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RESONANT VIBRATIONS AND DISSIPATIVE HEATING OF AN INFINITE PIEZOCERAMIC CYLINDER

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The monoharmonic radial vibrations and dissipative heating of an infinite hollow piezoceramic cylinder are studied in dynamic formulation, taking into account the temperature dependence of the complex electromechanical characteristics over a wide range of temperatures, including depolarization temperatures. The influence of the heat exchange conditions, the level of electric load, and geometry on the thermoelectromechanical characteristics is studied in the case of forced vibrations at the first resonance

Keywords: piezoceramic cylinder, monoharmonic radial vibrations, dissipative heating, heat exchange, thermoelectromechanical characteristics

Piezoelectric elements are widely used in modern technology as energy converters, transducers, exciters, etc., operating in the mode of harmonic oscillations. Dissipation causes heating of piezoelectric elements. Under real loading and heat-exchange conditions, the heating temperature may reach critical levels (Curie point) when the material is depolarized. The influence of inhomogeneities due to structural features or depolarization phenomena on the electromechanical behavior of piezoelectric elements under harmonic vibrations was studied in [4–6, 8–10, etc.]. Publications on the subject are reviewed in [3, 11].

The extended area of application and complicated operation conditions of piezoelectric elements require further in-depth research into the laws governing their electromechanical behavior, especially in the range of resonant frequencies. The present paper addresses the dynamic problem on monoharmonic vibrations and dissipative heating of a radially polarized infinite piezoceramic cylinder with temperature-dependent electromechanical properties over a wide range of temperatures, including depolarization temperatures.

Problem Formulation. Consider a hollow cylinder infinite along the z-axis. Its inner and outer radii are denoted by r_1 and r_2. The cylinder is made of a viscoelastic radially polarized piezoceramics. The radial surfaces are covered with solid electrodes. A harmonically varying potential difference $\tilde{\psi}(r_2, t) - \tilde{\psi}(r_1, t) = \mathrm{Re}\left(2V_0 e^{i\omega t}\right)$ ($V_0 = \text{const}$, ω is circular frequency, and t is time) applied to the electroded surfaces induces radial axisymmetric electromechanical vibrations and plane strain state ($\varepsilon_z = 0$), with the electric induction components D_z and D_θ perpendicular to the polarization direction being negligible [2]. The cylindrical surfaces ($r = r_1, r = r_2$) are free from stresses and participate in convective heat exchange with an ambient medium of temperature T_c. The dissipative properties of piezoceramics are described by the concept of complex electromechanical characteristics known over a wide range of temperatures, including depolarization temperatures.

The problem on monoharmonic radial vibrations and dissipative heating of the cylinder reduces to the following equations (the factor $e^{i\omega t}$ is omitted):

the equations of mechanical vibrations and electrostatics

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$$\frac{d\sigma_r}{dr}+\frac{1}{r}\left(\sigma_r-\sigma_\theta\right)+\rho_\bullet\omega^2 u=0,\qquad \frac{dD_r}{dr}+\frac{1}{r}D_r=0, \tag{1}$$

simplified (in view of the plane strain condition) electroviscoelastic relations

$$\varepsilon_r=\bar{s}_{33}\sigma_r+\bar{s}_{13}\sigma_\theta+\bar{d}_{31}E_r,\qquad \varepsilon_\theta=\bar{s}_{13}\sigma_r+\bar{s}_{11}\sigma_\theta+\bar{d}_{31}E_r, \tag{2}$$

$$D_r=\bar{d}_{33}\sigma_r+\bar{d}_{31}\sigma_\theta+\varepsilon_{33}^{p}E_r, \tag{3}$$

$$\varepsilon_r=\frac{du}{dr},\qquad \varepsilon_\theta=\frac{u}{r},\qquad E_r=-\frac{d\psi}{dr}, \tag{4}$$

the period-averaged heat-conduction equation

$$\frac{1}{a}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\overline{W} \tag{5}$$

with the following dissipative function:

$$\overline{W}=\frac{\omega}{2\lambda}\Big[\bar{s}''_{33}\left(\sigma_r'^2+\sigma_r''^2\right)+2\bar{s}''_{12}\left(\sigma'_r\sigma'_\theta+\sigma''_r\sigma''_\theta\right)+\bar{s}''_{11}\left(\sigma_\theta'^2+\sigma_\theta''^2\right)$$

$$+2\bar{d}''_{33}\left(\sigma'_rE'_r+\sigma''_rE''_r\right)+2\bar{d}''_{31}\left(\sigma'_\theta E'_r+\sigma''_\theta E''_r\right)+\varepsilon_{33}''^{p}\left(E_r'^2+E_r''^2\right)\Big], \tag{6}$$

the electromechanical and thermal boundary conditions

$$\sigma_r=0,\quad \psi=-V_0\quad \left(r=r_1\right),\quad \sigma_r=0,\quad \psi=V_0\quad \left(r=r_2\right), \tag{7}$$

$$\frac{\partial T}{\partial r}=\pm\frac{\alpha_{1,2}}{\lambda}\left(T-T_c\right)\quad \left(r=r_1,\ r=r_2\right),\quad T=T_0\quad (t=0), \tag{8}$$

where

$$\bar{s}_{33}=s_{33}^E\left(1-\nu_{13}^E\nu_{31}^E\right),\quad \bar{s}_{13}=s_{13}^E\left(1+\nu_{12}^E\right),\quad \bar{s}_{11}=s_{11}^E\left(1-\nu_{12}^{E\,2}\right),$$

$$\bar{d}_{33}=d_{33}+\nu_{13}^E d_{31},\quad \bar{d}_{31}=d_{31}\left(1+\nu_{12}^E\right),\quad \varepsilon_{33}^p=\varepsilon_{33}^T\left(1-k_p^2\right),$$

$$\nu_{12}^E=-\frac{s_{12}^E}{s_{11}^E},\quad \nu_{13}^E=-\frac{s_{13}^E}{s_{11}^E},\quad \nu_{31}^E=-\frac{s_{13}^E}{s_{33}^E},\quad k_p^2=\frac{d_{31}^2}{s_{11}^E\varepsilon_{33}^T},$$

$$s_{mn}^E=s'_{mn}-is''_{mn}=s'_{mn}\left(1-i\delta_{mn}^s\right),\quad d_{mn}=d'_{mn}-id''_{mn}=d'_{mn}\left(1-i\delta_{mn}^d\right),$$

$$\varepsilon_{33}^T=\varepsilon'_{33}-i\varepsilon''_{33}=\varepsilon'_{33}\left(1-i\delta_{33}^\varepsilon\right)\quad \left(m,n=1,\ldots,3\right) \tag{9}$$

are the temperature-dependent complex compliances, piezoelectric moduli, and electric permittivity, respectively; $u=u'+iu''$, $\sigma_r=\sigma'_r+i\sigma''_r$, and $\sigma_\theta=\sigma'_\theta+i\sigma''_\theta$ are the complex amplitudes of displacement and stresses; $D_r=D'_r+iD''_r$, $E_r=E'_r+iE''_r$, and $\psi=\psi'+i\psi''$ are the induction, strength, and potential of the electric field, respectively; $\rho_\bullet$ is the specific density of piezoceramics; λ and a are the thermal diffusivity and thermal conductivity; and α_1 and α_2 are the heat exchange factors. The prime and double prime mark the real and imaginary parts of complex quantities, respectively.

Solution. To solve the electrothermoviscoelastic problem (1)–(8), we write the electromechanics relations (1)–(4) as a system of ordinary differential equations

$$\frac{d\sigma_r}{dr}=-\frac{1-\bar{\nu}_p}{r}\sigma_r+\left(\frac{\bar{E}_s}{r^2}-\rho_\bullet\omega^2\right)u-\frac{\bar{d}_r}{r}D_r,$$

$$\frac{du}{dr}=\left(\bar{s}_{33}+\bar{\nu}_p\bar{s}_{13}-\bar{\nu}_\varepsilon\bar{d}_{33}\right)\sigma_r+\frac{1}{r}\left(\bar{s}_{13}\bar{E}_s-\bar{d}_{33}\bar{d}_r\right)u+\left(\bar{d}_{33}\bar{d}_\varepsilon-\bar{s}_{13}\bar{d}_r\right)D_r,$$

$$\frac{d\psi}{dr}=\bar{\nu}_\varepsilon\sigma_r+\frac{\bar{d}_r}{r}u-\bar{d}_\varepsilon D_r,\qquad \frac{dD_r}{dr}=-\frac{1}{r}D_r, \tag{10}$$

where $\bar{E}_s=\frac{1}{\bar{s}_{11}}\left(1+\bar{k}_p^2\right)$, $\bar{k}_p^2=\frac{\bar{d}_{31}^2}{\bar{s}_{11}\bar{\varepsilon}_{33}^p}$, $\bar{\varepsilon}_{33}^p=\varepsilon_{33}^p\left(1-\bar{k}_p^2\right)$, $\bar{\nu}_\varepsilon=\left(\bar{d}_{33}+\bar{\nu}_{13}^E\bar{d}_{31}\right)/\bar{\varepsilon}_{33}^p$, $\bar{\nu}_{13}^E=-\bar{s}_{13}/\bar{s}_{11}$, $\bar{d}_r=\bar{d}_{31}/\left(\bar{s}_{11}\bar{\varepsilon}_{33}^p\right)$, $\bar{\nu}_p=\bar{\nu}_{13}^E+\bar{\nu}_\varepsilon\bar{d}_{31}/\bar{s}_{11}$, and $\bar{d}_\varepsilon=1/\bar{\varepsilon}_{33}^p$.

Since the electromechanical characteristics (9) are temperature-dependent, system (10) with (7) should be solved simultaneously with the heat-conduction equation (5), (6) with (8), as a coupled nonlinear problem. A way to do this is to use the time-stepping method [2]. To this end, we will first split the electromechanics equations (10), (7) into real and imaginary parts to be solved by the orthogonalization method [7] at each time step, then calculate the dissipative function (6), and finally integrate the energy equation (5), (8) using an explicit finite-difference scheme. The temperature field found is used to calculate the electromechanical characteristics at the next time step Δt. At the first step, the electromechanics problem is solved for constant electromechanical characteristics. The time-stepping process runs until the heating temperature become steady-state. When the frequency ω or amplitude V_0 of excitation changes during the solution, the parameter continuation method is used to reduce the time the temperature reaches the steady-state value and to improve the convergence of the solution. This method suggests using the previous solution when the load parameter changes again, which represents the case where the load acts on the body with temperature observed at the previous step of loading.

Computed Results. The computations were based on an experimental temperature dependence of the electromechanical characteristics of TsTStBS-2 piezoceramics [1], which are approximated by the following dependences over a temperature range from 20 to 160 °C:

$$s'_{mn}=s_{mn}^0\left(A_{mn}+B_{mn}\tilde{T}+C_{mn}\tilde{T}^2\right),\ [\mathrm{m^2/N}],\qquad d'_{mn}=d_{mn}^0\left(P_{mn}+R_{mn}\tilde{T}+L_{mn}\tilde{T}^2\right),\ [\mathrm{K/m}],$$

$$\varepsilon'_{mn}=\varepsilon_\bullet\left(e_1+e_2\tilde{T}+e_3\tilde{T}^2\right),\ [\mathrm{F/m}],\qquad \delta_{mn}^s=\delta_{mn}^0\left(a_{mn}+b_{mn}\tilde{T}+s_{mn}^\bullet\tilde{T}^2+s_{mn}^{\bullet\bullet}\tilde{T}^3\right),$$

$$\delta_{mn}^d=\delta_{mn}^\bullet\left(p_{mn}+r_{mn}\tilde{T}+l_{mn}\tilde{T}^2\right),\qquad \delta_{33}^\varepsilon=\delta_\bullet\left(\delta_1+\delta_2\tilde{T}+\delta_3\tilde{T}^2\right),\qquad \tilde{T}=T-T_R\quad (m,n=1,2,3). \tag{11}$$

The coefficients in (11) corresponding to the reference temperature of the body ($T=T_R$) have the following values:

$$s_{11}^0=12.5\cdot10^{-12},\quad s_{13}^0=-5.42\cdot10^{-12},\quad s_{33}^0=14.8\cdot10^{-12},\quad s_{12}^0=-4.62\cdot10^{-12},$$

$$d_{33}^0=330\cdot10^{-12},\quad d_{31}^0=-160\cdot10^{-12},\quad \varepsilon_\bullet=21\cdot10^2\varepsilon_0,\quad \varepsilon_0=8.854\cdot10^{-12},$$

$$\delta_{11}^0=0.16\cdot10^{-2},\quad \delta_{13}^0=-0.15\cdot10^{-2},\quad \delta_{33}^0=0.125\cdot10^{-2},\quad \delta_{31}^\bullet=0.4\cdot10^{-2},$$

$$\delta_{33}^\bullet=0.3\cdot10^{-2},\quad \delta_\bullet=0.35\cdot10^{-2},\quad \delta_{12}^0=\delta_{11}^0,\quad T_R=20\,^\circ\mathrm{C}.$$

The remaining coefficients in (11) have the following values for the temperature range $(20\le T\le160)$ °C:

$$A_{mn}=a_{mn}=P_{mn}=p_{mn}=e_1=\delta_1=1,\quad B_{11}=0.3077\cdot10^{-3},\quad B_{33}=0.8085\cdot10^{-3},$$

$$B_{13}=-0.516167\cdot10^{-3},\quad C_{11}=0,\quad C_{33}=-0.784266\cdot10^{-5},\quad C_{13}=-0.779739\cdot10^{-5},$$

$$b_{11}=0.6155\cdot10^{-3},\quad b_{33}=0.290762\cdot10^{-2},\quad b_{13}=1.38333\cdot10^{-2},\quad s_{11}^\bullet=0.41575\cdot10^{-4},$$

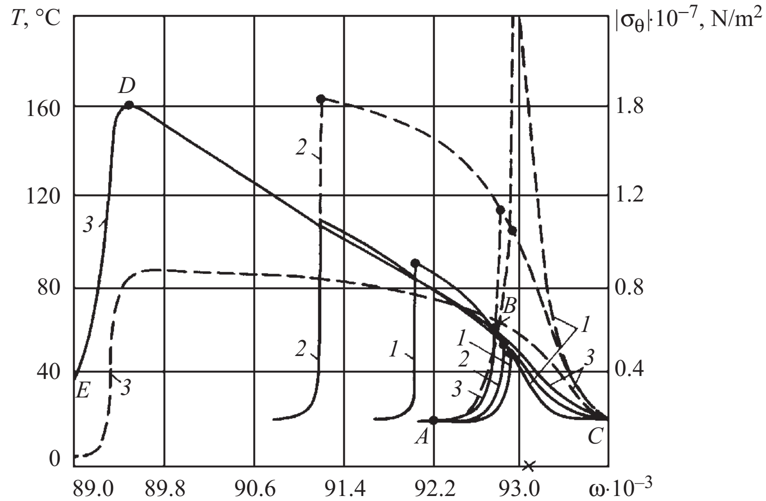

Fig. 1

$$s^{\bullet}_{33} = -0.441429\cdot 10^{-3}, \quad s^{\bullet}_{13} = -0.75\cdot 10^{-4}, \quad s^{\bullet\bullet}_{33} = 0.22381\cdot 10^{-5}, \quad s^{\bullet\bullet}_{11} = s^{\bullet\bullet}_{13} = 0,$$

$$R_{31} = 0.219\cdot 10^{-2}, \quad R_{33} = 0.35714\cdot 10^{-2}, \quad L_{31} = L_{33} = 0, \quad r_{31} = 1.198\cdot 10^{-2},$$

$$r_{33} = 0.670635\cdot 10^{-2}, \quad l_{31} = 1.823\cdot 10^{-4}, \quad l_{33} = 0.128968\cdot 10^{-3}, \quad e_2 = 0.111\cdot 10^{-3},$$

$$e_3 = 0.84256\cdot 10^{-4}, \quad \delta_2 = 0.119\cdot 10^{-1}, \quad \delta_3 = 0.119\cdot 10^{-3}.$$

In the range $(160 \le T \le 180)$ °C where the piezoelectric moduli drop abruptly with increase in temperature, the following coefficients should be used:

$$A_{11} = -21.92, \quad A_{33} = -10.52, \quad A_{13} = -13.69, \quad B_{11} = 0.29504, \quad B_{33} = 0.1625,$$

$$B_{13} = 0.206642, \quad C_{11} = -1024\cdot 10^{-2}, \quad C_{33} = -0.574324\cdot 10^{-3}, \quad C_{13} = -0.738\cdot 10^{-3},$$

$$a_{11} = -111.625, \quad a_{13} = 7.3, \quad a_{33} = 327.84, \quad b_{11} = 1.57619, \quad b_{33} = 4.488,$$

$$b_{13} = -6.5\cdot 10^{-2}, \quad s^{\bullet}_{11} = -5.46875\cdot 10^{-3}, \quad s^{\bullet}_{33} = 0.0152, \quad s^{\bullet}_{13} = 0.1667\cdot 10^{-3},$$

$$s^{\bullet\bullet}_{11} = s^{\bullet\bullet}_{33} = s^{\bullet\bullet}_{13} = 0, \quad P_{31} = 83.2205, \quad P_{33} = 90.91, \quad R_{31} = -1.167894,$$

$$R_{33} = -1.2644, \quad L_{31} = 0.4042\cdot 10^{-2}, \quad L_{33} = 0.4015\cdot 10^{-3}, \quad p_{31} = -21.75, \quad p_{33} = 26.8667,$$

$$r_{31} = 0.2, \quad r_{33} = 0.44, \quad l_{31} = 0, \quad l_{33} = 2\cdot 10^{3}, \quad e_1 = 9.66667, \quad e_2 = -0.5\cdot 10^{-2},$$

$$e_3 = 0.7143\cdot 10^{-3}, \quad \delta_1 = -130, \quad \delta_2 = -0.5\cdot 10^{-2}.$$

The characteristics of the piezoceramics are $\rho_{\bullet} = 7520$ kg/m^2 and $\lambda = 1.25$ W/(cm·°C), and the geometrical parameters of the cylinder are $r_2 = 0.05$ m and $H = r_2 - r_1 = 0.02$ m.

The numerical results cited below have been obtained on the assumptions that the surfaces are free from mechanical stresses (7), the inner boundary ($r = r_1$) is heat insulated ($\alpha_1 = 0$), and $T_0 = T_c = 20$ °C.

Curves *1*, *2*, and *3* in Fig. 1 have been plotted for $\gamma_2 = \alpha_2 H / \lambda = 10, 0.5, 0.1$ on the surface $r = r_2$ and $V_0 = 10$ V in view of thermomechanical coupling. These curves are the frequency dependence of the peak steady-state temperature $\tau = at / H = 1.5$ (solid lines) and the amplitude of hoop stresses $|\sigma_\theta| = (\sigma'^2 + \sigma''^2)^{1/2}$ (dashed lines) on the inner surface of the cylinder. The “×”

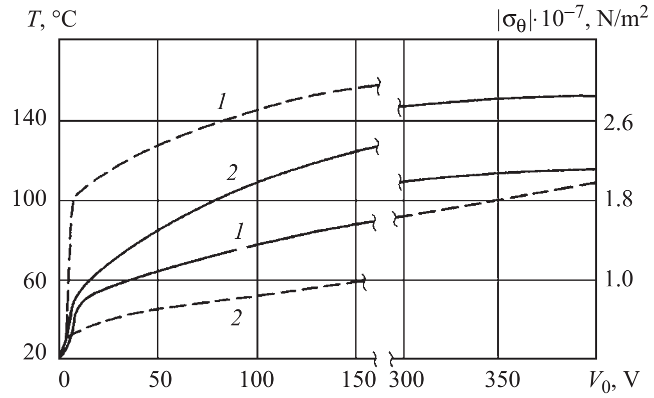

Fig. 2

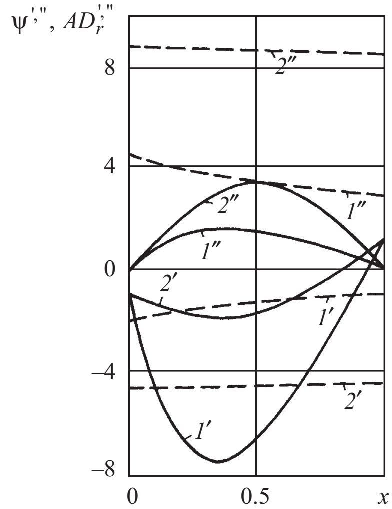

Fig. 3

symbol on the abscissa axis indicates the frequency of mechanical resonance in radial mode determined for the isothermal characteristics ($T = T_R$) of the piezoceramics. In numerical experiments, the amplitude– and temperature–frequency characteristics on the section *ABC* (curves *3*) have been obtained for both increasing frequency, $\omega \geq \omega_A$, and decreasing frequency, $\omega \leq \omega_C$. The high-temperature state corresponding to the section *BDE* was achieved with the help of the parameter continuation method only for decreasing frequency starting at the point *A*.

Curves *1*–*3* and computations reveal that intensive heat exchange on the cylinder surface (curves *1*) is accompanied by a moderate increase in the heating temperature (solid lines) at a considerable radial gradient and by significant increase in the amplitudes of mechanical stresses (dashed lines), which reach the limit of proportionality of piezoceramics $\sigma_p = 2 \cdot 10^7$ N/m^2. The better the heat insulation of the cylinder surface (curves *2* and *3*), the higher the heating temperature and the less the stress amplitudes. This process results in stronger nonlinearity of the temperature– and amplitude–frequency characteristics and wider frequency ranges for the low- and high-temperature modes. For small coefficients $\gamma_2 \leq 0.1$ (curves *3*), the temperature of the high-temperature mode reaches critical values $T \geq 160$ °C (solid lines) with decreasing frequency. In this case, radial smoothing of heating temperature and frequency stabilization of stress amplitudes (dashed lines) are observed.

Figure 2 shows the peak steady-state temperature $T_{r=r_1}$ (solid lines) and the amplitude of hoop stresses $|\sigma_\theta|_{r=r_1}$ (dashed lines) versus the electric potential V_0 for near-resonance frequency $\omega = 0.9294 \cdot 10^5$ sec^{-1} of loading and heat-exchange factors $\gamma_2 = 5.0, 0.05$ (curves *1* and *2*). The curves have been plotted with the help of the continuation method with V_0 as a continuation parameter. From Fig. 2 it is seen that for $V_0 \leq 10$ V heating temperature and stress amplitudes increase first impetuously and then monotonically. With intensive heat exchange on the outer surface (curves *1*), the increase in the stress amplitudes, which reach the limit of proportionality already for small values of V_0 (dashed line), predominates over the increase in the heating temperature (solid line). Curves *1* and *2* and numerical experiments show that possible loss of functionality of a piezoelectric element with increasing electric load is due mainly to mechanical failure (stresses reach the limit of proportionality) in the case of intensive heat exchange and due to thermal depolarization in the case of weak convective heat exchange.

Figure 3 shows the complex components of the electric potential $\overline{\psi}^{',''} = \psi^{',''} / V_0, V_0 = 1$ V (solid lines) and induction $AD_r^{',''}$ [F·V/m] (dashed lines) versus the dimensionless coordinate $x = (r - r_1)/H$ for a thick-walled cylinder ($H / r_2 = 2/5$, r_2 = 0.05 m) at $\omega = 0.9294 \cdot 10^5$ sec^{-1} and a thin-walled cylinder ($H / r_2 = 1/25$) at $\omega = 0.7280 \cdot 10^5$ sec^{-1}, both made of piezoceramics with temperature-independent properties. The scale coefficients for these cylinders are $A = 10^4$ and $A = 10^3$, respectively. Curves *1′*, *1″*, and *2′*, *2″* in Fig. 3 and numerical experiments reveal that as the thickness of the cylinder decreases, the distribution of the electric potential tends to be linear and the induction tends to be constant. The latter fact agrees with the assumption adopted in the theory of thickness-polarized piezoceramic shells that the component of the electric induction coinciding with the polarization direction is independent of the thickness coordinate.

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