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# VIBRATIONS OF CURVED AND TWISTED BLADES DURING COMPLEX ROTATION

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**The influence of the curving and twisting of an elongated blade on its vibrations during complex rotation is studied. It is shown that these geometrical factors may cause additional resonant vibrations**

**Keywords:** elongated blade, curving and twisting, additional resonant vibrations, abrupt increase in amplitude

**Introduction.** The blades of turbines in aircraft engines, helicopters, and windmills usually work in intensive centrifugal force fields and suffer the pressure and impacts of high-speed and high-temperature gas flows. Studies of curved and twisted beams in statics have resulted in a new formulation for the problem of optimizing the initial deflections of turbine blades to cancel the gas and centrifugal forces and reduce their mutual effect (momentless blades) [7, 10]. The theory of curved and twisted beams was applied in [9, 10, 12] to study the vibrations of blades. The operating conditions of blades are even more complicated in the case of complex rotation, when the rotation axis is forced to turn, which gives rise to periodic gyroscopic forces perpendicular to the plane of primary rotation. These forces may induce additional flexural stresses, which could reach high levels under resonant vibrations. Since the frequency of additional flexural vibrations coincides with the angular velocity of the blade, such resonances may occur only if its natural frequency is equal to the angular velocity.

By way of example of straight cantilever beams rotating with an angular velocity $\omega$, the author of [4] has established that the fundamental natural frequency $\omega_1^{\text{rot}}$ of straight untwisted blades determined with regard for the longitudinal centrifugal forces can be estimated by the Southwell theorem:

$$\omega_1^{\text{rot}} = \sqrt{\omega_1^2 + a\omega^2}, \tag{1}$$

where $\omega_1$ is the frequency of natural flexural vibrations of a stress-free blade and $a$ is a constant of proportionality equal to unity for the first vibration mode. According to this formula, for flapping (out-of-plane) motions of a blade, the frequency $\omega_1^{\text{rot}}$ is always higher than the frequency of forced vibrations equal to the angular velocity $\omega$. This precludes the possibility of resonance at this frequency during complex rotation.

However, the dynamic behavior of an elongated blade will drastically change if it is precurved and pretwisted. In this case, the centrifugal forces induce complex stress fields, which distort the frequency spectra and natural modes of the blade. Moreover, the frequencies of some modes (especially torsional) may even decrease, thus diminishing the dynamic resistance of the blade to external periodic forces. As will be shown below, the complex rotation of such blades may be accompanied by resonant vibrations.

**1. Problem Formulation and Original Relations.** Let us model the complex rotation of elongated twisted blades. Consider an elastic blade (Fig. 1). Initially, its centerline is a plane curve with a constant radius of curvature $R$ and its cross sections are twisted about this line. The blade is attached to a rigid disk of radius $a$. The disk rotates about its axis of symmetry with a constant angular velocity $\omega$. The symmetry axis turns with an angular velocity $\omega_0$ about a fixed straight line that passes through the center of the disk and is perpendicular to the rotation axis.

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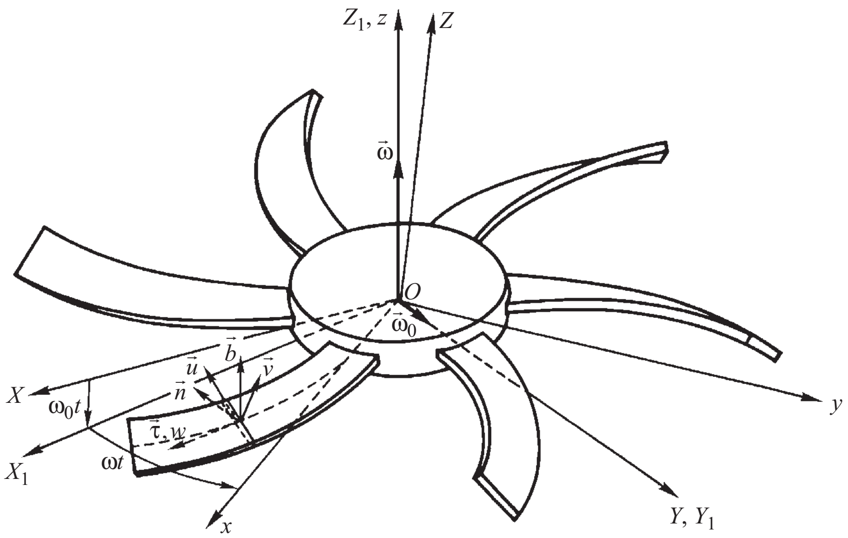


Fig. 1

Let us introduce the following right-hand coordinate systems:

(i) rectangular inertial system $OXYZ$ with the origin at the disk center and the $OY$-axis coinciding with the vector $\omega_0$;

(ii) coordinate system $OX_1Y_1Z_1$ with the $OZ_1$-axis coinciding with the rotation axis of the disk and turning with the angular velocity $\vec{\omega}_0$ about the $OY_1$-axis coinciding with the $OY$-axis;

(iii) disk-fixed coordinate system $Oxyz$ with the $Oz$-axis coinciding with $OZ_1$ and the $Ox$-axis directed along the tangent to the blade centerline at the blade root;

(iv) local coordinate system $(u, v, w)$ with the origin at the center of gravity of the blade's cross section, the $w$-axis directed along the tangent to the elastic centerline, and the $u$- and $v$-axes running along the principal central axes of inertia of the cross section;

(v) coordinate $s$ running along the elastic centerline of the blade and a natural trihedral of this line with unit principal normal $\vec{n}$, binormal $\vec{b}$, and tangent $\vec{\tau}$.

We will describe the small vibrations of the elongated curved blade excited by distributed inertial forces $\vec{f}$ using the dynamic equations of flexible curved bems [1, 8, 11]

$$\tilde{d}\vec{F} / ds = -\vec{\omega}_\chi \times \vec{F} - \vec{f}, \qquad \tilde{d}\vec{M} / ds = -\vec{\omega}_\chi \times \vec{M} - \vec{\tau} \times \vec{F} - \vec{m}, \qquad d\vec{\tau} / ds = K\vec{n},$$

$$d\vec{n} / ds = -K\vec{\tau} + \vec{b} / T, \qquad d\vec{b} / ds = -\vec{n} / T, \qquad d\vec{\rho} / ds = \vec{\tau},$$

$$M_u = A(p - p_0), \qquad M_v = B(q - q_0), \qquad M_w = C(r - r_0), \tag{2}$$

where $\vec{F}$ and $\vec{M}$ are the vectors of internal forces and moments with components $F_u, F_v, F_w$ and $M_u, M_v, M_w$, respectively; $\vec{\omega}_\chi$ is the Darboux vector; $T$ is the radius of torsion; $K$ is curvature; $\vec{\rho} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of points on the centerline in the coordinate system $Oxyz$ with unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$; $A$, $B$, and $C$ are parameters of flexural and torsional stiffness; $p$, $q$, and $r$ are the curvatures of the projection of the element $ds$ onto the planes $(v, w)$ and $(u, w)$ and the torsion of the centerline; and $\vec{m}$ is the vector of distributed moments of inertial forces.

To close Eqs. (2), we need to determine the distributed inertial forces $\vec{f}$ and moments $\vec{m}$ acting on the blade in complex rotation. To this end, we use the formula for the absolute acceleration of an element of the blade following from the Coriolis theorem [1, 5, 13, 14]:

$$\vec{a} = \vec{a}^e + \vec{a}^r + \vec{a}^c,$$

where $\vec{a}^e$, $\vec{a}^r$, and $\vec{a}^c$ are the vectors of translational, relative, and Coriolis accelerations, respectively, defined by the formulas

$$\vec{a}^e = \vec{\varepsilon}\times\vec{\rho} + \vec{\Omega}\times[\vec{\Omega}\times\vec{\rho}], \qquad \vec{a}^r = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}, \qquad \vec{a}^c = 2\vec{\Omega}\times\vec{V}^r,$$

where $\vec{V}^r = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$ is the vector of relative velocity of the element and $\vec{\Omega} = \vec{\omega} + \vec{\omega}_0$ and $\vec{\varepsilon} = \vec{\omega}_0 \times \vec{\omega}$ are the vectors of absolute angular velocity and acceleration of the moving coordinate system *Oxyz*.

In calculating the components of the vector $\vec{f}$ in the coordinate system *Oxyz*, we consider that $\omega >> \omega_0$ and neglect the terms containing $\omega_0^2$ and the product of $\omega_0$ and small elastic displacements. Finally, we obtain

$$f_x = -\gamma(\partial^2 x/\partial t^2 - 2\omega\cdot\partial y/\partial t - x\omega^2),$$

$$f_y = -\gamma(\partial^2 y/\partial t^2 + 2\omega\cdot\partial x/\partial t - y\omega^2),$$

$$f_z = -\gamma[\partial^2 z/\partial t^2 - 2\omega\omega_0(x\sin\omega t + y\cos\omega t)]. \tag{3}$$

To determine the vector of distributed moment $\vec{m}$ due to the inertia of deflection of the beam, we will write the vector of angular velocity of its element in terms of its projections onto the axes of the natural trihedral [1]:

$$\vec{\omega}^m = \omega_b^m \vec{b} + \omega_\tau^m \vec{\tau} + \omega_n^m \vec{n}, \tag{4}$$

where $\vec{\omega}^m$ is the vector of total angular velocity corrected for the elastic deflections of the blade.

Let us differentiate expression (4) with respect to time:

$$d\vec{\omega}^m/dt = \dot{\omega}_b^m \vec{b} + \omega_b^m d\vec{b}/dt + \dot{\omega}_\tau^m \vec{\tau} + \omega_\tau^m d\vec{\tau}/dt + \dot{\omega}_n^m \vec{n} + \dot{\omega}_n^m d\vec{n}/dt. \tag{5}$$

If $d\vec{\tau}/dt = \vec{\omega}^m \times \vec{\tau}$, $d\vec{b}/dt = \vec{\omega}^m \times \vec{b}$, and $d\vec{n}/dt = \vec{\omega}^m \times \vec{n}$, then

$$d\vec{\tau}/dt = -\omega_n^m \vec{b} + \omega_b^m \vec{n}, \qquad d\vec{b}/dt = -\omega_\tau^m \vec{n} + \omega_n^m \vec{\tau}, \qquad d\vec{n}/dt = -\omega_b^m \vec{\tau} + \omega_\tau^m \vec{b}. \tag{6}$$

Multiplying scalarwise the left- and right-hand sides of Eqs. (6) by $\vec{n}$, $\vec{\tau}$, and $\vec{b}$, respectively, we obtain

$$\omega_b^m = d\vec{\tau}/dt\cdot\vec{n} = \dot{\tau}_x n_x + \dot{\tau}_y n_y + \dot{\tau}_z n_z,$$

$$\omega_n^m = d\vec{b}/dt\cdot\vec{\tau} = \dot{b}_x \tau_x + \dot{b}_y \tau_y + \dot{b}_z \tau_z,$$

$$\omega_\tau^m = d\vec{n}/dt\cdot\vec{b} = \dot{n}_x b_x + \dot{n}_y b_y + \dot{n}_z b_z.$$

In calculating the vector of angular acceleration $\vec{\varepsilon}^m$, we assume that the distributed moments due to the translational angular accelerations are much less than the distributed moments due to the relative elastic angular accelerations and can be neglected. Then $\vec{\varepsilon}^m = \vec{\varepsilon}^r = d\vec{\omega}^m/dt$. Let us determine the projections of this vector onto the axes of the natural trihedral using formulas (5) and (6):

$$\varepsilon_b^r = d\vec{\omega}^m/dt\cdot\vec{b} = \dot{\omega}_b^m, \qquad \varepsilon_\tau^r = d\vec{\omega}^m/dt\cdot\vec{\tau} = \dot{\omega}_\tau^m, \qquad \varepsilon_n^r = d\vec{\omega}^m/dt\cdot\vec{n} = \dot{\omega}_n^m.$$

Since the contribution of the inertia of turn of cross sections about the *u*- and *v*-axes to the total balance of moments is insignificant, the engineering theory of beams usually neglects it. Therefore, we will consider only the inertia of turn of an element about the *w*-axis, which affects the torsional vibrations of the blade. Then [2], we obtain

$$m_w = -\lambda I_w \varepsilon_w^r, \tag{7}$$

where $\lambda$ is the density of the blade material, $I_w$ is the moment of inertia of its cross section about the *w*-axis, and

$$\varepsilon_w^r = \varepsilon_\tau^r = \ddot{n}_x b_x + \ddot{n}_y b_y + \ddot{n}_z b_z + \dot{n}_x \dot{b}_x + \dot{n}_y \dot{b}_y + \dot{n}_z \dot{b}_z.$$

**2. Numerical Method.** Let the vibrations of the blade under the inertial forces be steady. The condition $\omega >> \omega_0$ allows us to study the relative (elastic) vibrations of the blade in two stages [1, 2]. At the first stage, the rigid disk simply rotates with angular velocity $\vec{\omega}$, centrifugal forces act on the blade, and there are no vibrations. At the second stage, the prestressed blade executes small elastic vibrations about the equilibrium position with given angular velocities $\omega_0$ and $\omega$ under additional periodic inertial forces generated in complex rotation. To describe the vibrations, we derive the equations of small vibrations by linearizing, using Newton's method, Eqs. (2) about the state of simple rotation [1]:

$$d\Delta\vec{F} / ds = -\vec{\omega}_\chi \times \Delta\vec{F} - \Delta\vec{\omega}_\chi \times \vec{F} - \Delta\vec{f},$$

$$d\Delta\vec{M} / ds = -\vec{\omega}_\chi \times \Delta\vec{M} - \Delta\vec{\omega}_\chi \times \vec{M} - \vec{\tau} \times \Delta\vec{F} - \Delta\vec{\tau} \times \vec{F} - \Delta\vec{m},$$

$$d\Delta\vec{\tau} / ds = K\Delta\vec{n} + \Delta K\vec{n}, \qquad d\Delta\vec{n} / ds = -K\Delta\vec{\tau} - \Delta K\vec{\tau} + \Delta\vec{b} / T - \vec{b}\Delta T / T^2,$$

$$d\Delta\vec{b} / ds = -\Delta\vec{n} / T + \vec{n}\Delta T / T^2, \qquad d\Delta\vec{\rho} / ds = \Delta\vec{\tau}. \tag{8}$$

This system of six vector equations is equivalent to a system of eighteen scalar equations. To derive this system, we project the first two equations onto the *u*-, *v*-, and *w*-axes and the remaining four equations onto the *Ox*-, *Oy*-, and *Oz*-axes. To calculate the components of the vector of intensity of inertial forces and their moments, we will linearize relations (3) and (7):

$$\Delta f_x = -\gamma(\partial^2 \Delta x / \partial t^2 - 2\omega \cdot \partial\Delta y / \partial t - \Delta x \omega^2),$$

$$\Delta f_y = -\gamma(\partial^2 \Delta y / \partial t^2 + 2\omega \cdot \partial\Delta x / \partial t - \Delta y \omega^2),$$

$$\Delta f_z = -\gamma(\partial^2 \Delta z / \partial t^2 - 2\omega\omega_0 (x \sin\omega t + y \cos\omega t)),$$

$$\Delta m_w = -\lambda I_w (\Delta\ddot{n}_x b_x + \ddot{n}_x \Delta b_x + \Delta\ddot{n}_y b_y + \ddot{n}_y \Delta b_y + \Delta\ddot{n}_z b_z + \ddot{n}_z \Delta b_z$$
$$+ \Delta\dot{n}_x \dot{b}_x + \dot{n}_x \Delta\dot{b}_x + \Delta\dot{n}_y \dot{b}_y + \dot{n}_y \Delta\dot{b}_y + \Delta\dot{n}_z \dot{b}_z + \dot{n}_z \Delta\dot{b}_z). \tag{9}$$

Since the load acting on the blade during complex rotation is periodic in time, we can represent the scalar components of the unknown variables as

$$\Delta F_u(s,t) = \Delta F_u^s(s) \sin\omega t + \Delta F_u^c(s) \cos\omega t,$$

$$\Delta F_v(s,t) = \Delta F_v^s(s) \sin\omega t + \Delta F_v^c(s) \cos\omega t,$$

$$\Delta F_w(s,t) = \Delta F_w^s(s) \sin\omega t + \Delta F_w^c(s) \cos\omega t,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Delta x(s,t) = \Delta x^s(s) \sin\omega t + \Delta x^c(s) \cos\omega t,$$

$$\Delta y(s,t) = \Delta y^s(s) \sin\omega t + \Delta y^c(s) \cos\omega t,$$

$$\Delta z(s,t) = \Delta z^s(s) \sin\omega t + \Delta z^c(s) \cos\omega t. \tag{10}$$

Now, considering the functions $\sin\omega t$ and $\cos\omega t$ as coordinate ones and using the projection method, we obtain from (8) a system of ordinary differential equations of the 36th order for the unknown variables $\Delta F_u^s(s)$, $\Delta F_u^c(s)$, $\Delta F_v^s(s)$, $\Delta F_v^c(s)$, …, $\Delta z^s(s)$, and $\Delta z^c(s)$ with *s* as a unique independent variable.

Written in general form, this system reads

$$dX / ds = A(s)X + \omega_0 f(s), \tag{11}$$

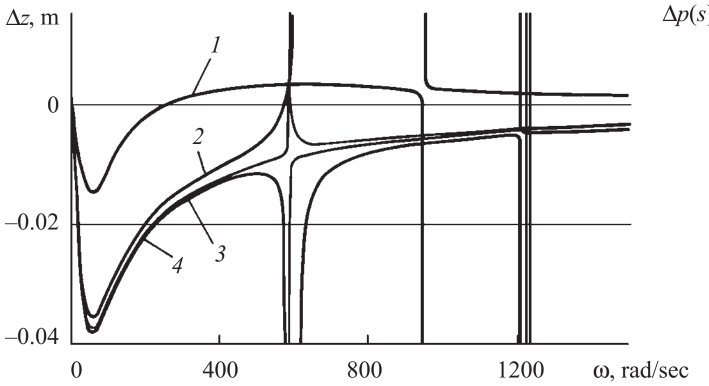

Fig. 2

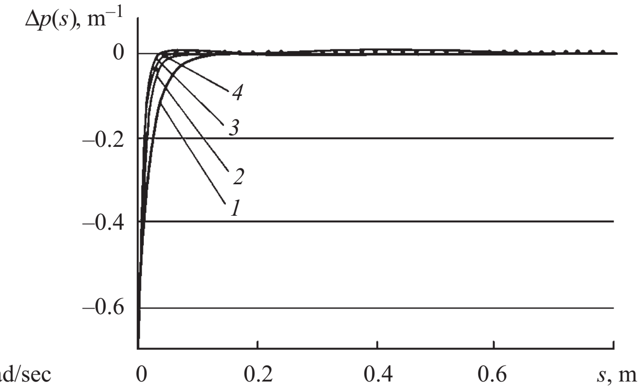

Fig. 3

where $X(s)$ is the vector of unknown variables of length 36, $A(s)$ is the matrix of coefficients defined by (8), and $f(s)$ is the given vector of right-hand sides determined by the inertial forces acting on the blade. System (11) is supplemented with the following boundary conditions: the blade is rigidly fixed at $s = 0$ and free at $s = S$.

System (11) is solved by the method of initial parameters; the equations of this system are integrated using the fourth-order Runge-Kutta method. Since some elements of the matrix $A$ contain the coefficients $\omega^2$ quickly increasing with $\omega$, system (11) is stiff and has rapidly increasing functions among partial solutions. Therefore, the orthogonalization method [1, 2] is additionally used to numerically integrate the system.

**3. Discussion of the Results.** The method outlined above has been implemented in algorithms and software to analyze the influence of the curving and twisting of blades on the interaction of flapping (out-of-plane) and lagging (in-plane) motions of blades during complex rotation. The blades are assumed to have constant cross section, twist, and curvature. The length ($S$) and cross-sectional dimensions ($a$, $b$) of the blades are the following: $S = 0.8$ m, $a = 0.005$ m, and $b = 0.1$ m. The elastic modulus, the shear modulus, and the density of the blade material are the following: $E = 2.11\cdot10^{11}$ Pa, $G = 7.9\cdot10^{10}$ Pa, and $\lambda = 7.8\cdot10^3$ kg/m$^3$. The range of variation in the angular velocity of the rotor $0 \le \omega \le 2500$ sec$^{-1}$, and the angular velocity of the rotation axis $\omega_0 = 1\,\text{sec}^{-1}$.

We have examined two cases where the centerline of the blade has a constant radius of curvature in the planes $xOy$ and $xOz$. The analysis has revealed that torsional vibrations occur in the former case (blade curved in the plane $xOy$), since the inertial forces are directed along the $Oz$-axis and generate not only bending moments in the plane $xOz$ but also moments about the $Ox$-axis, which twist the blade. Figure 2 shows the amplitude–frequency response $\Delta z(t)$ of the end of a blade curved in the plane $xOy$ and twisted through an angle of 10° at the end $s = S$. Curves *1*, *2*, *3*, and *4* correspond to the following radii of curvature: $R = 0.3$, 1, 1.5, 2 m. It can be seen that all the amplitude–frequency curves have discontinuities representing resonant vibrations. According to the Southwell theorem, a straight untwisted blade does not resonate during complex rotation. Therefore, we may conclude that the resonant states displayed by curves *1–4* are due to the curving and twisting of the blade.

We have also established that curves *2–4* group near a skeleton curve that turns out to be the amplitude–frequency response of the straight untwisted blade. Resonances in the neighborhood of the angular velocity $\omega = 560$ rad/sec occur in both pretwisted and untwisted blades, whereas additional narrow resonance bands appear at high angular velocities in twisted blades anole.

An analysis of the vibration modes of the blade has revealed that the amplitudes of flapping motions are an order of magnitude larger than those of lagging motions. Figure 3 shows the curvature increment $\Delta p(s)$ due to the flapping deflection of a blade with $R = 2$ m versus the axial coordinate $s$. Curves *1–4* correspond to $\omega = 500$, 1000, 1500, 2500 rad/sec. As is seen, $\Delta p$ (together with the bending moment $\Delta M_u = A\cdot\Delta p$) in the root section $s = 0$ increases with $\omega$. However, the amplitude values of $\Delta p(s)$ do not increase with $\omega$ within the interval $0 \le s \le S$. The greater the angular velocity $\omega$, the more abrupt the variation in $\Delta p(s)$ at the blade root and the larger the value $\Delta p(0)$, though the remaining portion of the blade bends insignificantly. This effect is attributed to the fact that the rotating blade is prestressed by the longitudinal inertial forces, which considerably increase its flexural stiffness. Therefore, as $\omega$ increases, the blade vibrates as a rigid body and experiences local bending at the root.

The forced motions of the untwisted blade curved in the plane *xOz* are purely flapping. Additional twisting of this blade results in excitation of lagging motions of very small amplitude. Note that these curving and twisting are responsible for resonant vibrations at some angular velocities.

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