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COMPLEX BEHAVIOR OF A TRAJECTORY IN SINGLE- AND DOUBLE-FREQUENCY SYSTEMS

A. A. Martynyuk and N. V. Nikitina

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The complex behavior of trajectories is investigated. A new approach is proposed to estimate the applicability limits of some models. Examples of real phenomena are considered

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This paper continues the studies into the complex behavior of closed phase trajectories [12–16]. Developing a general theory of complex oscillations is a very difficult task. In this connection, we will continue to search for ways of identifying complex motions [9–11]. The qualitative analysis employs variables ρ and θ, which allow us to describe evolution as the modulus of a complex quantity [12]. We will identify double-frequency oscillations on the basis of the symmetry of the solution $\rho^0(\psi)$, where ψ corresponds to the inphase or resonant state of two angular deviations, and on the basis of a generalization of the Nemytskii–Stepanov symmetry principle for double-frequency systems (the existence of quasiperiodic motions is proved using the symmetry principle). The novelty of the present study is in establishing the limits of applicability for certain models of complex oscillations in systems of dimension greater than two and in identifying what makes simple oscillations complex. We will discuss the relationship of the Bogolyubov–Mitropol'skii asymptotic theory [3, 6] and the symmetry principle.

1. Double-Frequency Oscillations. Synchronization. A nonlinear oscillator with periodic disturbance at a close frequency Ω is assumed to have a three-dimensional phase space. This system is expressed in terms of the variables ρ and θ as follows:

$$\frac{d\rho}{dt}=\mathcal{R}(\rho,\theta,\Omega t),\qquad \frac{d\theta}{dt}=\omega+\mathcal{T}(\rho,\theta,\Omega t), \tag{1}$$

where $\mathcal{R}(\rho,\theta,\Omega t)$ is a continuously differentiable function for $0\le\rho<+\infty$, $\mathcal{T}(\rho,\theta,\Omega t)$ is a continuously differentiable function for $0<\rho<+\infty$, and ω is a constant (the frequency of the corresponding linear oscillator). Let us analyze a near-resonance state of the system, $\theta\approx\pm\psi$, $\psi=\Omega t$. In double-frequency systems, oscillations can also be identified from the symmetry of the solution $\rho^0(\psi)$. In this case, we consider a near-resonance state $\theta\approx\psi$ and establish either the extension–contraction symmetry of a trajectory about $\rho^0(\psi)$ in the case of a conservative system or attraction symmetry in the case of a dissipative system. The equilibrium state $\theta\approx\psi$ holds until $\theta\approx-\psi$. The equilibrium state is symmetric too.

Let us split the interval $\theta\in[0,2\pi]$ into $2N$ subintervals and consider two realizations of the symmetry principle for double-frequency systems. The list of realizations of the symmetry principle in [5] is continued below.

4. Quasiperiodic motion exists in the conservative system (1) when for $\theta\approx\psi$ the trajectory is attracted to the solution $\rho^0(\psi)$, $\lambda(\psi)<0$ with a certain symmetry axis on N subintervals and repelled from zero since $d\rho/dt>0$ with the same symmetry axis on the remaining N subintervals. For $\theta\approx-\psi$, the situation is opposite.

Here $\lambda^0(\psi)=\partial\mathcal{R}(\rho,\psi)/\partial\rho|_{\rho=\rho^0(\psi)}$, $\psi\in(0,2\pi)$. The inphase oscillations $\theta\approx\psi$ in system (1) correspond to extension, i.e., the trajectory moves away from zero until antiphase oscillations $\theta\approx-\psi$ occur. The qualitative pattern reverses—antiphase

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S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, Kiev. Translated from Prikladnaya Mekhanika, Vol. 41, No. 3, pp. 108–118, March 2005. Original article submitted February 9, 2004.

oscillations correspond to contraction and the trajectory is attracted to zero. On the whole, the symmetric pattern represents a stable quasiperiodic motion.

Example 1. Let us consider the Lotka–Volterra model [5] with a disturbance frequency Ω close to the frequency of a nonlinear oscillator without disturbances. The systems is expressed in terms of variables (ρ, θ) as follows:

$$\frac{d\rho}{dt} = -2\rho^2 \sin\theta\cos\theta(\cos\theta - \sin\theta) + m\cos\theta\cos t/2,$$

$$\frac{d\theta}{dt} = 1 + 2\rho\sin\theta\cos\theta(\cos\theta + \sin\theta) - m\sin\theta\cos t/(2\rho). \quad (2)$$

For nearly synchronous motions $\theta \approx \psi$, $\psi = \Omega t$, the trajectory of system (2) is attracted to the solution

$$\rho^0(\psi) = \frac{1}{2}\sqrt{\frac{m\cos\psi}{\sin\psi(\cos\psi - \sin\psi)}}$$

on the subintervals $(0, \pi/4)$, $(\pi/2, 3\pi/4)$, $(3\pi/4, \pi)$, and $(5\pi/4, 3\pi/2)$ with a characteristic number $\lambda^0(\psi) = -2m\rho^0(\psi)\sin\psi\cos\psi \times$ $\times(\cos\psi - \sin\psi)$. On the subintervals $(\pi/4, \pi/2)$, $(\pi, 5\pi/4)$, and $(3\pi/2, 2\pi)$, the trajectory is repelled from zero ("+") since $d\rho/dt > 0$. The general symmetric pattern for the trajectory (symmetry axes $\psi = \pi/4$ and $\psi = 3\pi/4$) is as follows:

ψ	$0, \frac{\pi}{4}$	$\frac{\pi}{4}, \frac{\pi}{2}$	$\frac{\pi}{2}, \frac{3\pi}{4}$	$\frac{3\pi}{4}, \pi$	$\pi, \frac{5\pi}{4}$	$\frac{5\pi}{4}, \frac{6\pi}{4}$	$\frac{6\pi}{4}, \frac{7\pi}{4}$	$\frac{7\pi}{4}, 2\pi$
$d\rho/dt, \lambda^0$	$\lambda^0 < 0$	+	$\lambda^0 < 0$	$\lambda^0 < 0$	+	$\lambda^0 < 0$	+	+

For $\theta \approx -\psi$, the qualitative pattern is opposite and the model of conditionally periodic motion is realized. There are regular stable motions in system (2).

5. The quasiperiodic motion of the conservative system (1) exists because of the following symmetry conditions: for $\psi \approx \theta$, symmetric extension–contraction about $\rho_1^0(\psi)$ on the first half-period and symmetric extension–contraction about $\rho_2^0(\psi)$ on the second half-period. When $\psi \approx -\theta$, the symmetry of extension–contraction remains.

Symmetric extension–contraction in a double-frequency system causes stable quasiperiodic oscillations.

Example 2. Let us consider a bistable oscillator with periodic disturbance:

$$\frac{dx_1}{dt} = \frac{\mu}{m}x_2 - \frac{\mu\sigma}{m}x_2^3 + f(\Omega t), \qquad \frac{dx_2}{dt} = x_1, \quad (3)$$

where μ, σ, and m are positive parameters. The equations of motion (3) relative to a saddle take the following form [12]:

$$\frac{d\rho}{dt} = \rho\omega\sin\theta\cos\theta(3/2 - 2\sigma\rho^2\sin^2\theta) - \mathcal{B}\sin^2\Omega t\cos\theta/(2\omega),$$

$$\frac{d\theta}{dt} = \omega(1 - \sin^2\theta(3/2 - 2\sigma\rho^2\sin^2\theta)) + \mathcal{B}\sin^2\Omega t\sin\theta/(2\omega\rho). \quad (4)$$

Here periodic disturbance $f(\Omega t) = -\mathcal{B}\sin^2\Omega t$, $\omega = \sqrt{2\mu/m}$. Let $(\mu, m, \sigma) = (1; 2; 0.25)$. We will find the solutions of the algebraic equation for inphase oscillations from the first equation of system (4):

$$\rho^3 - \frac{3\rho}{\sin^2\psi} + \frac{\mathcal{B}}{\sin\psi} = 0. \quad (5)$$

In Eq. (5), the trajectory is attracted to and repelled from the quasistatic solution according to model 5, i.e., the trajectory is attracted on $\psi \in (0, \pi/2)$ and repelled on $\psi \in (\pi/2, \pi)$ relative to $\rho_1^0(\psi)$ with equal (in absolute value) $\lambda_1^0(\psi)$ and is attracted on

$\psi \in (\pi, 3\pi/2)$ and repelled on $\psi \in (3\pi/2, 2\pi)$ relative to $\rho_2^0(\psi)$ with equal (in absolute value) $\lambda_2^0(\psi)$. Qualitative patterns of neutral extension–contraction symmetry are considered for near-resonance motion. The solution $\rho^0(\psi)$ makes it possible to reveal how a regular trajectory behaves under relatively weak oscillations about zero. One of the conditions of weak oscillations is $d\theta / dt > 0$, $\psi \in (0, 2\pi)$ $(\rho \neq 0)$.

The phenomenon of synchronization occurs in dissipative single-frequency systems with periodic disturbance. Synchronization as a resonance phenomenon implies that two coupled systems oscillate at equal or multiple frequencies. This phenomenon is described by the following model.

6. A closed trajectory exists in a dissipative system (1) with periodic disturbance because of a smooth attractive solution $\rho^0(\psi)$, $\psi \in (0, 2\pi)$.

Model 6 interprets synchronization as a trajectory closing on the phase plane Ox_1x_2, as with model 1 for a limit cycle in [16]. In contrast to synchronization at a small parameter value, quasiperiodic motions are due to the nonsmooth attractive solution $\rho^0(\psi)$. At a large parameter value, aperiodic solutions appear on the phase trajectory, which may generate synchronous oscillations for nonsmooth $\rho^0(\psi)$.

7. Two-dimensional torus exists because of a nonsmooth attractive symmetric solution $\rho^0(\psi)$, $\psi \in (0, 2\pi)$.

Models 4–7 discussed here are associated with the Bogolyubov–Mitropol'skii asymptotic theory [3, 6]. Limit cycles are generated by an attractive solution $\rho^0(\theta)$ smooth in $\theta \in (0, 2\pi)$; double-frequency oscillations are generated by a nonsmooth solution $\rho^0(\psi)$, $\psi \in (0, 2\pi)$, where ψ corresponds to an inphase state of two angular deviations. We will call the parameter small (<< 1) when the phase trajectory of a limit cycle everywhere has an attractive *periodic* solution. Models 6 and 7 are realized only for small parameter values (<< 1). If the parameter value is less than unity, then aperiodic solutions appear on the trajectory, the synchronization limits extend, and the existence of a two-dimensional torus is not determined by model 7. Model 6 is sufficient in this case. For periodic trajectories with symmetry axes, the first approach of identifying closed trajectories based on the geometrical symmetry principle is applied [7]. We will now describe geometrically symmetry in multidimensional systems and generalize the symmetry principle to the multidimensional case.

2. Criterion of Trajectory Closing on a Torus. The oscillations of two coupled nonlinear oscillators are described by the vector equation

$$\frac{dx}{dt} = F(x), \tag{6}$$

where $x(t) \in R^4$ is the vector of state of the system at $t \in R$, $F: R^4 \to R^4$. Let the system

$$\frac{d\tilde{x}}{dt} = A\tilde{x} \tag{7}$$

be a linear approximation of system (6). The eigenvalues of the matrix A are simple and purely imaginary: $\lambda_j, \bar{\lambda}_j = \pm i\omega_j$ $(j = 1, 2)$. In [7], the principle of geometrical symmetry is applied to identify a center in a two-dimensional nonlinear system. Using this principle, we will specify sufficient conditions for the existence of a stable quasiperiodic motion in system (6).

System (6) has quasiperiodic motions if the functions F_k $(k = 2, 4)$ are even with respect to x_2 and x_4 and the functions F_j $(j = 1, 3)$ are odd with respect to x_2 and x_4, i.e.,

$$F_k(x_1, -x_2, x_3, -x_4) = F_k(x_1, x_2, x_3, x_4) \quad (k = 2, 4),$$
$$F_j(x_1, -x_2, x_3, -x_4) = -F_j(x_1, x_2, x_3, x_4) \quad (k = 1, 3). \tag{8}$$

Let us represent system (6) as connected equations

$$\frac{dx_j}{dt} = P_j(x_j, x_{j+1}) + S_j(x_1, x_2, x_3, x_4),$$

$$\frac{dx_{j+1}}{dt} = P_{j+1}(x_j, x_{j+1}) + S_{j+1}(x_1, x_2, x_3, x_4) \quad (j = 1, 3), \tag{9}$$

where $S_j(x_1, x_2, x_3, x_4)$ and $S_{j+1}(x_1, x_2, x_3, x_4)$ are functions of constraints, including nonlinear ones. If $S_j(x_1, x_2, x_3, x_4) = S_{j+1}(x_1, x_2, x_3, x_4) = 0$, then system (9) breaks up into two subsystems

$$\frac{dx_j}{dt} = P_j(x_j, x_{j+1}), \qquad \frac{dx_{j+1}}{dt} = P_{j+1}(x_j, x_{j+1}) \qquad (j = 1, 3), \tag{10}$$

each having a closed trajectory and symmetry axes. This follows from the symmetry principle.

Indeed, any integral curve emerging from a point quite close to the origin of coordinates $M_k(0, -\varepsilon_k), \varepsilon_k > 0$ $(k = 2, 4)$ of the lower half-plane $Ox_j x_{j+1}$ $(j = 1, 3)$ will again cross the Ox_k-axis at the point $N_k(0, \eta_k), \eta_k > 0$ $(k = 2, 4)$ of the upper half plane $Ox_j x_{j+1}$ $(j = 1, 3)$. By virtue of symmetry (8), an integral curve below the Ox_j-axes $(j = 1, 3)$ of each phase section is the mirror reflection of a curve above, i.e., symmetric about the Ox_j-axes $(j = 1, 3)$. The phase flow of linear nonconnected systems

$$\frac{d\hat{x}_1}{dt} = \Omega_1 \hat{x}_2, \qquad \frac{d\hat{x}_2}{dt} = -\Omega_1 \hat{x}_1, \qquad \frac{d\hat{x}_3}{dt} = \Omega_2 \hat{x}_4, \qquad \frac{d\hat{x}_4}{dt} = -\Omega_2 \hat{x}_3$$

corresponding to (10) is specified by rotations through angles $\Omega_1 t$ and $\Omega_2 t$, respectively, and the coordinates $\hat{x}_1$ and $\hat{x}_3$ determine the longitude and latitude of a two-dimensional torus. The phase trajectory on the torus is either closed or everywhere dense [2]. Let us construct an analog of the torus selecting the nonlinear system (10) so that closed trajectories form a manifold. Then the coordinates x_1 and x_3 are analogs of the latitude and longitude, respectively. The analog of torus is constructed in accordance with the nonconnected nonlinear subsystems of (10). Then any integral curve emerging from a point quite close to the origin of coordinates is either closed or everywhere dense. This is typical for relatively weak nonlinear oscillations about zero such that almost all solutions on the trajectory are periodic.

Thus, the evenness and oddness conditions (8) for the right-hand sides of system (6) are sufficient for the existence of a quasiperiodic trajectory in the form of a two-dimensional torus. Similar conditions can be formulated for other axes.

System (6) has quasiperiodic motions if the functions F_k $(k = 1, 3)$ are even with respect to x_1 and x_3 and the functions F_j $(j = 2, 4)$ are odd with respect to x_1 and x_3, i.e.,

$$F_k(-x_1, x_2, -x_3, x_4) = F_k(x_1, x_2, x_3, x_4) \quad (k = 1, 3),$$

$$F_j(-x_1, x_2, -x_3, x_4) = -F_j(x_1, x_2, x_3, x_4) \quad (j = 2, 4). \tag{11}$$

Example 3. Let us consider a chain of two simple pendulums. A point particle P_1 of mass m_1 moves in a vertical plane along a circle with radius l_1 and center O. A particle P_2 of mass m_2 moves in a vertical plane at a distance l_2 from P_1. The Lagrange coordinates are the angle φ_1 between the vertical and the interval OP_1 and the angle φ_2 between the vertical and the interval P_1P_2. Obviously, the motions of the system are Poisson stable. The Lagrange equations of the system are

$$(m_1 + m_2) l_1^2 \frac{d^2\varphi_1}{dt^2} + m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1) \frac{d^2\varphi_2}{dt^2} - m_2 l_1 l_2 \sin(\varphi_2 - \varphi_1) \left(\frac{d\varphi_2}{dt}\right)^2 = -(m_1 + m_2) g l_1 \sin\varphi_1,$$

$$m_2 l_1 l_2 \cos(\varphi_2 - \varphi_1) \frac{d^2\varphi_1}{dt^2} + m_2 l_2^2 \frac{d^2\varphi_2}{dt^2} + m_2 l_1 l_2 \sin(\varphi_2 - \varphi_1) \left(\frac{d\varphi_1}{dt}\right)^2 = -m_2 g l_2 \sin\varphi_2.$$

The equations of motion of a double simple pendulum reduced to a first-order system are

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = X_2(x_1, x_2, x_3, x_4),$$

$$\frac{dx_3}{dt} = x_4, \qquad \frac{dx_4}{dt} = X_4(x_1, x_2, x_3, x_4), \tag{12}$$

where

$$x_1 = \varphi_1, \qquad x_2 = \frac{d\varphi_1}{dt}, \qquad x_3 = \varphi_2, \qquad x_4 = \frac{d\varphi_2}{dt},$$

$$X_2 = -\frac{g(1+\kappa)\sin x_1}{l_1 \mathcal{B}} + \frac{g\sin x_3 \cos(x_3 - x_1)}{l_1 \mathcal{B}} + \frac{x_2^2 \sin 2(x_3 - x_1)}{2\mathcal{B}} + \frac{l x_4^2 \sin(x_3 - x_1)}{\mathcal{B}},$$

$$X_4 = -\frac{g(1+\kappa)\sin x_3}{l_2 \mathcal{B}} + \frac{g(1+\kappa)\sin x_1 \cos(x_3 - x_1)}{l_2 \mathcal{B}} - \frac{(1+\kappa) x_2^2 \sin(x_3 - x_1)}{l \mathcal{B}} - \frac{x_4^2 \sin 2(x_3 - x_1)}{2\mathcal{B}},$$

$$\kappa = \frac{m_1}{m_2}, \quad l = \frac{l_2}{l_1}, \quad \mathcal{B} = \kappa + \sin^2(x_3 - x_1).$$

Quasiperiodic motions exist in (12) since the equations satisfy conditions (8) and (11). A distinguishing feature of this motion is that closed curves forming a system of curvilinear coordinates have two symmetry axes, since they satisfy the symmetry conditions (8) and (11). The symmetry of the quasiperiodic motions is due to the symmetry of the curvilinear coordinates.

3. Numerical Analysis of the Characteristic Numbers of a Phase Trajectory and the Characteristic Numbers of a Periodic Solution. Chaos in single-frequency oscillating systems is known to be associated with the intersection of stable and unstable manifolds of such motions [1]. Thus, to identify the trajectory of complex or stochastic motions, it is necessary to establish the nature of motions originating in the neighborhood of solutions. Let us introduce a small deviation from the solution of the system:

$$\frac{dx}{dt} = Ax + X(x), \tag{13}$$

where $x(t) \in R^4$ is the vector of state of the system at $t \in R$, A is a constant 4×4 matrix, and X: $R^4 \to R^4$ is a vector polynomial of no less than the second degree. Let us introduce a small deviation in the neighborhood of the solutions $\bar{x}_i$ (i = 1, 2, 3, 4): $\delta x_i = x_i(t) - \bar{x}_i(t)$ (i = 1, 2, 3, 4). The linearized disturbed system (13) has the following form in the neighborhood of the solution $\bar{x}_i(t)$ (i = 1, 2, 3, 4):

$$\frac{d\delta x}{dt} = A(\bar{x})\delta x, \quad \delta x \in R^4. \tag{14}$$

The roots of the characteristic equation of system (14) are said to be the characteristic numbers of the phase trajectory $x_i(t)$ of the nonlinear system (13). The coefficients of system (14) linear in δx depend on fixed $\bar{x}_i$ at each point of the curve. Andronov was the first to prove that one of the characteristic numbers of a stable periodic solution is equal to zero.

Let us introduce a small deviation in the neighborhood of the solutions $\bar{\rho}_j(t)$ (j = 1, 2) of the system in variables ρ, θ: $\delta\rho_j = \rho_j(t) - \bar{\rho}_j(t)$ (j = 1, 2). The linearized system of disturbed equations has the following form in the neighborhood of the solutions $\bar{\rho}_j(t)$ (j = 1, 2):

$$\frac{d\delta\rho}{dt} = \varepsilon(\bar{\rho}, \theta)\delta\rho. \tag{15}$$

The coefficients of system (15) depend on fixed $\bar{\rho}_j$ at each point of the curve (solution). Let a non-singular transformation diagonalize the matrix $\varepsilon(\bar{\rho}, \theta)$. Then

$$\frac{d\delta z_i}{dt} = l_i(\bar{\rho}, \theta)\delta z_i \quad (i = 1, 2).$$

The characteristic number of a periodic solution has the form

$$\Lambda_j = \frac{1}{T}\int_0^T l_j(\bar{\rho})dt \quad (j = 1, 2).$$

The complex isolated closed trajectories of dissipative systems are mainly attractive on $t \in (0, T)$. Note that trajectories with periodic solutions are searched for when $d\theta_k / dt > 0$. The derivative $d\theta_k / dt$ may become negative if aperiodic solutions appear on the trajectory. There are saddle cycles among dissipative three-dimensional cycles. Trajectories tend to a saddle cycle, filling a set of dimension less than three [14].

4. Increase in the Period and Chaos in Dissipative Double-Frequency Systems and under Convergence. An increase in the parameter, making it large instead of small, makes it difficult to identify motion using $\rho(\psi)$. This is due primarily to the nonuniformity of the motion of the representative point, which results in aperiodic solutions on the phase trajectory. Nonuniformity in a dissipative system is manifested as drift-like slowdown of the representative point. The self-regulation of motion and synchronous oscillations are due to a multiple increase in the period. The system executes compensating loop (on the phase plane) motions with a negative derivative $d\theta/dt$. For small parameter values ($\mu << 1$ in the Van der Pol system), Eqs. (1) reflect the original sense of the variables: modulus ρ and argument θ of a complex quantity. Then an attractive smooth solution $\rho^0(\psi)$ induces closure of a synchronous phase trajectory and an attractive nonsmooth solution $\rho^0(\psi)$ induces quasiperiodic motion. When the phase trajectory has aperiodic solutions, Eqs. (1) are formal. Aperiodic solutions may cause the phase trajectory to synchronize in the case of an attractive nonsmooth solution $\rho^0(\psi)$. Model 7 is invalid in this case. Aperiodic solutions appear if the parameter is less than unity. The multiplicity of the period is associated with a parameter value greater than unity.

The synchronization of oscillations at multiple frequencies is due to aperiodic solutions on the trajectory, i.e., the closed trajectory consist of a finite union of submanifolds of the phase space. Recall that the existence of a smooth attractive solution $\rho^0(\psi)$ is just a sufficient condition of synchronism.

Chaotic motions are manifested as an analog of double-frequency oscillations when saddle solutions appear on the trajectory.

The well-known property of linear systems to produce periodic motions under periodic disturbances is also typical of some nonlinear systems and is called convergence. While the amplitude of forced oscillations in linear systems cannot be infinite because of the presence of resisting forces, chaotic loss of stability as infinite increase in coordinate deviations can be observed in nonlinear systems.

Let us address the classical results on nonlinear systems [4]. Consider a system

$$\frac{dx_s}{dt} = \sum_{i=1}^{2n} a_{si} x_i + X_s(x_1, \ldots, x_{2n}) + f_s(t) \qquad (s = 1, \ldots, 2n), \tag{16}$$

where a_{si} are real constants and the eigenvalues of the matrix of these quantities have negative real parts, $X_s(x_1, \ldots, x_{2n})$ are functions nonlinear in $x_1, \ldots, x_{2n}$, and $f_s(t)$ is a periodic disturbance.

Denote by Y the fundamental matrix of the solutions of the system of equations

$$\frac{dx_s}{dt} = \sum_{i=1}^{2n} a_{si} x_i \qquad (s = 1, \ldots, 2n), \qquad Y = (y_{si}(t)).$$

Definition 1. If system (16) has a unique periodic solution asymptotically stable in the large, then this system has the property of convergence.

The following statement has been proved in [4]:

Theorem 1. If the right-hand sides of system (16) satisfy the conditions:

(i) the real parts of the roots of the matrix $A = (a_{si})$ are negative,

$$\lambda < 0, \qquad \lambda = \max \operatorname{Re} \lambda_i \qquad (i = 1, \ldots, 2n),$$

(ii) the functions $X_s(x_1, \ldots, x_{2n})$ satisfy the inequalities

$$|X_s(x_1, \ldots, x_{2n}) - X_s(\hat{x}_1, \ldots, \hat{x}_{2n})| \le \sum_{i=1}^{2n} b_{si} |x_i - \hat{x}_i| \qquad (s = 1, \ldots, 2n),$$

where b_{si} are non-negative constants;

(iii) $\mu < |\hat{\lambda}|$, where μ is the maximum real part of the roots of the matrix $\mathcal{PB}$, $\mathcal{P} = (p_{si})$, $\mathcal{B} = (b_{si})$, and

$$p_{si} = \sup y_{si}(t-\tau)e^{-\hat{\lambda}(t-\tau)}, \qquad 0<\tau<t, \qquad t>0,$$

then system (16) possesses the property of convergence. Here $\hat{\lambda}$ denotes some negative constant such that $\hat{\lambda} > \lambda$.

Theorem 1 does not propose a way of searching for a periodic solution. Condition (iii) of Theorem 1, i.e., $\mu < |\hat{\lambda}|$, shows that the effect of the nonlinear function $X_s(x_1, \ldots, x_{2n})$ in system (16) is not so strong and is such that the evolution of the linear component dominates. If nonlinearity is so strong that condition (iii) fails, then aperiodic solutions appear on the trajectory and may cause the periodic trajectory to close not in a period but in a multiple number of periods. The closed trajectory is complex because it contains different kinds of solutions: periodic and aperiodic. In some cases, a change of the parameter leads to a cascade of doublings, resulting in chaos. In other cases, the trajectory falls into an infinite domain of saddle solutions. After that, the trajectory may go to infinity.

5. Determining the Boundary of the Domain of Aperiodic Solutions. The asymptotic solution technique is applicable when a closed trajectory is represented by a manifold. In the case of a union of submanifolds, it is necessary to perform a qualitative analysis. It is helpful to construct a boundary along which $d\theta_j / dt = 0$. The nonlinear oscillating system of the fourth order

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = X_2(x), \qquad \frac{dx_3}{dt} = x_4, \qquad \frac{dx_4}{dt} = X_4(x),$$

where $x(t) \in R^4$ is the vector of state of the system at $t \in R$ and $X_j(x)$ are the right-hand sides of differential equations that have a nonlinear component, can be written in terms of polar coordinates ρ and θ:

$$\frac{d\rho_j}{dt} = \mathcal{R}_j(\rho,\theta), \qquad \frac{d\theta_j}{dt} = \omega_j + \mathcal{T}_j(\rho,\theta) \qquad (j = 1, 2). \tag{17}$$

Constructing a plane curve along which $d\theta_j / dt = 0$, i.e., finding the solutions of the equation $\mathcal{F}_j(x_j, x_{j+1}) = 0$ $(j = 1, 2)$, where $\mathcal{F}_j(x_j, x_{j+1})$ is the right-hand side of Eqs. (17) in θ on the plane $Ox_j x_{j+1}$, is an important task in determining approximate solutions of nonlinear systems with several singular points and in determining solutions in the case where a singular point bifurcates and the trajectory consists of sets of joined solutions. The domain $Ox_j x_{j+1}$ with $d\theta_j / dt = 0$ on the boundary has a physical meaning. The curve $d\theta_j / dt(\rho,\theta) = 0$ may be a separatrix along which the trajectory goes to infinity.

Suppose that a domain of saddle solutions appears on the phase plane within a certain range of parameters. To find the projection of the separatrix of the saddle domain onto one of the coordinate planes, it is necessary to solve the scalar algebraic equation

$$\mathcal{F}_j(x_j, x_{j+1}) = 0. \tag{18}$$

Using an arc coordinate s, the boundary of the domain of aperiodic solutions can be represented in parametric form:

$$x_j^* = x_j^*(s), \qquad x_{j+1}^* = x_{j+1}^*(s). \tag{19}$$

The geometrical condition relates solutions (19):

$$\left(\frac{dx_j^*}{ds}\right)^2 + \left(\frac{dx_{j+1}^*}{ds}\right)^2 = 1. \tag{20}$$

From Eq. (18) we have

$$\frac{\partial \mathcal{F}}{\partial x_j^*}\frac{dx_j^*}{ds} + \frac{\partial \mathcal{F}}{\partial x_{j+1}^*}\frac{dx_{j+1}^*}{ds} = 0. \tag{21}$$

Equations (19) and (20) yield a system of differential equations:

$$\frac{dx_j^*}{ds} = \pm\sqrt{\frac{\left(\frac{\partial\mathcal{F}}{\partial x_{j+1}^*}\right)^2}{\left(\frac{\partial\mathcal{F}}{\partial x_j^*}\right)^2 + \left(\frac{\partial\mathcal{F}}{\partial x_{j+1}^*}\right)^2}}, \qquad \frac{dx_{j+1}^*}{ds} = \pm\sqrt{1 - \frac{\left(\frac{\partial\mathcal{F}}{\partial x_{j+1}^*}\right)^2}{\left(\frac{\partial\mathcal{F}}{\partial x_j^*}\right)^2 + \left(\frac{\partial\mathcal{F}}{\partial x_{j+1}^*}\right)^2}} \qquad (j = 1, 2). \tag{22}$$

The signs of the right-hand sides of system (22) can be identified by moving along the domain boundary in one or opposite direction. To integrate system (22), it is necessary to specify an initial condition. There is a point on the plane at which $d\theta_j / dt = 0$. If the boundary curve is symmetric, then it is desirable that the point lay on the symmetry axis.

6. Discussion of the Results. Section 1 is devoted to double-frequency oscillations. The conditions of symmetry are formulated for $\rho^0(\psi)$, where ψ corresponds to an inphase or resonant state of two angular deviations. Based on the symmetry of the solutions $\rho^0(\psi)$, it is possible to identify the quasiperiodic motion of conservative and dissipative systems and the synchronous motion of dissipative systems. It should be noted that the symmetry principle is associated with the asymptotic theory and makes it possible to identify trajectories in a wider class of problems. Model 6 is valid for any parameter values and model 7 for small parameter values. A small parameter (<< 1) has been defined to guarantee that only periodic solutions exist on the phase trajectory.

Section 2 establishes sufficient geometrical conditions for the existence of a quasiperiodic curve.

Section 3 describes a numerical method for determining the characteristic numbers of a phase trajectory and periodic solution. The coefficients of disturbed systems depend on fixed numerical solutions at each point of the integral curve. Note that manifolds of periodic solutions are searched for when $d\theta_k / dt > 0$.

Section 4 considers the multiplicity of the period in a double-frequency dissipative system as synchronous oscillations in a certain range of parameters. The multiple increase of the period is due to the self-regulation of the system in the case of nonuniform motion of the representative point. Drift-like nonuniformity is due to the aperiodic solutions and the proximity of the separating curve $\rho^*(\psi)$ to the solution $\rho(t)$. The role of aperiodic solutions was emphasized in [1]. Stable chaotic motions are realized as double-frequency oscillations whose trajectories have saddle solutions. In the parameter space, chaotic motions of disturbed limit cycles alternate with period doubling.

The section also examines the increase in the period of forced oscillations and transition of convergent oscillations into chaos. Zubov's theorem [4] establishes existence conditions for convergent oscillations. Convergent oscillations with the period of disturbing force are associated with the presence of only periodic solutions on the phase trajectory. Degeneration begins when aperiodic solutions appear on the phase curve. In contrast to periodically disturbed limit cycles, period doubling forms cascades of doublings that end in chaos in the case of stable oscillations.

Section 5 addresses a general method for determining the boundary (on the phase plane) along which $d\theta_j / dt = 0$. When the right-hand sides of equations are polynomials, polar coordinates can be used. In the general case, the boundary of the domain of aperiodic solutions is found by solving a system of differential equations, as in the parameter continuation method ingeniously outlined in [18].

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