International Applied Mechanics, Vol. 41, No. 4, 2005

NONLINEAR VIBRATIONS OF A CYLINDRICAL SHELL CONTAINING A FLOWING FLUID

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UDC 539.3

The Bogolyubov–Mitropolsky method is used to find approximate periodic solutions to the system of nonlinear equations that describes the large-amplitude vibrations of cylindrical shells interacting with a fluid flow. Three quantitatively different cases are studied: (i) the shell is subject to hydrodynamic pressure and external periodical loading, (ii) the shell executes parametric vibrations due to the pulsation of the fluid velocity, and (iii) the shell experiences both forced and parametric vibrations. For each of these cases, the first-order amplitude–frequency characteristic is derived and stability criteria for stationary vibrations are established

Keywords: cylindrical shell, perfect incompressible fluid, nonlinear vibrations, single-frequency method, critical velocity, amplitude–frequency characteristic, stability of vibrations

Introduction. The problem of dynamic interaction between elastic cylindrical shells and the fluid they contain attracted the attention of many noted scientists in the field of mechanics. The most significant results in this area have been obtained by V. V. Bolotin [2], A. S. Vol'mir [4], J. Gorachek and I. A. Zolotarev [6], M. A. Il'gamov [7], N. A. Kil'chevskii and his disciples [10], and other authors. Most studies were devoted to the buckling of shells under the nonconservative hydrodynamic forces exerted by the fluid flow. As a rule, two qualitatively different types of buckling were examined: static (divergent type) and dynamic (flutter type). The moment of buckling of one type or another was determined from an analysis of the corresponding linearized equations of shell deformation.

Much less attention was given to the dynamic deformation of these shells after buckling, i.e., postbuckling deformation. The complexity of the associated problems is in the necessity of accounting for nonlinear factors such as geometrical nonlinearity and nonlinear damping and of using multidimensional design models of shells [2, 4]. The nonlinearities limit the increase of vibration amplitudes after buckling, and the multidimensionality is necessary for the adequate description of nonconservative (or, to put it differently, pseudo-gyroscopic [2]) forces, which are the major cause of buckling. M. Amabili, F. Pellicano, and M. P. Paidoussis [11, 12, etc.] set forth several modern approaches to the analysis of multimode (seven modes) free and forced nonlinear vibrations of cylindrical shells containing a flowing fluid. They also briefly reviewed scientific papers devoted to the stability and large-amplitude vibrations of shells with fluid. In most cases, the nonlinear problems mentioned above were solved by the following scheme:

(i) geometrically nonlinear equations of the classical theory (such as the Donnell–Mushtari–Vlasov equations) are used as the original equations describing the motion of shells;

(ii) hydrodynamic pressure of the fluid is determined by the linearized theory of potential flow along the shell;

(iii) the Bubnov–Galerkin method is used to reduce the dynamic (partial differential) equations of shell with fluid to a finite-dimensional system of ordinary differential equations;

(iv) this system is solved by numerical methods (collocation, Runge-Kutta, etc.) because of the difficulty of applying analytical methods of nonlinear mechanics.

S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Kiev. Translated from Prikladnaya Mekhanika, Vol. 41, No. 4, pp. 75–84, April 2005. Original article submitted July 14, 2004.

In the present paper, we outline a method for finding single-frequency periodic solutions of the system of differential equations describing the multimode nonlinear vibrations of cylindrical shells conveying a fluid. The method is based on the asymptotic Bogolyubov–Mitropolsky method intended for analysis of the single-frequency vibrations of quasilinear multidegree-of-freedom systems [1]. Applying this method, we can derive analytical expressions for the amplitudes and frequencies of vibrations depending on the velocities of fluid flow, the physical and geometrical parameters of the shell, waveformation parameters, and external load. We will obtain expressions for the following three practically important cases: (i) the shell is subject to an external periodic force (the velocity of the fluid is constant), (ii) the shell interacts with the fluid moving with a variable (pulsing) velocity, and (iii) both factors (external load and varying velocity) act simultaneously.

1. The original dynamic equations of a shell conveying a fluid have a mixed form [3, 4]:

$$\frac{D}{h}\nabla^4 w=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2\Phi}{\partial x^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\Phi}{\partial x\partial y}+\frac{1}{R}\frac{\partial^2\Phi}{\partial x^2}-\bar{\varepsilon}_k\rho\frac{\partial w}{\partial t}-\rho\frac{\partial^2 w}{\partial t^2}+\frac{q}{h}-\frac{P_{\text{h}}}{h},$$

$$\frac{1}{E}\nabla^4\Phi=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\frac{1}{R}\frac{\partial^2 w}{\partial x^2}, \tag{1.1}$$

where $w = w(x, y, t)$ is the radial deflection of the shell (positive toward the center of curvature); Φ is the function of stresses in the median surface; h and R are the thickness and radius of the shell; $D=Eh^3/12(1-\mu^2)$ is cylindrical stiffness (E is the elastic modulus and μ is Poisson's ratio); ρ is the density of the shell material; $\bar{\varepsilon}_k$ are the damping factors (generally different for different vibration modes); $q = q(x, y, t)$ is the external load of the form $q(x, y, t) = q_0(x, y)\cos\Omega t$, where q_0 is some function of spatial coordinates x and y; and P_{h} is the hydrodynamic pressure on the shell walls. The fluid is perfect and incompressible; its motion is potential.

The pressure P_{h} and the velocity potential $\varphi = \varphi(x, r, \bar{\theta}, t)$ are related by the following approximate formula [2, 4, 11]:

$$P_{\text{h}}=-\rho_0\left(\frac{\partial\varphi}{\partial t}+U\frac{\partial\varphi}{\partial x}\right)\bigg|_{r=R}, \tag{1.2}$$

where ρ_0 is the density of the fluid; U is the velocity of the fluid; and x, r, and $\bar{\theta}$ are the cylindrical coordinates. The potential φ should be determined by solving the following boundary-value problem [3, 11]:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\bar{\theta}^2}=0, \tag{1.3}$$

$$\frac{\partial\varphi}{\partial r}\bigg|_{r=R}=-\left(\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}\right),\qquad \frac{\partial\varphi}{\partial r}\bigg|_{r=0}<\infty \tag{1.4}$$

($0 \le r \le R$, $0 \le x \le l$, $0 \le \bar{\theta} \le 2\pi$, l is the length of the shell).

Assuming the shell is simply supported, we represent the dynamic deflection w by truncated two-parameter series:

$$w=\sum_{n=1}^{N_1}\sum_{m=1}^{N_2}\left(f_1^{nm}\cos s_n y+f_2^{nm}\sin s_n y\right)\sin\lambda_m x, \tag{1.5}$$

where $f_{1,2}^{nm}$ are some functions of time having the sense of generalized coordinates; $s_n=n/R$ and $\lambda_m=m\pi/l$ are the waveformation parameters; and N_1 and N_2 are the number of axial and circumferential modes retained in the expansion of the deflection function.

Substituting (1.5) into (1.4) and using (1.3), we obtain

$$\varphi=\sum_{n=1}^{N_1}\sum_{m=1}^{N_2}\frac{I_n(\lambda_m r)}{\lambda_m I_n'(\lambda_m r)}\left[\left(\dot{f}_1^{nm}\cos s_n y+\dot{f}_2^{nm}\sin s_n y\right)\sin\lambda_m x\right.$$

$$+\lambda_m U\left(f_1^{nm}\cos s_n y+f_2^{nm}\sin s_n y\right)\cos\lambda_m x\Big], \tag{1.6}$$

where I_n are the modified Bessel functions of the nth kind.

Substituting (1.5) into the second equation in (1.1), we determine the stress function Φ. Applying then the Bubnov–Galerkin method to the first equation in (1.1), taking (1.2)–(1.6) into account, we derive a system of equations for $f_{1,2}^{nm}$ appearing in (1.5):

$$\ddot{f}_1^{nm}+\left(\omega_{nm}^2-\alpha_{nm}U^2\right)f_1^{nm}+\varepsilon_1^{nm}\dot{f}_1^{nm}+\sum_{k=1}^{N_2}\beta_k^{nm}U\dot{f}_1^{nk}=\varepsilon_0 F_1^{nm}\left(\left\{f_1^{pq}\right\},\left\{f_2^{pq}\right\}\right)+Q_1^{nm}\cos\Omega t,$$

$$\ddot{f}_2^{nm}+\left(\omega_{nm}^2-\alpha_{nm}U^2\right)f_2^{nm}+\varepsilon_2^{nm}\dot{f}_2^{nm}+\sum_{k=1}^{N_2}\beta_k^{nm}U\dot{f}_2^{nk}=\varepsilon_0 F_2^{nm}\left(\left\{f_1^{pq}\right\},\left\{f_2^{pq}\right\}\right)+Q_2^{nm}\cos\Omega t$$

$$(n,p=1,2,\ldots,N_1,\quad m,q=1,2,\ldots,N_2), \tag{1.7}$$

where ω_{nm} are the natural frequencies of the shell–fluid system, and [11, 15]

$$\omega_{nm}^2=\frac{1}{\rho m_0^{nm}}\left[\frac{D}{h}\Delta\left(\lambda_m,s_n\right)+\frac{E\lambda_m^4}{R^2\Delta\left(\lambda_m,s_n\right)}\right],\qquad m_0^{nm}=1+\frac{\rho_0}{\rho}\frac{1}{\lambda_m h}\frac{I_n(\lambda_m R)}{I_n'(\lambda_m R)} \tag{1.8}$$

($\Delta(a, b) = (a^2 + b^2)^2$ is an operator); α_{nm} and β_k^{nm} are constant coefficients,

$$\alpha_{nm}=\frac{\rho_0}{\rho h}\frac{\lambda_m}{m_0^{nm}}\frac{I_n(\lambda_m R)}{I_n'(\lambda_m R)},$$

$$\beta_k^{nm}=\frac{4\rho_0 U}{\rho hl}\sum_{q=1}^{N_2}\frac{I_n(\lambda_q R)}{I_n'(\lambda_q R)}\frac{\lambda_m[1-(-1)^{m-q}]}{\lambda_m^2-\lambda_q^2}\quad\text{for}\quad m\neq q,$$

$$\beta_k^{nm}=\quad\text{for}\quad m=q\qquad(\lambda_q=q\pi/l). \tag{1.9}$$

Moreover, $\varepsilon_k^{nm}=\dfrac{\bar{\varepsilon}_k}{m_0^{nm}}$, $\left\{f_k^{pq}\right\}=\left\{f_k^{11},f_k^{12},\ldots,f_k^{N_1N_2}\right\}$ $(k = 1, 2)$,

$$\left.\begin{matrix}Q_1^{nm}\\ Q_2^{nm}\end{matrix}\right\}=\frac{2}{\pi Rl\rho h m_0^{nm}}\int_0^l\int_0^{2\pi R}q_0(x,y)\left.\begin{matrix}\cos s_n y\cdot\sin\lambda_m x\\ \sin s_n y\cdot\sin\lambda_m x\end{matrix}\right\}dxdy, \tag{1.10}$$

$\varepsilon_0 F_{1,2}^{nm}$ are nonlinear functions up to the third power in $f_{1,2}^{pq}\left(p=\overline{1,N};q=\overline{1,N_2}\right)$ [8, 9], ε_0 is a small positive parameter.

The system of equations (1.7) is used to analyze the stability and nonlinear dynamic deformation of shells conveying a fluid. Here we will outline a method for finding the periodic solutions of this system corresponding to single-frequency forced, parametric, and combined vibrations of shells. It is convenient to represent system (1.7) in the form

$$\ddot{f}_k+\left(\omega_k^2-\alpha_k U^2\right)f_k+\varepsilon_k\dot{f}_k+\sum_{j=1}^{N}\beta_{jk}U\dot{f}_j=\varepsilon_0 F_k\left(\left\{f_p\right\}\right)+Q_k\cos\Omega t$$

$$(k,p=1,2,\ldots,N,\ \ N=2N_1\cdot N_2), \tag{1.11}$$

where the index k is used instead of m and n in (1.7).

Before constructing periodic solutions, it is necessary to analyze the shell for stability, using the linear system (1.11) with $\varepsilon_0 = 0$. Assuming that $Q_k = 0$ $(k=\overline{1,N})$, we set up a characteristic equation for this system. Let

$$f_k = P_k e^{i\lambda t} \qquad (P_k = \text{const},\ k = \overline{1,N},\ i = \sqrt{-1}). \tag{1.12}$$

Then we obtain the equation

$$||(\omega_j^2 - \alpha_j U^2 - \lambda^2 + \varepsilon_j i\lambda)\delta_{jk} + \beta_{jk} U i\lambda|| = 0 \tag{1.13}$$

(j = 1, 2,…, N; δ_{jk} is the Kronecker delta).

The nondisturbed mode shape of the shell is stable if all the characteristic exponents $s_0 = i\lambda$ in (1.12) fall onto the left half-plane of the complex variable [2]. If at least one of the exponents is in the right half-plane, then either static or dynamic (oscillatory type) instability may occur. The exponent s_0 passes to the right half-plane through the origin of coordinates (Im $[s_0] = 0$) in the former case and through the point at which Im $[s_0] \neq 0$ in the latter case. We will consider the second type of instability (flutter), which is of practical importance for the dynamic design of piping systems [4, 11]. This type of instability occurs at the minimum velocity $U_{\min} = U_*$ at which one of the exponents s_0 passes to the right half-plane, remaining complex. Thus, U_* is a critical velocity of the fluid at which flexural vibrations with increasing amplitudes are excited in the shell. The frequency of these vibrations, $\lambda = \lambda_*$, is determined from Eq. (1.13) where $U = U_*$. Note that the flexural modes corresponding to the frequency λ_* are characterized by certain values of m and n [2, 6, 11].

2. To find the single-frequency periodic solutions of the general system (1.11) (in the case of different damping factors ε_k, this system satisfies all the existence conditions for single-frequency vibrations [1]), we assume that the fluid velocity U (U = const) is close to the critical value U_* and, moreover, the frequency of external excitation Ω is close to the natural frequency λ_* of the shell–fluid system. Thus, we are considering the worst (from the dynamic-strength viewpoint) situation where the shell is about to buckle and is at principal harmonic resonance ($\Omega \approx \lambda_*$). The corresponding periodic solution can be represented, according to [1], as follows (hereafter the subscript k takes integer values from the interval $1 \le k \le N$):

$$f_k = a\left(\varphi_k e^{i\psi} + \bar{\varphi}_k e^{-i\psi}\right), \tag{2.1}$$

where $\psi = \Omega t + \theta$; φ_k are eigenfunctions, which are the nontrivial solutions of the system of homogeneous algebraic equations

$$\sum_{j=1}^{N} \left[\left(\omega_j^2 - \alpha_j U_*^2 - \lambda_*^2 + \varepsilon_j i\lambda_*\right)\delta_{jk} + \beta_{jk} U_* i\lambda_* \right] \varphi_j = 0; \tag{2.2}$$

$\bar{\varphi}_k$ are the complex conjugate eigenfunctions; and a and θ are slowly varying functions of time, which define the amplitude and phase of single-frequency vibrations and are determined as a first approximation from the equations

$$\frac{da}{dt} = \varepsilon_0 A(a,\theta), \qquad \frac{d\theta}{dt} = \varepsilon_0 \Delta_0 + \varepsilon_0 B(a,\theta), \tag{2.3}$$

where $\varepsilon_0 \Delta_0 = \lambda_* - \Omega$; $\varepsilon_0 A$ and $\varepsilon_0 B$ are functions constructed by a special procedure [1]. Let us determine these functions. To this end, we find the following derivatives:

$$\frac{df_k}{dt} = \left(\varphi_k e^{i\psi} + \bar{\varphi}_k e^{-i\psi}\right)\frac{da}{dt} + \left[ai\left(\varphi_k e^{i\psi} - \bar{\varphi}_k e^{-i\psi}\right)\right]\frac{d\theta}{dt} + ai\Omega\left(\varphi_k e^{i\psi} - \bar{\varphi}_k e^{-i\psi}\right),$$

$$\frac{d^2 f_k}{dt^2} = 2\left[i\Omega\left(\varphi_k e^{i\psi} - \bar{\varphi}_k e^{-i\psi}\right)\right]\frac{da}{dt} - 2\left[a\Omega\left(\varphi_k e^{i\psi} + \bar{\varphi}_k e^{-i\psi}\right)\right]\frac{d\theta}{dt} - \left[a\Omega^2\left(\varphi_k e^{i\psi} + \bar{\varphi}_k e^{-i\psi}\right)\right] + \varepsilon_0^2(\cdots). \tag{2.4}$$

The right-hand sides of (1.11) are represented as

$$\varepsilon_0 F_k(\cdots) = \varepsilon_0 F_k^{(1)}(a) e^{i\psi} + \varepsilon_0 F_k^{(2)}(a) e^{-i\psi} + \ldots,$$

$$Q_k \cos\Omega t = \frac{Q_k}{2}\left(e^{-i\theta} e^{i\psi} + e^{i\theta} e^{-i\psi}\right), \tag{2.5}$$

where the omitted terms correspond to higher harmonics ($\pm 2i\psi$, $\pm 3i\psi$, etc.) of the Fourier series.

Substituting (2.1), (2.3), and (2.4) into (1.11) and collecting the terms with $e^{i\psi}$, we obtain the system of equations

$$\left(\omega_k^2-\alpha_k U_*^2-\lambda_*^2+i\lambda_*\varepsilon_k\right)a\varphi_k+i\lambda_* a\sum_{j=1}^{N}\beta_{jk}U_*\varphi_j$$

$$=-\varepsilon_0\,(A+iaB)\left[\left(2i\lambda_*+\varepsilon_k\right)\varphi_k+U_*\sum_{j=1}^{N}\beta_{jk}\varphi_j\right]+\varepsilon_0 F_k^{(1)}(a)$$

$$+\frac{1}{2}Q_k e^{-i\theta}+\varepsilon_0\Delta_1 U_* a\left[2\alpha_k\varphi_k-i\lambda_*\sum_{j=1}^{N}\beta_{jk}\varphi_j\right]=M_k(a,\theta), \tag{2.6}$$

where $\varepsilon_0\Delta_1=U-U_*$ and $U^2-U_*^2\approx 2U_*\varepsilon_0\Delta_1$.

Similar equations will result if we equate the coefficients of $e^{-i\psi}$.

Since Eq. (2.2) where $\varphi_j\neq 0\,(j=\overline{1,N})$ is valid, for the existence of the periodic solutions f_k and the unique definition of the functions A and B in (2.3), it is necessary and sufficient that the orthogonality condition [1]

$$\sum_{k=1}^{N}M_k(a,\theta)\chi_k=0 \tag{2.7}$$

be satisfied, where χ_k are nontrivial solutions of an algebraic system conjugate to (2.2), i.e.,

$$\sum_{k=1}^{N}\left[\left(\omega_k^2-\alpha_k U_*^2-\lambda_*^2-\varepsilon_k i\lambda_*\right)\delta_{jk}-\beta_{jk}U_* i\lambda_*\right]\chi_k=0. \tag{2.8}$$

If

$$\sum_{k=1}^{N}\left[\left(2i\lambda_*+\varepsilon_k\right)\varphi_k\chi_k+U_*\sum_{j=1}^{N}\beta_{jk}\varphi_j\chi_k\right]=d_1+id_2,$$

$$\sum_{k=1}^{N}\varepsilon_0 F_k^{(1)}(a)\chi_k=\varepsilon_0 H_1(a)+i\varepsilon_0 H_2(a),$$

$$\sum_{k=1}^{N}\left[2\alpha_k\varphi_k\chi_k-i\lambda_*\sum_{j=1}^{N}\beta_{jk}\varphi_j\chi_k\right]=d_3+id_4,\qquad \frac{1}{2}\sum_{k=1}^{N}Q_k\chi_k=\varepsilon_0\quad\left(Q_{01}+iQ_{02}\right), \tag{2.9}$$

then from Eq. (2.7) with (2.6) we obtain the following expressions for $A(a,\theta)$ and $B(a,\theta)$:

$$A(a,\theta)=G_0\Delta_1 U_* a+G_1(a)+R_1\cos\theta+S_1\sin\theta,$$

$$B(a,\theta)=G_2\Delta_1 U_*+G_3(a)+\frac{1}{a}\left(S_1\cos\theta-R_1\sin\theta\right), \tag{2.10}$$

where

$$G_0=\frac{d_1d_3+d_2d_4}{d_0^2},\qquad G_1(a)=\frac{d_1H_1(a)+d_2H_2(a)}{d_0^2},\qquad G_2=\frac{d_1d_4-d_2d_3}{d_0^2},$$

$$G_3(a)=\frac{d_1H_2(a)-d_2H_1(a)}{d_0^2}, \quad R_1=\frac{Q_{01}d_1+Q_{02}d_2}{d_0^2}, \quad S_1=\frac{Q_{02}d_1-Q_{01}d_2}{d_0^2} \quad \left(d_0^2=d_1^2+d_2^2\right). \tag{2.11}$$

Returning to system (2.3) and assuming that $da/dt=0$ and $d\theta/dt=0$, we derive the amplitude–frequency characteristic of steady-state vibrations of the shell with flowing fluid:

$$\Delta_0=-G_2U_*\Delta_1-G_3(a)\pm\sqrt{\frac{K_1^2}{a^2}-\left(G_0\Delta_1U_*+\frac{G_1(a)}{a}\right)^2}, \quad K_1^2=R_1^2+S_1^2=\frac{Q_{01}^2+Q_{02}^2}{d_0^2}. \tag{2.12}$$

Whether the stationary solutions $a=a(\Omega)$ found from (2.12) are stable can be established from an analysis of the corresponding variational equations for (2.3). The stability criteria will be the following [5, 8]:

$$\frac{d}{da}\left[G_1(a)a\right]<-2G_0\Delta_1U_*a, \quad \left[G_0\Delta_1U_*+\frac{dG_1(a)}{da}\right]\left[G_0\Delta_1U_*+\frac{G_1(a)}{a}\right]$$

$$+\left[G_2\Delta_1U_*+\Delta_0+G_3(a)\right]\left[a\frac{dG_3(a)}{da}+G_2\Delta_1U_*+\Delta_0+G_3(a)\right]>0. \tag{2.13}$$

Analyzing Eqs (2.12) and (2.13), we can study the nonlinear prebuckling deformation of the shell, i.e., at $U\approx U_*$. It is significant that the deformation mode of the shell may be either "standing" waves with a complex spatial pattern or bending waves "traveling" in the circumferential direction. Which of these modes occurs depends on the relationship between the phases of the functions f_1^{nm} and f_2^{nm} in (1.5) [8, 9, 12–16]. In particular, the waves in the shell are traveling if $\dot{\alpha}_{nm}(t)\neq 0$ in the deflection function w (1.5) represented in a wave form [9]:

$$w=\sum_{n=1}^{N_1}\sum_{m=1}^{N_2}a_{nm}(t)\cos\left(s_ny-\alpha_{nm}(t)\right)\sin\lambda_mx. \tag{2.14}$$

Otherwise (α_{nm} = const), each term in (1.5) will characterize a standing wave.

3. Let us consider the case where the velocity U_1 of fluid is not constant but "pulsing" in time by the formula

$$U_1=U(1+\varepsilon_0\mu_0\cos\nu t), \tag{3.1}$$

where ν and $\varepsilon_0\mu_0$ (μ_0 = const) are the frequency and small amplitude of pulsations. As in the previous section, we assume that $U\approx U_*$ and, moreover, $\lambda_*\approx\nu/2$, which is a precondition for excitation of parametric vibrations.

If no lateral force acts on the shell (i.e., $q\equiv 0$), then the periodic solution of system (1.11) can also be represented in the form (2.1) [5, 8], where $\psi=\nu t/2+\theta$. In view of (2.11), Eqs. (2.3) take the form

$$\frac{da}{dt}=\varepsilon_0\left[G_0\Delta_1U_*a+G_1(a)+\left(R_2\cos 2\theta+S_2\sin 2\theta\right)a\right],$$

$$\frac{d\theta}{dt}=\varepsilon_0\left[\Delta_0+G_2\Delta_1U_*+G_3(a)+\left(S_2\cos 2\theta-R_2\sin 2\theta\right)\right], \tag{3.2}$$

where

$$R_2=\frac{d_1d_5+d_2d_6}{d_0^2}, \quad S_2=\frac{d_1d_6+d_2d_5}{d_0^2}, \quad \varepsilon_0\Delta_0=\lambda_*-\frac{\nu}{2}, \tag{3.3}$$

and d_5 and d_6 are determined from

$$\frac{\mu_0}{2}\sum_{k=1}^{N}\left[2\alpha_k\varphi_k^*U^2+i\sum_{j=1}^{N}\frac{\nu}{2}\beta_{jk}\varphi_j^*U\right]\chi_k=d_5+id_6. \tag{3.4}$$

The amplitude–frequency characteristic of the steady-state vibrations of the shell is

$$\Delta_0 = -G_2 U_* \Delta_1 - G_3(a) \pm \sqrt{K_2^2 - \left(G_0 \Delta_1 U_* + \frac{G_1(a)}{2}\right)^2}, \quad K_2^2 = R_2^2 + S_2^2 = \frac{d_5^2 + d_6^2}{d_0^2}, \tag{3.5}$$

and the stationary solutions $a = a(\nu)$ found from (3.5) are stable if

$$\frac{d}{da}\left[G_1(a)a\right] + 2G_0 \Delta_1 U_* a < 0,$$

$$\left[G_1(a) + G_0 \Delta_1 U_* a\right] \frac{d}{da}\left[\frac{G_1(a)a}{a}\right] + \left[\Delta_0 + G_2 \Delta_1 U_* + G_3(a)\right] a \frac{dG_3(a)}{da} > 0. \tag{3.6}$$

4. When the shell is simultaneously subjected to both factors mentioned above (external periodic load and pulsation of fluid velocity), of practical importance is the case where both frequencies Ω and ν are close to the natural frequency λ_* of the shell–fluid system [5, 8, 9], i.e., $\Omega \approx \lambda_*$ and $\nu \approx 2\lambda_*$. In this case, the periodic solution of system (2.1) can be represented, as a first approximation, in the form (2.1), where $\psi = \Omega t + \theta$ (without loss of generality, we hereafter assume $\Omega = \nu/2$). To determine the amplitude a and the phase θ of single-frequency resonant vibrations of the shell, we derive from (2.7) the following equations:

$$\frac{da}{dt} = \varepsilon_0 \left[G_0(a) + R_1 \cos\theta + S_1 \sin\theta + \left(R_2 \cos 2\theta + S_2 \sin 2\theta\right) a\right],$$

$$\frac{d\theta}{dt} = \varepsilon_0 \left[\Delta_0 + H(a) + \frac{1}{a}\left(S_1 \cos\theta - R_1 \sin\theta\right) + S_2 \cos 2\theta - R_2 \sin 2\theta\right], \tag{4.1}$$

where

$$G(a) = G_0 \Delta_1 U_* a + G_1(a), \quad H(a) = \Delta_0 + G_2 \Delta_1 U_* + G_3(a) \tag{4.2}$$

and the notation (2.11), (3.3) is used.

To derive the amplitude–frequency characteristics, we will eliminate (assuming that $da/dt = 0$ and $d\theta/dt = 0$) the phase θ from Eqs. (4.1). As a result, we obtain the amplitude–frequency equation

$$\left(T(a) - K_2^2 a^2\right)^2 \left\{4a^2 K_1^2 K_2^2 - \left[T(a) - \left(K_1^2 + K_2^2 a^2\right)\right]\right\} = 4a^2 K_1^2 K_2^2 \left[R_1 H(a) a - S_1 G(a)\right]^2, \tag{4.3}$$

where $T(a) = G^2(a) + H^2(a) a^2$.

The solutions $a = a(\Omega)$ found from this equation will be stable if

$$\frac{d}{da}[G(a)a] < 0,$$

$$\left[\frac{d}{da}\left(\frac{G(a)}{a}\right) - \frac{1}{a^2}\left(R_1 \cos\theta + S_1 \sin\theta\right)\right]\left[2G(a) + \left(R_1 \cos\theta + S_1 \sin\theta\right)\right]$$

$$+ \left[2aH(a) + \left(S_1 \cos\theta - R_1 \sin\theta\right)\right]\left[\frac{dH(a)}{da} - \frac{1}{a^2}\left(S_1 \cos\theta - R_1 \sin\theta\right)\right] > 0, \tag{4.4}$$

where

$$\cos\theta = \frac{T(a) - (K_1^2 + K_2^2 a^2)}{2a\left(R_1 R_2 + S_1 S_2\right)}, \quad \sin\theta = \frac{R_1 H(a) a - S_1 G(a)}{T(a) - K_2^2 a^2}. \tag{4.5}$$

Conclusions. We have outlined a method for analysis of the single-frequency nonlinear vibrations of cylindrical shells interacting with the fluid flowing inside them. The method can be used to analyze the dynamic stability and nonlinear vibrations of elastic shells conveying a fluid in specific cases.

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