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NUMERICAL SOLUTION OF DYNAMIC PROBLEMS FOR REINFORCED ELLIPSOIDAL SHELLS UNDER NONSTATIONARY LOADS

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A dynamic problem for reinforced ellipsoidal shells under nonstationary loads is formulated and solved numerically. The results obtained are analyzed

Keywords: ellipsoidal shell, dynamic behavior, nonstationary load, numerical algorithm

The forced axisymmetric vibrations of reinforced shells of revolution of canonical shapes (cylindrical, spherical, and conical) under nonstationary loads [2, 4, 9–11] have been investigated adequately. The axisymmetric vibrations of reinforced shells of revolution were studied in [4] in geometrically nonlinear formulation. The physical nonlinearity of rib-reinforced cylindrical shells was accounted for in [5]. The dynamic processes in reinforced laminated shells under nonstationary loads were considered in [9, 10]. The complexity of original problems and associated physical processes necessitates the use and development of numerical approaches for their solution [2, 4].

The present paper studies forced vibrations of discretely reinforced ellipsoidal shells. Variational and differential formulations of the corresponding problem will be given. The numerical algorithm to be used is based on the approximation of the original variational functional. Numerical examples will be given.

Problem Formulation. Consider an ellipsoidal shell. The geometry of its median surface is defined by the following relations [3]:

$$x = R\sin\alpha_1 \sin\alpha_2, \qquad y = R\sin\alpha_1\cos\alpha_2, \qquad z = kR\cos\alpha_1, \tag{1}$$

where α_1 and α_2 are the Gaussian curvilinear coordinates (meridional and circumferential, respectively) on the shell surface, and $k = a/b$ is the aspect ratio of the ellipse (a and b are the ellipse semiaxes).

In view of (1), the components of the metric and form of the median surface are

$$a_{11} = R^2(\cos^2\alpha_1 + k^2\sin^2\alpha_1), \qquad a_{22} = R^2\sin^2\alpha_1,$$

$$b_{11} = kR(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{-1/2}, \qquad b_{22} = kR\sin^2\alpha_1(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{-1/2}. \tag{2}$$

According to formulas (2), the coefficients of the first quadratic form and curvature of the median surface are

$$A_1 = a(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{1/2}, \qquad A_2 = a\sin\alpha_1,$$

$$k_1 = \frac{a}{b^2}(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{-3/2}, \qquad k_2 = \frac{a}{b^2}(\cos^2\alpha_1 + k^2\sin^2\alpha_1)^{-1/2}. \tag{3}$$

The inhomogeneous shell structure under consideration consists of an ellipsoidal shell and a system of rings (ribs) rigidly bound to it (the ribs are assumed to be aligned along the coordinate line α_2). As a mathematical model describing the

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forced vibrations of this structure, we will use the hyperbolic system of nonlinear differential equations of the theory of shells and curvilinear Timoshenko-type rods. The variation of the displacements across the shell thickness is described by the formulas

$$u_1^z(s,z)=u_1(s)+z\varphi_1(s), \qquad u_3^z(s,z)=u_3(s), \tag{4}$$

where u_1, u_3, and φ_1 are the components of the generalized displacement vector, and $s=\alpha_1 A_1$.

To determine the strain state of the jth rib, we will use the generalized displacement vector of the center of gravity of its cross section $\overline{U}_j=(u_{1j},u_{3j},\varphi_{1j})$. The contact conditions relating the components of the vectors $\overline{U}=(u_1,u_3,\varphi_1)$ and $\overline{U}_j$ have the form

$$u_{1j}(t)=u_1(s_j,t)\pm h_{cj}\varphi_1(s_j,t), \qquad u_{3j}(t)=u_3(s_j,t), \qquad \varphi_{1j}(t)=\varphi_1(s_j,t), \tag{5}$$

where s is the coordinate of the point of contact between the jth rib and the shell; t is the time coordinate; and $h_{cj}=0.5(h+b_j)$, h is the thickness of the shell and b_j is the height of the jth rib.

To determine the stress–strain state of the structure, we will use the quadratically nonlinear theory of shells [6]. In this case, the deformation relations for the shell and jth rib are

$$\varepsilon_{11}=\frac{\partial u_1}{\partial s}+\frac{1}{2}\theta_1^2+k_1u_3, \qquad \varepsilon_{22}=\frac{u_1}{A_2}\frac{\partial A_2}{\partial s}+k_2u_3, \qquad \varepsilon_{13}=\varphi_1+\theta_1,$$

$$\theta_1=\frac{\partial u_3}{\partial s}-k_1u_1, \qquad \kappa_{11}=\frac{\partial \varphi_1}{\partial s}, \qquad \kappa_{22}=\frac{\varphi_1}{A_2}\frac{\partial A_2}{\partial s}, \qquad \varepsilon_{22j}=k_{2j}u_{3j}. \tag{6}$$

To derive the equations describing the vibrations of the shell structure, we will use the Hamilton–Ostrogradskii variational principle:

$$\int_{t_1}^{t_2}[\delta(\Pi-K)-\delta A]\,dt=0, \tag{7}$$

where Π and K are the total potential and kinetic energies of the structure and A is the work done by external forces.

We will also use the contact conditions for the shell and the jth rib in the following integral form [1]:

$$u_{1j}(s_j)=\int_{s_0}^{s_N}[u_1(s)\pm h_{cj}\varphi_1(s)]\delta(s-s_j)\,ds,$$

$$u_{3j}(s_j)=\int_{s_0}^{s_N}u_3(s)\delta(s-s_j)\,ds, \qquad \varphi_{1j}(s_j)=\int_{s_0}^{s_N}\varphi_1(s)\delta(s-s_j)\,ds, \tag{8}$$

where $\delta(s-s_j)$ is the Dirac delta function.

After standard transformations of (7) using (5) and (8), we reduce the variational equation to the form

$$\int_{s_0}^{s_N}\left\{\left[L_1(\overline{U})+\sum_{j=1}^{J}L_{1j}(\overline{U})\delta(s-s_j)\right]\delta u_1+\left[L_2(\overline{U})+\sum_{j=1}^{J}L_{2j}(\overline{U})\delta(s-s_j)\right]\delta u_3\right.$$

$$\left.+\left[L_3(\overline{U})+\sum_{j=1}^{J}L_{3j}(\overline{U})\delta(s-s_j)\right]\delta\varphi_1-P_1\delta u_1-P_2\delta u_3-m_1\delta\varphi_1\right\}A_2\,ds=0, \tag{9}$$

where

$$L_1(\overline{U})=-\frac{1}{A_2}\left[\frac{\partial}{\partial s}(A_2T_{11})-\frac{\partial A_2}{\partial s}T_{22}\right]-k_1\overline{T}_{13}+\rho h\frac{\partial^2 u_1}{\partial t^2},$$

$$L_2(\bar{U}) = -\frac{1}{A_2}\frac{\partial}{\partial s}(A_2\bar{T}_{13}) + k_1 T_{11} + k_2 T_{22} + \rho h \frac{\partial^2 u_3}{\partial t^2},$$

$$L_3(\bar{U}) = -\frac{1}{A_2}\left[\frac{\partial}{\partial s}(A_2 M_{11}) - \frac{\partial A_2}{\partial s} M_{22}\right] + T_{13} + \rho \frac{h^3}{12}\frac{\partial^2 \varphi_1}{\partial t^2},$$

$$\bar{T}_{13} = T_{13} + T_{11}\theta_1, \qquad L_{1j}(\bar{U}) = \rho_j F_j \left(\frac{\partial^2 u_1}{\partial t^2} \pm h_{cj}\frac{\partial^2 \varphi_1}{\partial t^2}\right),$$

$$L_{2j}(\bar{U}) = k_{2j} T_{22j} + \rho_j F_j \frac{\partial^2 u_3}{\partial t^2}, \qquad L_{3j}(\bar{U}) = \rho_j F_j \left[\pm h_{cj}\frac{\partial^2 u_1}{\partial t^2} + \left(h_{cj}^2 + \frac{I_{crj}}{F_j}\right)\frac{\partial^2 \varphi_1}{\partial t^2}\right],$$

where ρ and ρ_j are the densities of the materials of the shell and jth rib, F_j and I_{crj} are the geometrical parameters of the jth rib, and P_1, P_3, and m_1 are the components of the generalized load vector.

The forces and moments are expressed in terms of the strains as follows:

$$T_{11} = B_{11}(\varepsilon_{11} + \nu_{21}\varepsilon_{22}), \qquad T_{22} = B_{22}(\varepsilon_{22} + \nu_{12}\varepsilon_{11}), \qquad T_{13} = B_{13}k^2\varepsilon_{13},$$

$$M_{11} = D_{11}(\kappa_{11} + \nu_{21}\kappa_{22}), \qquad M_{22} = D_{22}(\kappa_{22} + \nu_{12}\kappa_{11}), \qquad T_{22j} = B_{22j}\varepsilon_{22j}, \tag{10}$$

where

$$B_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \qquad B_{22} = \frac{E_2 h}{1 - \nu_{12}\nu_{21}}, \qquad B_{13} = G_{13}h,$$

$$D_{11} = \frac{E_1 h^3}{12(1 - \nu_{12}\nu_{21})}, \qquad D_{22} = \frac{E_2 h^3}{12(1 - \nu_{12}\nu_{21})}, \qquad B_{22j} = E_j F_j,$$

$E_1, E_2, G_{13}, \nu_{12}$, and ν_{21} are the physicomechanical parameters of the shell material; k^2 is the integral coefficient of transverse shear in the theory of plates and shells; and E_j and F_j are Young's modulus and the cross-sectional area of the jth rib.

By virtue of the independence of the variations δu_1, δu_3, and $\delta\varphi_1$, Eqs. (9) yield the differential vibration equations

$$\frac{1}{A_2}\left[\frac{\partial}{\partial s}(A_2 T_{11}) - \frac{\partial A_2}{\partial s} T_{22}\right] + k_1\bar{T}_{13} + P_1 = \rho h \frac{\partial^2 u_1}{\partial t^2} + \sum_{j=1}^{J} \rho_j F_j \left.\left(\frac{\partial^2 u_1}{\partial t^2} \pm h_{cj}\frac{\partial^2 \varphi_1}{\partial t^2}\right)\right|_{s=s_j},$$

$$\frac{1}{A_2}\frac{\partial}{\partial s}(A_2\bar{T}_{13}) - k_1 T_{11} - k_2 T_{22} - \left. k_{2j}T_{22j}\right|_{s=s_j} + P_3 = \rho h \frac{\partial^2 u_3}{\partial t^2} + \sum_{j=1}^{J} \rho_j F_j \left.\frac{\partial^2 u_3}{\partial t^2}\right|_{s=s_j},$$

$$\frac{1}{A_2}\left[\frac{\partial}{\partial s}(A_2 M_{11}) - \frac{\partial A_2}{\partial s} M_{22}\right] - T_{13} + m_1 = \rho \frac{h^3}{12}\frac{\partial^2 \varphi_1}{\partial t^2} + \sum_{j=1}^{J} \rho_j F_j \left.\left[\pm h_{cj}\frac{\partial^2 u_1}{\partial t^2} + \left(h_{cj}^2 + \frac{I_{crj}}{F_j}\right)\frac{\partial^2 \varphi_1}{\partial t^2}\right]\right|_{s=s_j}, \tag{11}$$

which should be supplemented with boundary conditions (for forces or displacements) and zero initial conditions.

Numerical Algorithm. The algorithm is based on the finite-difference approximation of the variational equation (9):

$$\int_{s_0}^{s_{1/2}} \left\{[L_1(\bar{U}) + P_1]\delta u_1 + [L_2(\bar{U}) + P_3]\delta u_3 + [L_3(\bar{U}) + m_1]\delta\varphi_1\right\} A_2\, ds$$

$$+\sum_{i=1}^{N-1}\int_{s_{i-1/2}}^{s_{i+1/2}}\Big\{[L_1(\overline{U})+L_{1j}\delta(s-s_j)+P_1]\delta u_1+[L_2(\overline{U})+L_{2j}\delta(s-s_j)+P_3]\delta u_3$$

$$+[L_3(\overline{U})+L_{3j}\delta(s-s_j)+m_1]\delta\varphi_1\Big\}A_2\,ds$$

$$+\int_{s_{n-1/2}}^{s_N}\Big\{[L_1(\overline{U})+P_1]\delta u_1+[L_2(\overline{U})+P_3]\delta u_3+[L_3(\overline{U})+m_1]\delta\varphi_1\Big\}A_2\,ds=0.$$

Evaluating the integrals using explicit approximation in time coordinate, we obtain the following three groups of difference equations.

1. Difference vibration equations in the smooth domain

$$\frac{1}{A_{2,i}}\left(\frac{A_{2,i+1/2}T^n_{11,i+1/2}-A_{2,i-1/2}T^n_{11,i-1/2}}{\Delta s}\right)-\frac{1}{A_{2,i}}\frac{A_{2,i+1/2}-A_{2,i-1/2}}{\Delta s}T^n_{22,i}+k_{1,i}\overline{T}^n_{13,i}+P^n_{1,i}=\rho h(u^n_{1,i})_{\bar{t}t},$$

$$\frac{1}{A_{2,i}}\left(\frac{A_{2,i+1/2}\overline{T}^n_{13,i+1/2}-A_{2,i-1/2}\overline{T}^n_{13,i-1/2}}{\Delta s}\right)-k_{1,i}T^n_{11,i}-k_{2,i}T^n_{22,i}+P^n_{3i}=\rho h(u^n_{3,i})_{\bar{t}t},$$

$$\frac{1}{A_{2,i}}\left(\frac{A_{2,i+1/2}M^n_{11,i+1/2}-A_{2,i-1/2}M^n_{11,i-1/2}}{\Delta s}\right)-\frac{1}{A_{2,i}}\frac{A_{2,i+1/2}-A_{2,i-1/2}}{\Delta s}M^n_{22,i}-T^n_{13,i}+m^n_{1,i}=\frac{\rho h^3}{12}(\varphi^n_{1,i})_{\bar{t}t},$$

$$\overline{T}^n_{13,i\pm1/2}=T^n_{13,i\pm1/2}+T^n_{11,i\pm1/2}\theta^n_{1,i\pm1/2}. \quad (12)$$

2. Difference vibration equations on the jth discontinuity line

$$\frac{A_{2,j+1/2}}{A_{2,j}}T^n_{11,j+1/2}-\frac{A_{2,j-1/2}}{A_{2,j}}T^n_{11,j-1/2}+k_{1,j}\tilde{\tilde{T}}^n_{13,j}+P^n_{1,j}\Delta s=\rho h\Delta s(u^n_{1,j})_{\bar{t}t}+\rho_jF_j[(u^n_{1,j})_{\bar{t}t}\pm h_{cj}(\varphi^n_{1,j})_{\bar{t}t}],$$

$$\frac{A_{2,j+1/2}}{A_{2,j}}\overline{T}^n_{13,j+1/2}-\frac{A_{2,j-1/2}}{A_{2,j}}\overline{T}^n_{13,j-1/2}-k_{1,j}\tilde{T}^n_{11,j}-k_{2,j}\tilde{T}^n_{22,j}+k_{2j}T^n_{22j}+P^n_{3,j}\Delta s=(\rho h\Delta s+\rho_jF_j)(u^n_{3,j})_{\bar{t}t},$$

$$\frac{A_{2,j+1/2}}{A_{2,j}}M^n_{11,j+1/2}-\frac{A_{2,j-1/2}}{A_{2,j}}M^n_{11,j-1/2}$$

$$-\tilde{T}^n_{13,j}+m^n_{1,j}\Delta s=\rho\frac{h^3\Delta s}{12}(\varphi^n_{1,j})_{\bar{t}t}+\rho_jF_j\left[\pm h_{cj}(u^n_{1,j})_{\bar{t}t}+\left(h^2_{cj}+\frac{I_{crj}}{F_j}\right)\Big(\varphi^n_{1,j}\Big)_{\bar{t}t}\right], \quad (13)$$

where $\tilde{T}^n_{11,j}=\Delta s\Big(B_{11}k_{1,j}u^n_{3,j}+B_{12}k_{2,j}u^n_{3,j}\Big)$, $\tilde{T}^n_{22,j}=\Delta s\Big(B_{21}k_{1,j}u^n_{3,j}+B_{22}k_{2,j}u^n_{3,j}\Big)$, and $T^n_{13,j}=B_{13}\Delta sk^2\varphi^n_{1,j}$.

3. With the following boundary conditions at $s=s_0$:

$$T_{11}=F_1(t),\qquad \overline{T}_{13}=F_2(t),\qquad M_{11}=F_3(t),$$

the difference vibration equations

$$\frac{2}{A_{2,0}}\left(\frac{A_{2,1/2}T^n_{11,1/2}-A_{2,0}F^n_1}{\Delta s}\right)-\frac{1}{A_{2,0}}\frac{A_{2,1/2}-A_{2,-1/2}}{\Delta s}T^n_{22,0}+k_{1,0}\overline{T}^n_{13,0}+P^n_{1,0}=\rho h(u^n_{1,0})_{\bar{t}t},$$

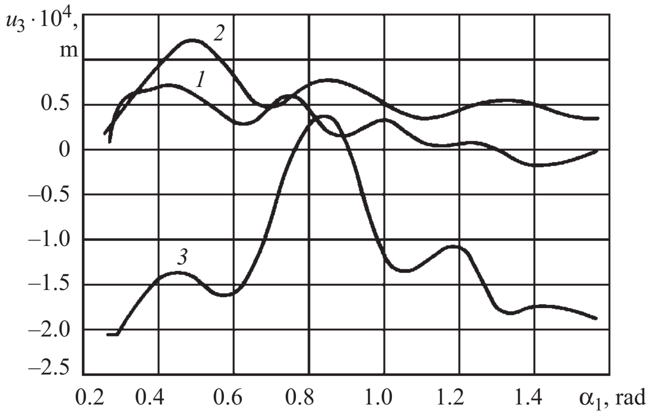

Fig. 1

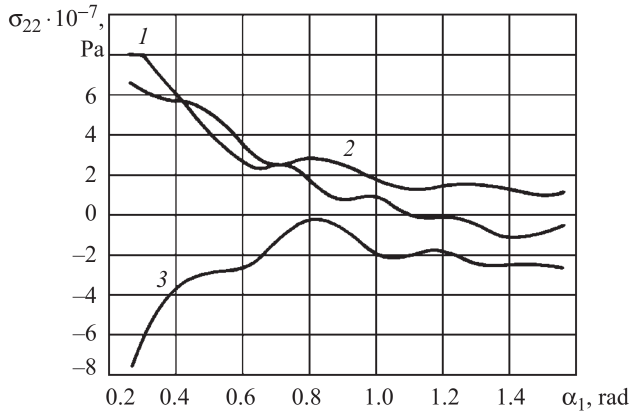

Fig. 2

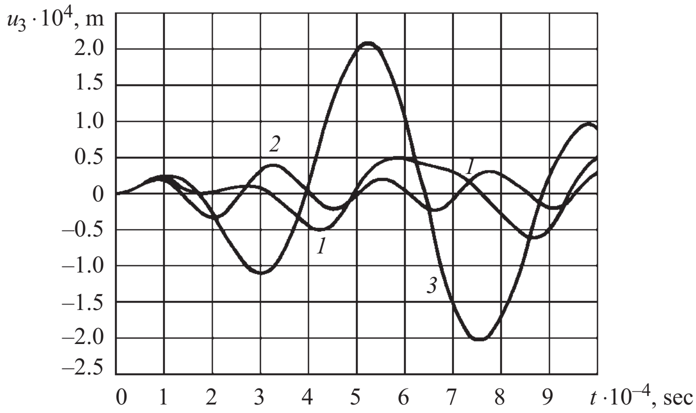

Fig. 3

$$\frac{2}{A_{1,0}}\left(\frac{A_{2,1/2}\bar{T}^n_{13,1/2}-A_{2,0}F^n_2}{\Delta s}\right)-k_{1,0}T^n_{11,0}-k_{2,0}T^n_{22,0}+P^n_{3,0}=\rho h(u^n_{3,0i})_{\bar{t}t},$$

$$\frac{2}{A_{2,0}}\left(\frac{A_{2,1/2}M^n_{11,1/2}-A_{2,0}F^n_3}{\Delta s}\right)-\frac{1}{A_{2,0}}\frac{A_{2,1/2}-A_{2,-1/2}}{\Delta s}M^n_{22,0}-T^n_{13,0}+m^n_{1,0}=\frac{\rho h^3}{12}(\varphi^n_{1,0})_{\bar{t}t}. \tag{14}$$

The functions and derivatives in (12)–(14) are denoted as in [8].

Numerical Example. Consider an ellipsoidal shell with symmetric openings at $\alpha_{1,0}=\pi/12$ and $\alpha_{1,N}=\pi-\alpha_{1,0}$. The shell is subjected to a nonstationary normal internal load. The free edge of each opening is reinforced with a ring rib at $s_1 \approx s_0$ (the edge of the rib is flush with the free edge of the shell, and the center of gravity of the rib's cross section is projected into an internal point of the median surface, i.e., $s_j = s_0 + 0.5b_j$, where b_j is the rib width). The boundary conditions at $s = s_0$ are: $T_{11}=0$, $\bar{T}_{13}=0$, and $M_{11}=0$. The initial conditions are zero.

The nonstationary load is defined by

$$P_3 = A\cdot\sin\frac{\pi t}{T}\left[\eta(t)-\eta(t-T)\right],$$

where A and T are the amplitude and duration of the load ($A=10^6$ Pa and $T=50\cdot10^{-6}$ sec).

The geometrical and physicomechanical parameters are: $a/b=2/3$ (for $a=0.3$ m and $b=0.45$ m), $a/b=1$ (for $a=b=$ 0.45 m), $a/b=3/2$ (for $a=0.45$ m and $b=0.3$ m), $h/h_j=0.5$, $F_j=2\cdot10^{-4}$ m^2, $\alpha_{1,0}=\pi/12$, $E_1=7\cdot10^{10}$ Pa, $\rho=2.7\cdot10^3$ kg/m^3, $E_j=E_1$, and $\rho_j=\rho$.

The numerical results obtained allow us to analyze the stress–strain state of the shell at any instant of time (on the interval $0 \le t \le 100T$). In particular, Figs. 1 and 2 show the dependences of u_3 and σ_{22}, respectively, on the spatial coordinate α_1 at fixed instants (time points at which the plotted quantities peak). In Fig. 1, curve *1* corresponds to $a/b = 2/3$, $a/h = 30$ at $t = 8.5T$; curve *2* to $a/b = 1$, $a/h = 45$ (spherical shell) at $t = 6T$; and curve *3* to $a/b = 3/2$, $a/h = 45$ at $t = 6T$. In Fig. 2, curves *1*, *2*, and *3* correspond to $t = 8.5T$, $6T$, and $14.5T$ and the same geometrical parameters as in Fig. 1. Figure 3 shows the dependence of u_3 on time on the interval $0 \le t \le 20T$ at the edge of the opening ($\alpha_1 = \alpha_{1,0}$) for the three above-mentioned cases. The curves allow us to estimate the effect of the geometry and ribs on the distribution of the displacements u_3 and stresses σ_{22} over the meridional section of an inhomogeneous ellipsoidal shell.

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