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BIFURCATIONS OF A SINGLE-SUPPORT ELASTIC THIN-WALLED ROTOR

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The bifurcations of a thin-walled shell rotor during simple and complex rotation are analyzed. The similarity and difference of the problem formulations and solution techniques are pointed out. In both cases, the buckling mode is described by the first circumferential harmonic. The dependence of rotor bifurcations on natural frequencies is studied

Keywords: shell rotor, bifurcation, instability, simple and complex rotations

Introduction. It was established in [2–4, 10, 11, 15] that bifurcations (in the sense defined in [1, 6, 7]) of rotating conical, spherical, paraboloidal, and compound shells occur under centrifugal forces as static buckling if the rotation is simple and as precession resonance if the rotation is complex. It was shown that the critical states of simply rotating shells are due to instability of dynamic equilibrium, resulting in buckling caused by inertial forces, which depend on the position of the elastic element relative to the axis of rotation.

The second type of bifurcations of a rotating shell mounted on a carrier body may occur when the body changes its attitude, forcing the rotation axis to move. In this case, the shell experiences precessions that may become resonant (bifurcation) at certain natural frequencies and angular velocity.

We will demonstrate below that the natural frequencies and critical angular velocities of a thin-walled elastic rotor are in certain relationships during simple and complex rotations. The rotation of the rotor changes substantially the natural frequencies and modes, since multiple frequencies split and vibration modes transform into waves traveling (precessing) in the circumferential direction. In this case, one of the split frequencies corresponds to a wave running in the direction of rotation (direct regular precession) and the other to a wave running in the opposite direction (retrograde regular precession).

The angular velocities at which the split frequencies combine and take zero values are the velocities of static (in the rotating coordinate system) buckling. During complex rotation, the rotor reaches critical states when the frequency of retrograde precession becomes equal to the velocity of rotation.

It is of interest to find out which of the bifurcations of a compound shell occurs earlier and how they are related to the natural frequencies. The features of bifurcations of elastic shells during simple and complex rotation can be used to analyze the dynamic behavior of thin-walled rotors in the engine of a maneuvering aircraft.

1. Equations of Complex Rotation of a Shell Element. According to [8, 9, 13, 14], the natural vibrations of rotating shells turn out to be precession. To analyze the relationship between these vibrations and critical states occurring during simple and complex rotations, we will select natural modes in which precession occurs in the first circumferential harmonic. Since the equation of static buckling and natural precessions are special cases of the equations of precessions during complex rotation of a shell, let us first determine the precession resonances of thin axisymmetric shells in complex rotation, analyzing natural vibrations and static buckling as simpler cases.

Assume that one end of a thin compound shell rotor is rigidly fixed to a perfectly rigid carrier body rotating together with its coordinate system $Oxyz$ at a constant angular velocity $\vec{\omega}$ about the symmetry axis Oz, which, in turn, is forced to turn with a constant velocity $\vec{\omega}_0$ in a plane (Fig. 1). The other end of the shell is free from external forces and constraints. Let us now introduce right coordinate systems: an inertial coordinate system $OX^*Y^*Z^*$ with the origin at the center of the support contour

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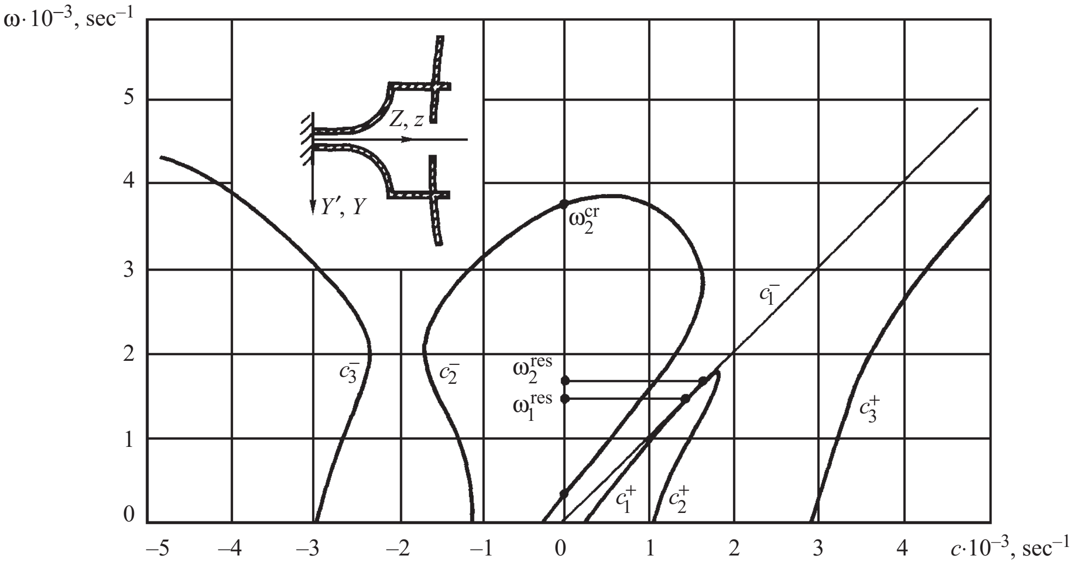

Fig. 1

of the rotor and the OY-axis collinear to the vector $\vec{\omega}_0$ and a rotating coordinate system $OXYZ$ with the fixed OY-axis coinciding with the OY^*-axis and the OZ-axis coinciding with and moving together with the Oz-axis. On the median surface of the shell, we introduce an orthogonal curvilinear coordinate system $Ox^1x^2x^3$ with the coordinate line x^1 lying in the generating section, x^2 directed in the circumferential direction, and x^3 running along the inward normal to the shell surface.

Initially, the rotor is stressed by stationary axisymmetric centrifugal forces and does not vibrate. Let us examine the possibility of instability of dynamic equilibrium or forced precessions in the form of a harmonic wave precessing with a frequency $\vec{\omega}$ relative to the initial stress state.

Let us use the geometrically nonlinear equations of dynamic equilibrium of a shell element in general form [3, 12]:

$$\nabla_\alpha \vec{T}^\alpha + \vec{p} = 0, \quad \nabla_\alpha \vec{M}^\alpha + (\vec{e}_\alpha \times \vec{T}^\alpha)\sqrt{a} = 0 \quad (\alpha = 1, 2), \tag{1}$$

where $\vec{T}^\alpha$ is the vector of internal forces, $\vec{M}^\alpha$ is the vector of internal moments, ∇_α denotes covariant derivative, and $\vec{p}$ is the vector of intensity of external forces.

Using the relations between the contravariant components of the functions of internal forces $\vec{T}^{ij}$ and moments $\vec{M}^{ij}$ and the covariant components of the functions of strain ε_{ij} and changes of curvatures μ_{ij},

$$T^{ij} = Eh\varepsilon_{\alpha\beta}(a^{ij}a^{\alpha\beta} + (1-\nu)a^{i\alpha}a^{j\beta})/(1-\nu^2),$$

$$M^{ij} = Eh^3\mu_{\alpha\beta}(a^{ij}a^{\alpha\beta} + (1-\nu)a^{i\alpha}a^{j\beta})/12(1-\nu^2), \tag{2}$$

expressing these functions in terms of the covariant components u_1, u_2, and u_3 of the displacement vector $\vec{u}$ and angles of rotation ϑ_i

$$\varepsilon_{ij} = (e_i \cdot \partial\vec{u}/\partial x^j + \vec{e}_j \cdot \partial\vec{u}/\partial x^i + \vartheta_i \cdot \vartheta_j)/2, \qquad \vartheta_i = (\partial\vec{u}/\partial x^i)\vec{e}_3,$$

$$\mu_{ij} = (\vec{\boldsymbol{e}}^k / c^{ik} \cdot \partial\vec{\boldsymbol{\Omega}}/\partial x^j + \vec{\boldsymbol{e}}^k / c^{jk} \cdot \partial\vec{\boldsymbol{\Omega}}/\partial x^i)/2, \qquad \vec{\boldsymbol{\Omega}} = c^{ij}\vartheta_i\vec{\boldsymbol{e}}_j \qquad (i, j, k = 1, 2) \tag{3}$$

and considering, in expanding the vector operations in (1), the change of the parameters b_i^j of the second quadratic form during deformation, we obtain nonlinear equations of dynamic equilibrium.

The only active forces acting on the shell are inertial forces. They are determined from the equality $\vec{p} = -\gamma h\vec{a}$, where γ is the density of the shell material, h is the shell thickness, and $\vec{a}$ is the vector of absolute acceleration defined by the formula

$\vec{a} = \vec{a}_t + \vec{a}_r + \vec{a}_C$. To calculate the vectors of translational ($\vec{a}_t$), relative ($\vec{a}_r$), and Coriolis ($\vec{a}_C$) accelerations, we use the kinematic relations describing the complex motion of a point [5]:

$$\vec{a}_t = \vec{\boldsymbol{\epsilon}} \times \vec{\boldsymbol{\rho}} + \vec{\boldsymbol{\Omega}} \times (\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\rho}}), \qquad \vec{a}_r = d^2 \vec{\boldsymbol{\rho}} / dt^2, \qquad \vec{a}_C = 2\vec{\boldsymbol{\Omega}} \times (d\vec{\boldsymbol{\rho}} / dt), \tag{4}$$

where $\vec{\boldsymbol{\Omega}} = \vec{\boldsymbol{\omega}}_0 + \vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\epsilon}} = \vec{\boldsymbol{\omega}}_0 \times \vec{\boldsymbol{\omega}}$ are the vectors of absolute angular velocity and angular acceleration of the moving coordinate system *Oxyz*, and $\vec{\boldsymbol{\rho}}$ is the position vector of a shell element in this system.

Performing the vector operations (4) and neglecting ω_0^2, we determine the contravariant components of the acceleration vectors:

$$a_t^1 = -\omega^2 r \sin\varphi / \sqrt{a_{11}} + 2\omega_0 \omega r \sin(\omega t + x^2) \cos\varphi / \sqrt{a_{11}} - \omega^2 u_1 \sin^2\varphi / a_{11} + \omega^2 u_3 \sin\varphi \cos\varphi / \sqrt{a_{11}},$$

$$a_t^2 = -\omega^2 u_2 / a_{22},$$

$$a_t^3 = \omega^2 r \cos\varphi + 2\omega_0 \omega r \sin(\omega t + x^2) \sin\varphi + \omega^2 (u_1 \sin\varphi / \sqrt{a_{11}} - u_3 \cos\varphi) \cos\varphi,$$

$$a_C^1 = -2\omega \dot{u}_2 \sin\varphi / \sqrt{a_{11} a_{22}},$$

$$a_C^2 = 2\omega \dot{u}_1 \sin\varphi / \sqrt{a_{11} a_{22}} - 2\omega \dot{u}_3 \cos\varphi / \sqrt{a_{22}}, \qquad a_C^3 = 2\omega \dot{u}_2 \cos\varphi / \sqrt{a_{22}},$$

$$a_r^1 = \ddot{u}_{11} / a_{11}, \qquad a_r^2 = \ddot{u}_2 / a_{22}, \qquad a_r^3 = \ddot{u}_3. \tag{5}$$

These accelerations, except for the terms depending on the independent variable t alone, include terms with dependent variables $u_1, u_2, u_3, \dot{u}_1, \dot{u}_2$, and $\dot{u}_3$ brought in the left-hand sides of the governing equations. This leads to restructuring of the equations and the possibility of their degeneration in critical states.

The equations of complex rotation follow from relations (1)–(3) transformed using (4) and (5). However, the equalities containing large ω^2 account for changes in the geometry of the shell due to loading and include the quantities $r + \Delta r$ and $\varphi + \Delta\vartheta_1^*$ instead of r and φ, which are the distance from the rotation axis to the section of interest and the angle between the tangent and the generatrix.

The factors $\sin(\omega t + x^2)$ on the right-hand sides of these equations are associated with the type of the inertial load, which is harmonic in x^2 and t and runs around the shell with frequency ω, generating precessions with the same frequency. In simulating precessions excited by these forces, we assume they are small since $\omega >> \omega_0$. Therefore, we first consider the stress–strain state of the rotor simply rotating with frequency ω and then analyze the precessions using the equations of motion linearized about the first state [10]:

$$\partial\Delta T^{11} / \partial x^1 + \partial\Delta T^{12} / \partial x^2 + (2\Gamma_{11}^1 + \Gamma_{21}^2) \Delta T^{11} + \Gamma_{22}^1 \Delta T^{22} - b_1^1 \Delta T^{13}$$

$$-\gamma h[-\omega^2 \sin\varphi \Delta r / \sqrt{a_{11}} - \omega^2 r \cos\varphi \Delta\vartheta_1^* / \sqrt{a_{11}}$$

$$-2\omega \sin\varphi \Delta\dot{u}_2 / \sqrt{a_{11} a_{22}} + \Delta\ddot{u}_1 / a_{11}] = 2\gamma h \omega_0 \omega r \sin(\omega t + x^2) \cos\varphi / \sqrt{a_{11}},$$

$$\partial\Delta T^{12} / \partial x^1 + \partial\Delta T^{22} / \partial x^2 + (3\Gamma_{12}^2 + \Gamma_{11}^1) \Delta T^{12} - b_2^2 \Delta T^{23}$$

$$-\gamma h[-\omega^2 r \cos\varphi \Delta\vartheta_2^* / \sqrt{a_{22}} + 2\omega \sin\varphi \Delta\dot{u}_1 / \sqrt{a_{11} a_{22}} - 2\omega \cos\varphi \Delta\dot{u}_3 / \sqrt{a_{22}} + \Delta\ddot{u}_2 / a_{22} - \omega^2 \Delta u_2 / a_{22}] = 0,$$

$$\partial\Delta T^{13} / \partial x^1 + \partial\Delta T^{23} / \partial x^2 + (\Gamma_{12}^2 + \Gamma_{11}^1) \Delta T^{13} + b_{11} \Delta T^{11} + \Delta b_{11} T^{11} + b_{22} \Delta T^{22}$$

$$+ \Delta b_{22} T^{22} - \gamma h[\omega^2 \cos\varphi \Delta r - \omega^2 r \sin\varphi \Delta\vartheta_1^* + 2\omega \cos\varphi \Delta\dot{u}_2 / \sqrt{a_{22}} + \Delta\ddot{u}_3] = 2\gamma h \omega_0 \omega r \sin\varphi \sin(\omega t + x^2). \tag{6}$$

From the structure of these equations, it follows that the unknown functions vary as $\sin(\omega t + x^2)$ if they are odd in x^2 and t and vary as $\cos(\omega t + x^2)$ if they are even in these variables. This fact allows us to eliminate the variables x^2 and t from Eqs. (6) and to reduce partial differential equations to ordinary differential equations with the axial coordinate x^1 as an independent variable. The resulting system of equations is solved using the initial-parameter and Runge-Kutta methods [3].

2. Free Vibrations of Simply Rotating Shells. In the case of simple rotation, the inertial forces acting on the shell include terms that do not have the factor ω_0. Since we are considering the critical states of simply and complexly rotating shells in the minimum-energy deformation mode described by the first circumferential harmonic in x^2, we will approximate the unknown functions by the first harmonics $\sin(ct + x^2)$ and $\cos(ct + x^2)$, where $ct + x^2$ is a phase coordinate and c is the natural frequency [13, 14]. With this simplification, we can derive from (6) equations describing a partial mode of free vibrations:

$$dT^{(11)}/dx^1 - T^{(12)} + (2\Gamma_{11}^1 + \Gamma_{21}^2)T^{(11)} + \Gamma_{22}^1 T^{(22)} - b_1^1 T^{(13)}$$

$$-\gamma\, h\left(-\omega^2 r\cos\varphi\, \vartheta_{(1)}/a_{11} - \omega^2 u_{(1)} \sin^2\varphi / a_{11} + \omega^2 u_{(3)} \sin\varphi\cos\varphi/\sqrt{a_{11}} + 2\omega\, c u_{(2)} \sin\varphi/\sqrt{a_{11}a_{22}} - c^2 u_{(1)}/a_{11}\right) = 0,$$

$$dT^{(12)}/dx^1 + T^{(22)} + (3\Gamma_{12}^2 + \Gamma_{11}^1)T^{(12)} - b_2^2 T^{(23)}$$

$$-\gamma h\left(-\omega^2 u_{(2)}/a_{22} + 2\omega c\, u_{(1)} \sin\varphi/\sqrt{a_{11}a_{22}} - 2\omega c\, u_{(3)} \cos\varphi/\sqrt{a_{22}} - c^2 u_{(2)}/a_{22} - \omega^2 r\cos\varphi\vartheta_{(2)}/a_{22}\right) = 0,$$

$$dT^{(13)}/dx^1 - T^{(23)} + (\Gamma_{12}^2 + \Gamma_{11}^1)T^{(13)} + b_{11}T^{(11)} - \mu_{(11)}T_0^{11} + b_{22}T^{(22)} - \mu_{(22)}T_0^{22}$$

$$-\gamma h\left(-\omega^2 r\sin\varphi\vartheta_{(1)}/\sqrt{a_{11}} + \omega^2 u_{(1)} \sin\varphi\cos\varphi/\sqrt{a_{11}} - \omega^2 u_{(3)}\cos^2\varphi - 2\omega\, c u_{(2)} \cos\varphi/\sqrt{a_{22}} - c^2 u_{(3)}\right) = 0. \qquad (7)$$

The free vibrations described by these equations have the form of a harmonic wave running in the circumferential direction with the angular velocity c. When $\omega = 0$, each of the frequencies is double and the corresponding waves are standing. When $\omega \neq 0$, the multiple frequencies split into two and the two modes start precessing in the direction of rotation (direct regular precession) if the frequency c_i is negative and in the opposite direction (retrograde regular precession) if the frequency c_i is positive.

3. Instability of the Dynamic Equilibrium of Rotating Shells. Assume that the rotating shell buckles in a minimum-energy mode described by the first circumferential harmonic in x^2. Since the shell does not vibrate in this case, only positional forces of inertia act on its elements during buckling:

$$a_t^1 = -\omega^2 r\sin\varphi/\sqrt{a_{11}}, \qquad a_t^3 = \omega^2 r\cos\varphi. \qquad (8)$$

Using these equalities in (1)–(3) and linearizing them about the states of simple rotation with angular velocity ω, we derive a homogeneous system of equations of neutral equilibrium:

$$\partial\Delta T^{11}/\partial x^1 + \partial\Delta T^{12}/\partial x^2 + (2\Gamma_{11}^1 + \Gamma_{21}^2)\Delta T^{11} + \Gamma_{22}^1 \Delta T^{22} - b_1^1 \Delta T^{13}$$

$$-\gamma h\left[-\omega^2 r\cos\varphi\Delta\vartheta_1^*/\sqrt{a_{11}} - \omega^2 \Delta u_1 \sin^2\varphi/a_{11} + \omega^2 \Delta u_3 \sin\varphi\cos\varphi/\sqrt{a_{11}}\right] = 0,$$

$$\partial\Delta T^{12}/\partial x^1 + \partial\Delta T^{22}/\partial x^2 + (3\Gamma_{12}^2 + \Gamma_{11}^1)\Delta T^{12} - b_2^2 \Delta T^{23} + \gamma h\omega^2 r\cos\varphi\Delta\vartheta_2^* = 0,$$

$$\partial\Delta T^{13}/\partial x^1 + \partial\Delta T^{23}/\partial x^2 + (\Gamma_{12}^2 + \Gamma_{11}^1)\Delta T^{13} + b_{11}\Delta T^{11} + \Delta b_1^1 T^{11} + b_{22}\Delta T^{22} + \Delta b_{22} T^{22}$$

$$-\gamma h\left[-\omega^2 r\sin\varphi\Delta\vartheta_1^* + \omega^2 \Delta u_1 \sin\varphi\cos\varphi/\sqrt{a_{11}} - \omega^2 \Delta u_3 \cos^2\varphi\right] = 0. \qquad (9)$$

In contrast to Eqs. (6) and (7), these equations are referred to the coordinate system $Ox^1x^2x^3$ fixed to the shell, the phase variables $\omega t + x^2$ and $ct + x^2$ are not used, and the unknown variables are approximated using the function $\sin x^2$ or $\cos x^2$, depending on whether the variable being approximated is odd or even. Such substitutions reduce Eqs. (9) to a system of homogeneous ordinary differential equations. Their partial solutions are found by the initial-parameter method. The critical

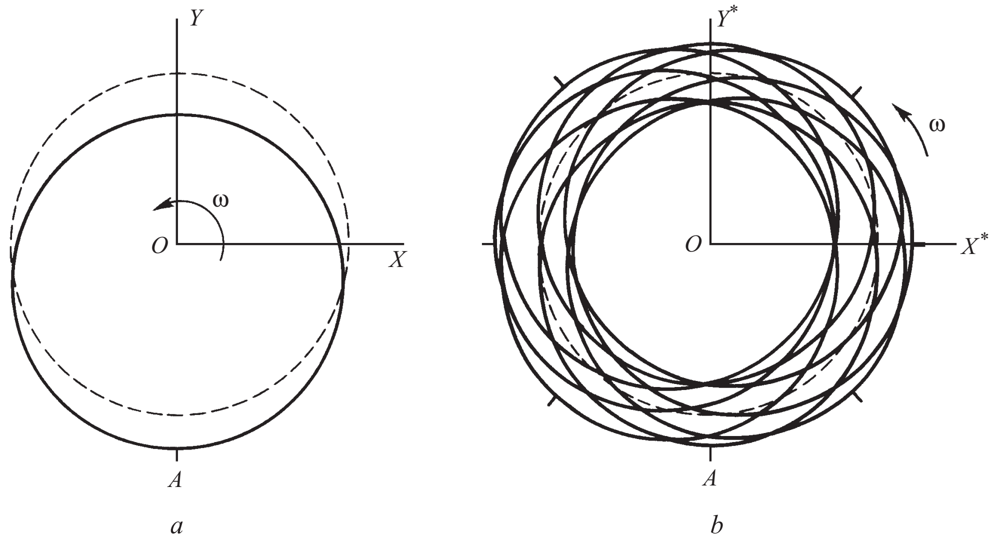

Fig. 2

values ω_{cr} of the angular velocity at which this system has a nontrivial solution are found by enumerating the values of ω from some range. A critical state is considered to set in when the matrix of the corresponding equations, which follow from the boundary conditions, degenerate.

4. Identifying the Critical States of a Rotating Rotor from Its Frequency Curves. Consider a thin-walled rotor consisting of cylindrical and conical shells and a shell of negative Gaussian curvature (Fig. 1). Its geometrical and mechanical characteristics are given in [11]. We used the equations of free vibrations from Sect. 2 to plot the dependence of the natural frequency c_i on the angular velocity ω of simple rotation (Fig. 1). Let us now analyze the deformation modes described by the first harmonics. As indicated above, they are forced-precession modes of a complexly rotating shell and minimum-energy static modes of a simply rotating shell. First, we determined, by enumeration, the natural frequencies $c_i^{+,-}$ of a nonrotating rotor (ω = 0). For each value of c obtained by adding a small increment Δc, we evaluated the determinant of the matrix of the corresponding system of algebraic equations, which follows from the boundary conditions of the initial-parameter method. The values of c at which the determinant vanishes are eigenvalues. It turned out that the frequencies come in pairs where they are equal in absolute value and opposite in sign, $|c_i^-| = c_i^+$ (i = 1, 2, 3). Such frequencies may be considered multiple, since they correspond to standing (nonprecession) modes that differ only in phase in x^2. After that, for each pair of frequencies $c_i^-(0), c_i^+(0)$, we plotted characteristic curves $c_i^-(\omega)$ and $c_i^+(\omega)$ (i = 1, 2, 3) by varying the angular velocity ω. The curves located in the right-hand quadrant ($c_i^+ > 0$) represent retrograde precession and the curves in the left-hand quadrant ($c_i^- < 0$), direct precession. Since the dependence of c on ω is strong, the method, sometimes used in rotor design, of determining the resonant frequencies of rotating rotors from the natural frequencies of the corresponding nonrotating shells may produce significant errors.

As ω increases, the curves $c_i^-(\omega)$ and $c_i^+(\omega)$ cease to be symmetric about the ordinate axis, i.e., multiple frequencies split. Therefore, at the same ω, the velocities c of direct and retrograde precessions are different.

The curves $c_i(\omega)$ originating at the points $c_1^-(0)$ and $c_2^-(0)$ close each other, crossing the ordinate axis at the points ω_1^{cr} = 314 sec^{-1} and ω_2^{cr} = 3770 sec^{-1} when $c_i(\omega_1^{cr}) = c_i(\omega_2^{cr}) = 0$. This fact indicates that the rotor does not vibrate at these angular velocities, i.e., divergent buckling occurs. A numerical analysis of static buckling performed by the method from Sect. 3 confirmed this conclusion with adequate accuracy. Figure 2 shows buckling modes of the rotor's free end in the rotating (*Oxyz*, Fig. 2*a*) and inertial ($OX^*Y^*Z^*$, Fig. 2*b*) coordinate systems. The mark *A* allows us to trace the orientation of the section relative to the coordinate system chosen.

The curves $c_i(\omega)$ originating at the points $c_1^+(0)$ and $c_2^+(0)$ also close each other, crossing the bisector in the first quadrant. At the intersection points: $c_1^+(\omega_1^{res}) = \omega_1^{res} = 1432\,\text{sec}^{-1}$ and $c_1^+(\omega_2^{res}) = \omega_2^{res} = 1675\,\text{sec}^{-1}$. These angular velocities are

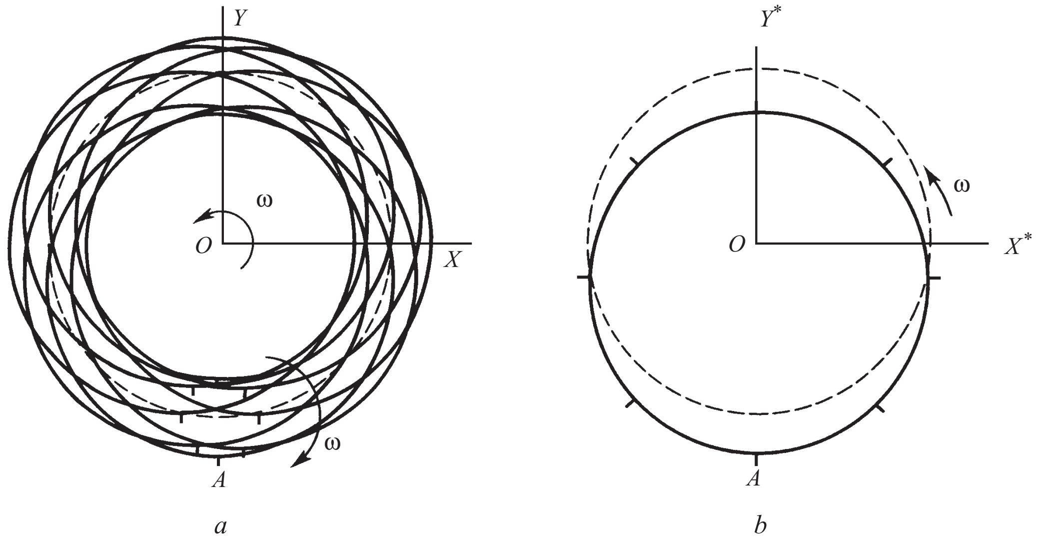

Fig. 3

resonant, because the precession frequency is equal to the angular velocity of the rotor and to the natural frequency at each of the intersection points. A numerical analysis performed by the method from Sect. 1 reliably confirmed this effect.

Figure 3*a*, *b* shows the modes of preresonant vibrations of the rotor's free end in the coordinate systems *Oxyz* and *OXYZ* coordinate systems.

Thus, the frequency curves in Fig. 1 describe the vibratory motion of the rotor. They may be used to establish critical states of the rotor during simple and complex rotation.

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