\lambda I_1 + 2\mu\varepsilon_{22} + A\frac{1}{r^2}\left[\frac{1}{r^2}(\varepsilon_{22})^2 + \frac{1}{3}(\varepsilon_{12})^2 + \frac{1}{3r^2}(\varepsilon_{23})^2\right] + B\left(\frac{2}{r^2}\varepsilon_{22}I_1 + I_2\right) + CI_1^2\right], \tag{33}$$

$$\sigma^{33} = \lambda I_1\frac{\partial I_1}{\partial\varepsilon_{33}} + \mu\frac{\partial I_2}{\partial\varepsilon_{33}} + \frac{1}{3}A\frac{\partial I_3}{\partial\varepsilon_{33}} + B\left(I_1\frac{\partial I_2}{\partial\varepsilon_{33}} + I_2\frac{\partial I_1}{\partial\varepsilon_{33}}\right) + CI_1^2\frac{\partial I_1}{\partial\varepsilon_{33}}$$

$$= \lambda I_1 + 2\mu\varepsilon_{33} + A\left[(\varepsilon_{33})^2 + \frac{1}{3r^2}(\varepsilon_{23})^2 + \frac{1}{3}(\varepsilon_{13})^2\right](\varepsilon_{33})^2 + B\left(2\varepsilon_{33}I_1 + I_2\right) + CI_1^2, \tag{34}$$

$$\sigma^{12} = \lambda I_1\cancel{\frac{\partial I_1}{\partial\varepsilon_{12}}} + \mu\frac{\partial I_2}{\partial\varepsilon_{12}} + \frac{1}{3}A\frac{\partial I_3}{\partial\varepsilon_{12}} + B\left(I_1\frac{\partial I_2}{\partial\varepsilon_{12}} + I_2\cancel{\frac{\partial I_1}{\partial\varepsilon_{12}}}\right) + CI_1^2\cancel{\frac{\partial I_1}{\partial\varepsilon_{12}}}$$

$$= \frac{1}{r^2}\left[2\mu\varepsilon_{12} + \left(\frac{2}{3}A + 2B\right)\varepsilon_{12}I_1 + \frac{2}{3r^2}A\varepsilon_{13}\varepsilon_{23}\right], \tag{35}$$

$$\sigma^{13} = \lambda I_1\cancel{\frac{\partial I_1}{\partial\varepsilon_{13}}} + \mu\frac{\partial I_2}{\partial\varepsilon_{13}} + \frac{1}{3}A\frac{\partial I_3}{\partial\varepsilon_{13}} + B\left(I_1\frac{\partial I_2}{\partial\varepsilon_{13}} + I_2\cancel{\frac{\partial I_1}{\partial\varepsilon_{13}}}\right) + CI_1^2\cancel{\frac{\partial I_1}{\partial\varepsilon_{13}}}$$

$$= \left[2\mu\varepsilon_{13} + \left(\frac{2}{3}A + 2B\right)\varepsilon_{13}I_1 + \frac{2}{3r^2}A\varepsilon_{12}\varepsilon_{23}\right], \tag{36}$$

$$\sigma^{23} = \lambda I_1\cancel{\frac{\partial I_1}{\partial\varepsilon_{23}}} + \mu\frac{\partial I_2}{\partial\varepsilon_{23}} + \frac{1}{3}A\frac{\partial I_3}{\partial\varepsilon_{23}} + B\left(I_1\frac{\partial I_2}{\partial\varepsilon_{23}} + I_2\cancel{\frac{\partial I_1}{\partial\varepsilon_{23}}}\right) + CI_1^2\cancel{\frac{\partial I_1}{\partial\varepsilon_{23}}}$$

$$= \frac{1}{r^2}\left[2\mu\varepsilon_{23} + \left(\frac{2}{3}A + 2B\right)\varepsilon_{23}I_1 + \frac{2}{3r^2}A\varepsilon_{12}\varepsilon_{13}\right], \tag{37}$$

where

$$\frac{\partial I_1}{\partial \varepsilon_{11}}=1, \quad \frac{\partial I_1}{\partial \varepsilon_{22}}=\frac{1}{r^2}, \quad \frac{\partial I_1}{\partial \varepsilon_{33}}=1, \quad \frac{\partial I_1}{\partial \varepsilon_{12}}=\frac{\partial I_1}{\partial \varepsilon_{13}}=\frac{\partial I_1}{\partial \varepsilon_{23}}=0,$$

$$\frac{\partial I_2}{\partial \varepsilon_{11}}=2\varepsilon_{11}, \quad \frac{\partial I_2}{\partial \varepsilon_{22}}=\frac{2}{r^2}\varepsilon_{22}, \quad \frac{\partial I_2}{\partial \varepsilon_{33}}=2\varepsilon_{33}, \quad \frac{\partial I_2}{\partial \varepsilon_{12}}=\frac{2}{r^2}\varepsilon_{12}, \quad \frac{\partial I_2}{\partial \varepsilon_{23}}=\frac{2}{r^2}2\varepsilon_{23}, \quad \frac{\partial I_2}{\partial \varepsilon_{13}}=2\varepsilon_{13},$$

$$\frac{\partial I_3}{\partial \varepsilon_{11}}=3(\varepsilon_{11})^2, \quad \frac{\partial I_3}{\partial \varepsilon_{22}}=\frac{3}{r^2}\left(\varepsilon_{22}\frac{1}{r^2}\right)^2, \quad \frac{\partial I_3}{\partial \varepsilon_{33}}=3(\varepsilon_{33})^2,$$

$$\frac{\partial I_3}{\partial \varepsilon_{12}}=\frac{2}{r^2}\left[\varepsilon_{12}\left(\varepsilon_{11}+\frac{1}{r^2}\varepsilon_{22}\right)+\varepsilon_{13}\varepsilon_{23}\right], \quad \frac{\partial I_3}{\partial \varepsilon_{13}}=2\left[\varepsilon_{13}\left(\varepsilon_{11}+\varepsilon_{33}\right)+\frac{1}{r^2}\varepsilon_{12}\varepsilon_{23}\right],$$

$$\frac{\partial I_3}{\partial \varepsilon_{23}}=\frac{2}{r^2}\left[\varepsilon_{23}\left(\frac{1}{r^2}\varepsilon_{22}+\varepsilon_{33}\right)+\varepsilon_{12}\varepsilon_{13}\right].$$

As follows from Eqs.(32)–(37), the procedure of expressing the stress components in terms of the displacement components begins with determining $(I_1)^2$, $(\varepsilon_{kk})^2$, and $(\varepsilon_{ik}\varepsilon_{lm})$ up to quadratically nonlinear terms.

Let us write the equations of motion for each of the three states.

State I. The second equation degenerates into an identity, and the first and third equations have the form

$$\sigma_{rr,r}+\sigma_{rz,z}+\frac{1}{r}\left(\sigma_{rr}-\sigma_{\vartheta\vartheta}\right)-\rho\ddot{u}_r=-\frac{1}{r}\left(u_{r,r}+ru_{r,rr}\right)\sigma_{rr}-\frac{1}{r}\left(2u_{r,zr}+u_{r,z}\right)\sigma_{rz}-u_{r,zz}\sigma_{zz}-\frac{u_r}{r^2}\sigma_{\vartheta\vartheta}$$

$$-u_{r,r}\sigma_{rr,r}-u_{r,z}\sigma_{rz,r}-u_{r,r}\sigma_{rz,z}-u_{r,z}\sigma_{zz,z}, \tag{38}$$

$$\sigma_{rz,r}+\sigma_{zz,z}+\frac{1}{r}\sigma_{rz}-\rho\ddot{u}_z=-\frac{1}{r}\left(u_{z,r}+ru_{z,rr}\right)\sigma_{rr}-\frac{1}{r}\left(2u_{z,zr}+u_{z,z}\right)\sigma_{rz}-u_{z,zz}\sigma_{zz}$$

$$-u_{z,r}\sigma_{rr,r}-u_{z,z}\sigma_{rz,r}-u_{z,r}\sigma_{rz,z}-u_{z,z}\sigma_{zz,z}. \tag{39}$$

State II. The third equation degenerates into an identity, and the first and second equations have the form

$$-\left(u_\vartheta\sigma_{r\vartheta}\right)_{,r}+u_{\vartheta,r}\sigma_{r\vartheta}-\left(u_\vartheta\sigma_{\vartheta z}\right)_{,z}+u_{\vartheta,z}\sigma_{\vartheta z}=-u_\vartheta\left(\cancel{\sigma_{r\vartheta,r}+\sigma_{\vartheta z,z}}\right)=0, \tag{40}$$

$$\cancel{\sigma_{r\vartheta,r}+\sigma_{\vartheta z,z}}+\frac{2}{r}\sigma_{r\vartheta}-\rho\ddot{u}_\vartheta$$

$$=-\left(u_{\vartheta,rr}+\frac{1}{r}u_{\vartheta,r}\right)\sigma_{rr}-u_{\vartheta,rz}\sigma_{rz}-u_{\vartheta,zz}\sigma_{zz}-\frac{1}{r^2}u_\vartheta\sigma_{\vartheta\vartheta}-u_{\vartheta,r}\sigma_{rr,r}-u_{\vartheta,r}\sigma_{rz,z}-u_{\vartheta,z}\sigma_{zz,z}. \tag{41}$$

Remark 2. The left-hand side of Eq. (41) includes a linear part that corresponds to linear theory. With Eq. (40), however, this linear part (without the crossed term) no longer corresponds to classical theory. This situation is unusual and should be commented on.

State III. The last two equilibrium equations are satisfied identically, and the first equation has the form

$$\sigma_{rr,r}+\frac{1}{r}\left(\sigma_{rr}-\sigma_{\vartheta\vartheta}\right)-\rho\ddot{u}_r=-\left(u_{r,rr}+\frac{1}{r}u_{r,r}\right)\sigma_{rr}-\frac{1}{r^2}u_r\sigma_{\vartheta\vartheta}-u_{r,r}\sigma_{rr,r}. \tag{42}$$

Remark 3. The nonlinear equation (42) is simpler than the previous ones; therefore, it is pertinent to comment on exactly this equation. Our comment is concerned with the conditional separation of physical and geometrical nonlinearities in such equations (i.e., Eqs. (38)–(41)). In fact, before the stress tensor expressed in terms of the displacement vector (constitutive relations) is substituted into these equations, their right-hand sides display only geometrical nonlinearity. Physical nonlinearity is introduced by the constitutive relations. Our experience of analyzing similar equations in Cartesian coordinates suggests that four cases are possible here: (i) the linear Cauchy–Green tensor in the nonlinear constitutive relations and nonlinear right-hand sides neglected in Eqs. (38)–(41) (nonlinearity is purely physical); (ii) the nonlinear Cauchy–Green tensor in the nonlinear constitutive relations and nonlinear right-hand sides neglected in Eqs. (38)–(41) (nonlinearity is purely physical); (iii) the nonlinear Cauchy–Green tensor in the nonlinear constitutive relations and nonlinear right-hand sides present in Eqs. (38)–(41) (all nonlinearities are taken into account); and (iv) the nonlinear Cauchy–Green tensor in the linear constitutive relations and nonlinear right-hand sides present in Eqs. (38)–(41) (nonlinearity is purely geometrical). Cases 1 and 2 were used in nonlinear acoustics and in the analysis of nonlinear waves in microstructural materials. We will further show the difference among the cases for state IV.

Let us now calculate the necessary stress components as functions of displacement components for each of the states.

State I. The algebraic invariants are

$$I_1 = \varepsilon_{11} + \frac{\varepsilon_{22}}{r^2} + \varepsilon_{33} = u_{r,r} + \frac{u_r}{r} + u_{z,z} + \frac{1}{2}(u_{r,r})^2 + \frac{1}{2r^2}(u_r)^2 + \frac{1}{2}(u_{z,z})^2 + \frac{1}{2}(u_{r,z})^2. \tag{43}$$

Remark 4. The first three terms are the linear part of the invariant and the other terms are quadratically nonlinear ones of five types. Since the strain tensor is quadratically nonlinear, the first invariant does not include higher nonlinearities,

$$I_2 = (\varepsilon_{11})^2 + (\varepsilon_{22})^2 + (\varepsilon_{33})^2 + (\varepsilon_{13})^2 = (u_{r,r})^2 + \frac{1}{r^2}(u_r)^2 + (u_{z,z})^2 + \frac{1}{4}(u_{r,z} + u_{z,r})^2. \tag{44}$$

Let us write some necessary expressions

$$(I_1)^2 = (u_{r,r})^2 + (u_{z,z})^2 + 2u_{z,z}u_{r,r}, \quad (\varepsilon_{11})^2 = (u_{r,r})^2, \quad (\varepsilon_{33})^2 = (u_{z,z})^2, \quad (\varepsilon_{13})^2 = \frac{1}{4}(u_{r,z} + u_{z,r})^2,$$

$$2\varepsilon_{11}I_1 = 2u_{r,r}(u_{r,r} + u_{z,z}), \quad 2\varepsilon_{22}I_1 = 2u_{z,z}(u_{r,r} + u_{z,z}), \quad 2\varepsilon_{13}I_1 = (u_{r,z} + u_{z,r})(u_{r,r} + u_{z,z})$$

and expressions for the stress components

$$\sigma_{11} = \sigma_{rr} = (\lambda + 2\mu)\, u_{r,r} + \lambda\left(\frac{u_r}{r} + u_{z,z}\right)$$

$$+ \left[\frac{1}{2}(\lambda + 2\mu) + A + 3B + C\right](u_{r,r})^2 + \left(\frac{1}{2}\lambda + B + C\right)\left[\frac{(u_r)^2}{r^2} + (u_{z,z})^2\right] + \frac{1}{6}(A + 3B)\, u_{r,z}u_{z,r}$$

$$+ \frac{1}{12}\left[6(\lambda + 2\mu) + A + 3B\right](u_{r,z})^2 + \frac{1}{12}[6\lambda + A + 3B](u_{z,r})^2 + 2C\left(u_{r,r}u_{z,z} + \frac{1}{r}u_r u_{r,r} + \frac{1}{r}u_r u_{z,z}\right), \tag{45}$$

$$\sigma_{22} = \frac{1}{r^2}\sigma_{\vartheta\vartheta} = (\lambda + 2\mu)\frac{u_r}{r} + \lambda\left(u_{r,r} + u_{z,z}\right)$$

$$+ \left[\frac{1}{2}(\lambda + 2\mu) + A + 3B + C\right]\frac{1}{r^2}(u_r)^2 + \left(\frac{1}{2}\lambda + B + C\right)\left[(u_{r,r})^2 + (u_{z,z})^2\right] + \frac{1}{2}Bu_{r,z}u_{z,r}$$

$$+ \frac{1}{4}(2\lambda + B)\left[(u_{r,z})^2 + (u_{z,r})^2\right] + 2Cu_{r,r}u_{z,z}, \tag{46}$$

$$\sigma_{33} = \sigma_{zz} = (\lambda + 2\mu)\, u_{z,z} + \lambda\left(u_{r,r} + \frac{u_r}{r}\right)$$

$$+ \left[\frac{1}{2}(\lambda + 2\mu) + A + 3B + C\right](u_{z,z})^2 + \left(\frac{1}{2}\lambda + B + C\right)\left[(u_{r,r})^2 + \frac{1}{r^2}(u_r)^2\right] + \frac{1}{6}(A + 3B)\, u_{r,z}u_{z,r}$$

$$+\frac{1}{12}\left[6(\lambda+2\mu)+A+3B\right](u_{z,r})^2+\frac{1}{12}[6\lambda+A+3B](u_{r,z})^2+2C\left(u_{r,r}u_{z,z}+\frac{1}{r}u_r u_{r,r}+\frac{1}{r}u_r u_{z,z}\right), \quad (47)$$

$$\sigma_{13}=\sigma_{rz}=\mu\left(u_{r,z}+u_{z,r}\right)+\left(\mu+\frac{1}{3}A+B\right)\left(u_{r,r}u_{r,z}+u_{z,r}u_{z,z}\right)+\left(\frac{1}{3}A+B\right)\left(u_{r,r}u_{z,r}+u_{r,z}u_{z,z}\right). \quad (48)$$

Remark 5. These expressions include 12 types of nonlinearity: 1) $(u_{r,r})^2$; 2) $(u_{z,z})^2$; 3) $(u_{r,z})^2$; 4) $(u_{z,r})^2$; 5) $(u_{r,r}u_{r,z})$; 6) $(u_{r,r}u_{z,r})$; 7) $(u_{r,z}u_{z,z})$; 8) $(u_{z,r}u_{z,z})$; 9) $(u_{r,r}u_{z,z})$; 10) $\frac{1}{r^2}(u_r)^2$; 11) $\left(\frac{1}{r^2}u_r u_{r,r}\right)$; 12) $\left(\frac{1}{r^2}u_r u_{z,z}\right)$.

State II. The algebraic invariants are

$$I_1=\frac{1}{2}\left[(u_{\vartheta,r})^2+\frac{1}{r^2}(u_\vartheta)^2+(u_{\vartheta,z})^2\right], \qquad I_2=\left(u_{\vartheta,r}-\frac{u_\vartheta}{r}\right)^2+(u_{\vartheta,z})^2. \quad (49)$$

We also need the following expressions:

$$(I_1)^2=0, \quad \varepsilon_{kk}I_1=0, \quad \varepsilon_{11}=\frac{1}{2}(u_{\vartheta,r})^2\to(\varepsilon_{11})^2=0, \quad \varepsilon_{22}=\frac{1}{2}(u_\vartheta)^2\to(\varepsilon_{22})^2=0,$$

$$\varepsilon_{33}=\frac{1}{2}(u_{\vartheta,z})^2\to(\varepsilon_{33})^2=0, \qquad \varepsilon_{12}=(ru_{\vartheta,r}-u_\vartheta)\to(\varepsilon_{12})^2=(ru_{\vartheta,r}-u_\vartheta)^2,$$

$$\varepsilon_{23}=ru_{\vartheta,z}\to(\varepsilon_{23})^2=r^2(u_{\vartheta,z})^2, \qquad \varepsilon_{13}=\frac{1}{2}u_{\vartheta,r}u_{\vartheta,z}\to(\varepsilon_{13})^2=0 \quad (50)$$

and the expressions for the stress components

$$\sigma_{11}=\sigma_{rr}=\frac{1}{6}\left[3(\lambda+2\mu)+2(A+3B)\right](u_{\vartheta,r})^2+\frac{1}{6}\left[3\lambda+2(A+3B)\right]\frac{1}{r^2}(u_\vartheta)^2$$

$$+\frac{2}{3}(A+3B)\frac{1}{r}u_\vartheta u_{\vartheta,r}+\left(\frac{\lambda}{2}+B\right)(u_{\vartheta,z})^2, \quad (51)$$

$$\sigma_{22}=\frac{1}{r^2}\sigma_{\vartheta\vartheta}=\frac{1}{6}\left[3(\lambda+2\mu)+2(A+3B)\right]\frac{1}{r^2}(u_\vartheta)^2+\frac{1}{6}\left[3\lambda+2(A+3B)\right](u_{\vartheta,r})^2$$

$$+\frac{1}{6}\left[3\lambda+2(A+3B)\right](u_{\vartheta,r})^2+\frac{2}{3}(A+3B)\frac{1}{r}u_\vartheta u_{\vartheta,r}, \quad (52)$$

$$\sigma_{33}=\sigma_{zz}=\frac{1}{6}\left[3(\lambda+2\mu)+2(A+3B)\right](u_{\vartheta,z})^2+\left(\frac{\lambda}{2}+B\right)\left[(u_{\vartheta,r})+\frac{1}{r^2}(u_\vartheta)^2\right]+2B\frac{1}{r}u_\vartheta u_{\vartheta,r}, \quad (53)$$

$$\sigma_{12}=\frac{1}{r}\sigma_{r\vartheta}=\frac{1}{r^2}\left[2\mu(ru_{\vartheta,r}-u_\vartheta)+\frac{2}{3}A\left(u_{\vartheta,r}u_{\vartheta,z}-\frac{1}{r}u_{\vartheta,z}u_\vartheta\right)\right], \quad (54)$$

$$\sigma_{23}=\frac{1}{r}\sigma_{\vartheta z}=\frac{2}{r^2}\mu u_{\vartheta,z}, \qquad \sigma_{13}=\sigma_{rz}=\left(\mu+\frac{2}{3}A\right)u_{\vartheta,r}u_{\vartheta,z}-\frac{2}{3}A\frac{1}{r}u_{\vartheta,z}u_\vartheta. \quad (55)$$

Remark 6. As follows from (51)–(55), this state is characterized by six types of nonlinearity: 1) $(u_\vartheta)^2$; 2) $(u_{\vartheta,r})^2$; 3) $(u_{\vartheta,z})^2$; 4) $(u_{\vartheta,r}u_{\vartheta,z})$; 5) $(u_\vartheta u_{\vartheta,r})$; 6) $(u_\vartheta u_{\vartheta,z})$.

State III. Let us calculate the components of the stress tensor $\sigma_{11}=\sigma_{rr}$ and $\sigma_{22}=\frac{1}{r^2}\sigma_{\vartheta\vartheta}$,

$$\sigma_{11} = \sigma_{rr} = \lambda I_1 + 2\mu\varepsilon_{11} + A\,(\varepsilon_{11})^2 + B\left(2\varepsilon_{11} I_1 + I_2\right) + C I_1^2, \tag{56}$$

$$\sigma_{22} = \frac{1}{r^2}\sigma_{\vartheta\vartheta} = \frac{1}{r^2}\left[\lambda I_1 + 2\mu\varepsilon_{22} + A\left(\varepsilon_{22}\frac{1}{r^2}\right)^2 + B\left(\frac{2}{r^2}\varepsilon_{22} I_1 + I_2\right) + C I_1^2\right]. \tag{57}$$

To this end, we write the expressions of the algebraic invariants

$$I_1 = u_{r,r} + \frac{u_r}{r} + \frac{1}{2}(u_{r,r})^2 + \frac{1}{2}(u_r)^2, \qquad I_2 = (u_{r,r})^2 + \frac{(u_r)^2}{r^2} \tag{58}$$

and some necessary expressions

$$(I_1)^2 = (u_{r,r})^2 + \frac{u_r^2}{r^2} + 2\frac{u_r u_{r,r}}{r}, \qquad (\varepsilon_{11})^2 = (u_{r,r})^2,$$

$$2\varepsilon_{11} I_1 = 2\left[(u_{r,r})^2 + \frac{1}{r}u_{r,r}u_r\right], \qquad 2\varepsilon_{22} I_1 = 2\left[(u_r)^2 + r u_{r,r} u_r\right].$$

Finally, we get

$$\sigma^{11} = \sigma^{rr} = \lambda\left(u_{r,r} + \frac{u_r}{r}\right) + 2\mu\, u_{r,r}$$

$$+\left[\frac{1}{2}(\lambda + 2\mu) + A + 3B + C\right](u_{r,r})^2 + \left(\frac{1}{2}\lambda + B + C\right)\frac{(u_r)^2}{r^2} + \frac{2}{r}(B + C)\, u_{r,r} u_r, \tag{59}$$

$$\sigma^{22} r^2 = \sigma^{\vartheta\vartheta} = \lambda\left(u_{r,r} + \frac{u_r}{r}\right) + 2\mu\frac{u_r}{r}$$

$$+\left(\frac{1}{2}\lambda + 3B + C\right)(u_{r,r})^2 + \left[\frac{1}{2}(\lambda + 2\mu) + A + B + C\right]\frac{(u_r)^2}{r^2} + \frac{2}{r}(B + C)\, u_{r,r} u_r. \tag{60}$$

Remark 7. Since this state is the simplest of those mentioned above, the number of nonlinearities in Eqs. (59), (60) is minimum: 1) $(u_{r,r})^2$; 2) $(u_r)^2$; 3) $(u_{r,r} u_r)$.

Thus, we have derived wave equations in terms of the stress tensor and displacement vector and second-order representations of the stress tensor in terms of the displacement vector for three states: Eqs. (38), (39), and (45)–(48) for state I; Eqs. (40), (41), and (51)–(55) for state II; and Eqs. (42), (59), and (60) for state III.

However, the wave equations expressed only in terms of displacement components are more customary. Let us also write such equations for cases 2, 3, and 4 (see Remark 3).

State I, *Case 4* (geometrical nonlinearity is taken into account):

$$\mu\left(u_{r,rr} + u_{r,zz} + \frac{1}{r}u_{r,r} - \frac{1}{r^2}u_r\right) + (\lambda + \mu)\left(u_{r,rr} + u_{z,zr} + \frac{1}{r}u_{r,r} - \frac{1}{r^2}u_r\right) - \rho\ddot{u}_r$$

$$\equiv \mu\,\Delta u_r + (\lambda + \mu)\, I_{1,r} - \rho\ddot{u}_r = -2(\lambda + 2\mu)\, u_{r,r}u_{r,rr} - 2(\lambda + 2\mu)\frac{1}{r}(u_{r,r})^2$$

$$-(\lambda + 2\mu)\frac{1}{r^3}(u_r)^2 - (\lambda + 2\mu)\, u_{r,z}u_{z,zz} - (\lambda + 2\mu)\, u_{z,z}u_{r,zz} - (\lambda + \mu)\, u_{r,r}u_{z,rz}$$

$$-(\lambda + 2\mu)\, u_{r,r}u_{r,zr} - (\lambda + \mu)\frac{1}{r}(u_{r,z})^2 - (\lambda + \mu)\, u_{r,r}u_{r,zz} - 3\mu u_{r,r}u_{r,zr}$$

$$-2\mu u_{z,r}u_{r,zr} - \mu u_{r,z}u_{z,rr} - \mu\frac{1}{r}u_{r,z}u_{z,r} - \lambda u_{r,r}u_{z,z} - \lambda\frac{1}{r}u_r u_{r,zz}$$

$$-\lambda\frac{1}{r}u_r u_{r,rr}-\lambda\frac{1}{r^2}u_r u_{r,r}-\lambda\frac{1}{r^2}u_r u_{z,z}-\lambda u_{z,z}u_{r,rr}-\lambda u_{r,z}u_{r,rz}, \tag{61}$$

$$\mu\left(u_{z,rr}+u_{z,zz}+\frac{1}{r}u_{z,r}\right)+(\lambda+\mu)\left(u_{r,rz}+u_{z,zz}+\frac{1}{r}u_{r,z}\right)-\rho\ddot{u}_z$$

$$\equiv\mu\,\Delta u_z+(\lambda+\mu)I_{1,z}-\rho\ddot{u}_z=-2(\lambda+2\mu)\,u_{z,z}u_{z,zz}-(\lambda+2\mu)\frac{1}{r}u_{r,r}u_{z,r}$$

$$-(\lambda+2\mu)\,u_{r,r}u_{z,rr}-(\lambda+2\mu)\,u_{z,r}u_{r,rr}-3\mu u_{z,r}u_{z,zr}-2\mu u_{r,z}u_{z,zr}$$

$$-(\lambda+\mu)\,u_{z,z}u_{r,rz}-(\lambda+\mu)\frac{1}{r}u_{z,z}u_{z,r}-(\lambda+\mu)\frac{1}{r}u_{z,r}u_{z,z}$$

$$-\mu u_{z,r}u_{r,zz}-\mu\frac{1}{r}u_{z,z}u_{r,z}-\lambda u_{r,r}u_{z,zz}-\lambda\frac{1}{r}u_r u_{z,rr}$$

$$-\lambda\frac{1}{r}u_{z,r}u_{r,r}-\lambda\frac{1}{r}u_{z,z}u_{r,z}-\lambda\frac{1}{r}u_r u_{z,zz}-\lambda u_{r,r}u_{z,zz}-\lambda u_{z,r}u_{z,zr}. \tag{62}$$

State II, *Case 4* (geometrical nonlinearity is taken into account).

Since all the stress components appearing on the right-hand side of Eq. (39) have only quadratically nonlinear components (no linear components), case 4 leads to zero right-hand side of Eq. (39). Therefore, there is no quadratically nonlinear equation, yet there is a cubically nonlinear equation. Hence, in quadratic approximation, case 2 coincides with case 3.

Sate II, *Case 2* (physical and partially geometrical nonlinearities are taken into account):

$$2\mu\left(u_{\vartheta,rr}+\frac{1}{r}u_{\vartheta,r}-\frac{1}{r^2}u_\vartheta+u_{\vartheta,zz}\right)-\rho\ddot{u}_\vartheta=-\frac{2}{3}A\frac{1}{r^2}\left[2(u_{\vartheta,r}u_{\vartheta,z})+(u_{\vartheta,rz}u_\vartheta)+(u_{\vartheta,rr}u_{\vartheta,z})+(u_{\vartheta,rz}u_{\vartheta,z})\right]. \tag{63}$$

Remark 8. The classical linear solution for a cylindrical wave (when the right-hand side of Eq. (63) is equal to zero) has the form $u_\vartheta(r,z,t)=u_\vartheta^0\cdot J_1(\beta r)\cdot e^{i(kz-\omega t)}$ ($u_\vartheta^0=\mathrm{const}$) [5, 10, 13]. Therefore, in view of the form of the nonlinear right-hand side, it is easy to solve the nonlinear wave equation (63) by the method of successive approximations and to predict that the second harmonic will appear and the form of amplitude will become more complex.

State III, *Case 3* (physical and geometrical nonlinearities are taken into account):

$$(\lambda+2\mu)\left(u_{r,r}+\frac{u_r}{r}\right)_{,r}-\rho\ddot{u}_r=-\left[3(\lambda+2\mu)+2(A+3B+C)\right]u_{r,rr}u_{r,r}-(\lambda+2B+2C)\frac{1}{r}u_{r,rr}u_r$$

$$-\frac{\lambda}{r^2}u_{r,r}u_r-\left[2\lambda+3\mu+A+2B+2C\right]\frac{1}{r}(u_{r,r})^2-\left[2\lambda+3\mu+A+2B+C\right]\frac{1}{r^3}(u_r)^2. \tag{64}$$

Remark 9. The first row in (64) is the linear part of the equation and can be calculated from (42) as follows:

$$\lambda\left(u_{r,r}+\frac{u_r}{r}\right)_{,r}+2\mu\,(u_{r,r})_{,r}+\frac{1}{r}\left[\lambda\left(u_{r,r}+\frac{u_r}{r}\right)+2\mu u_{r,r}-\lambda\left(u_{r,r}+\frac{u_r}{r}\right)-2\frac{u_r}{r}\right]-\rho\ddot{u}_r$$

$$=\lambda\left(u_{r,r}+\frac{u_r}{r}\right)_{,r}+2\mu\left(u_{r,rr}+\frac{u_{r,r}}{r}-\frac{u_r}{r^2}\right)-\rho\ddot{u}_r=(\lambda+2\mu)\left(u_{r,r}+\frac{u_r}{r}\right)_{,r}-\rho\ddot{u}_r.$$

State III, *Case 2* (physical and partially geometrical nonlinearities are taken into account):

$$(\lambda+2\mu)\left(u_{r,r}+\frac{u_r}{r}\right)_{,r}-\rho\ddot{u}_r=-\left[\lambda+2\mu+2(A+3B+C)\right]u_{r,rr}u_{r,r}-2(B+C)\frac{1}{r}u_{r,rr}u_r$$

$$-\frac{\lambda}{r^2}u_{r,r}u_r-\left[\mu+A+2B+2C\right]\frac{1}{r}(u_{r,r})^2-\left[\lambda+\mu+A+2B+C\right]\frac{1}{r^3}(u_r)^2. \tag{65}$$

State III, *Case 4* (geometrical nonlinearity is taken into account):

$$(\lambda+2\mu)\left(u_{r,r}+\frac{u_r}{r}\right)_{,r}-\rho\ddot{u}_r$$

$$=-2(\lambda+2\mu)\,u_{r,rr}u_{r,r}-\lambda\frac{1}{r}u_{r,rr}u_r-\frac{\lambda}{r^2}u_{r,r}u_r-2(\lambda+\mu)\frac{1}{r}(u_{r,r})^2-(\lambda+2\mu)\frac{1}{r^3}(u_r)^2. \tag{66}$$

Thus, we have derived systems of nonlinear wave equations in cylindrical (orthogonal) coordinates. Three special configurations (states) of an elastic medium have been considered:

(i) axisymmetric configuration dependent on the coordinates r and z and independent of coordinate ϑ with symmetry axis Oz;

(ii) configuration dependent only on the angular coordinate ϑ with symmetry axis Oz;

(iii) and axisymmetric configuration dependent only on the coordinate r with symmetry axis Oz.

We have used a rigorous approach based on the concepts of modern nonlinear continuum mechanics. Nonlinearities, corresponding to the quadratic nonlinearity of all basic relationships, have been introduced by means of metric coefficients, the Cauchy–Green tensor, and the Murnaghan potential. Four ways of introducing physical and geometrical nonlinearities to the wave equations have been considered. We have written six different systems of wave equations corresponding to different states and cases.

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