DAMPING OF FREE ELASTIC VIBRATIONS IN LINEAR SYSTEMS

K. V. Avramov and Yu. V. Mikhlin UDC 531.39

The possibility of using a Mises truss as an absorber of free elastic vibrations in a linear elastic system is examined. The nonlinear normal mode method is used to analyze nonlinear vibrations. A local nonlinear normal mode is shown to be favorable for vibration damping

Keywords: linear elastic system, free elastic vibrations, Mises truss, nonlinear normal mode method, vibration damping

1. Introduction and Problem Formulation. Vibration absorbers were described and analyzed in many publications. We will cite only those that study the dynamics of systems with nonlinear passive absorbers. The paper [8] addresses a one-degree-of-freedom system with a vibration-absorbing rod. A torsional vibration absorber was examined in [12] using Melnikov functions. In [11] an oscillator with a nonlinear spring was considered an absorber of forced vibrations of a Duffing oscillator, and the differential equations describing this system were solved by the small-parameter method. Various vibration-damping devices and their application to systems with a finite number of degrees of freedom were addressed in [2, 5]. A detailed review on vibration damping theory is given in [1].

We propose to attach a Mises truss to a linear system, which is subject to free linear longitudinal vibrations. The truss, which moves between two positions of static balance, would absorb a portion of the energy of linear vibrations. Let the linear system be a one-degree-of-freedom oscillator. Free vibrations in a two-degree-of-freedom system are analyzed by the nonlinear normal mode method [3, 13]. This method is especially efficient when applied to essentially nonlinear systems vibrating with large amplitudes. An optimum condition for vibration damping is large-amplitude vibrations of the Mises truss and small-amplitude vibrations of the linear system. Such a motion is a local normal mode [13]. We assume that the Mises truss is flat, and its mass and stiffness are much less than those of the linear subsystem, which is a design requirement to the vibration absorber.

The equations of motion of the system shown in Fig. 1 are

$$M\ddot{U}+\kappa_1 U+\kappa\left[U-L\cos\varphi+L\left\{1+\frac{W^2}{(L\cos\varphi-U)^2}\right\}^{-1/2}\right]=0,$$

$$m\ddot{W}+\kappa W\left[2-L\left\{(L\cos\varphi-U)^2+W^2\right\}^{-1/2}-L\left\{L^2\cos^2\varphi+W^2\right\}^{-1/2}\right]=0, \tag{1}$$

where (U, W) are the generalized coordinates; L is the length of the Mises truss springs; κ is the stiffness of these springs; and κ_1 is the stiffness of the linear subsystem spring. Note that the forced small-amplitude vibrations of system (1) about the static balance position were examined in [6, 7].

Let us introduce dimensionless variables and parameters:

$$u=\frac{U}{L},\qquad w=\frac{W}{L},\qquad t=\sqrt{\frac{M}{\kappa_1}}\tau,\qquad \gamma=\frac{\kappa}{\kappa_1},\qquad \mu=\frac{m}{M}$$

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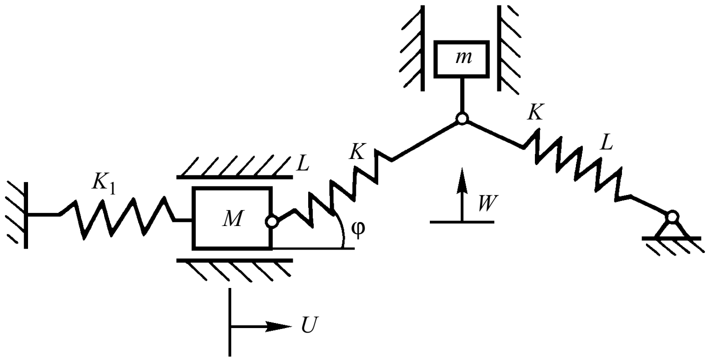

Fig. 1

and change variables: $u_1 = u + \frac{\gamma(1-\kappa)}{1+\gamma}$. Vibration damping occurs in system (1) when $u_1 << w$. Design considerations suggest that the dimensionless parameters of the vibration absorber should be small: $\mu = \varepsilon\bar{\mu}$, $\gamma = \varepsilon\bar{\gamma}$, $\varepsilon << 1$. Retaining the linear, quadratic, and cubic terms in series in powers of the variables, we represent the equations of motion (1) as

$$\ddot{u}_1 + (1+\varepsilon\bar{\gamma})u_1 - \frac{\varepsilon\bar{\gamma}}{\rho^3} u_1 w^2 - \frac{\varepsilon\bar{\gamma}}{2\rho^2} w^2 = 0, \tag{2}$$

$$\bar{\mu}\ddot{w} - \bar{\gamma}\alpha^2 w - \frac{\bar{\gamma}}{\rho^2} w u_1 + \frac{\bar{\gamma}\beta^2}{2} w^3 = 0, \tag{3}$$

where

$$\rho = \frac{\gamma+\kappa}{1+\gamma}, \quad \alpha^2 = \frac{1}{\rho} + \frac{1}{\kappa} - 2, \quad \beta^2 = \frac{1}{\rho^3} + \frac{1}{\kappa^3}. \tag{4}$$

2. Analysis of the Local Nonlinear Normal Mode. To apply the nonlinear normal mode method, we rearrange the equation of motion (2), (3) as

$$\ddot{u}_1 + \frac{\partial \Pi}{\partial u_1} = 0, \quad \mu\ddot{w} + \frac{\partial \Pi}{\partial w} = 0,$$

$$\Pi = (1+\varepsilon\bar{\gamma})\frac{u_1^2}{2} - \frac{\varepsilon\bar{\gamma}u_1^2}{2\rho^3} - \frac{\varepsilon\bar{\gamma}w^2 u_1}{2\rho^2} - \frac{\varepsilon\bar{\gamma}\alpha^2 w^2}{2} + \frac{\varepsilon\bar{\gamma}\beta^2 w^4}{8}, \tag{5}$$

where Π is the potential energy of the system.

Following the nonlinear normal mode method [1], we represent the trajectories of the system in the configuration space as $u_1 = u_2(w_1)$. Let us eliminate the time t from the equations of motion with the help of

$$\frac{d(\circ)}{dt} = \dot{w}_1 \frac{d(\circ)}{dw_1}, \quad \frac{d^2(\circ)}{dt^2} = \dot{w}_1^2 \frac{d^2(\circ)}{dw_1^2} + \ddot{w}_1 \frac{d(\circ)}{dw_1}. \tag{6}$$

Using relations (6), system (5), and the energy integral

$$\frac{\dot{u}_2^2}{2} + \varepsilon\bar{\mu}\frac{\dot{w}_1^2}{2} + \tilde{\Pi} = h, \tag{7}$$

where h is the total energy of the system, we obtain the following equations for the unknown trajectories:

$$u_1''(w)\frac{2(h-\Pi)}{u_1'^2+\mu}-\frac{1}{\mu}\Pi_w'u_1'=-\Pi_{u_1}'. \tag{8}$$

Equation (8) has a singularity on the maximum constant-energy surface $\widetilde{\Pi}=h$. Therefore, Eq. (10) should be supplemented with boundary conditions that would analytically continue the trajectory to the maximum constant-energy surface [6]

$$\left.\left\{\frac{1}{\mu}\Pi_w'u_1'-\Pi_{u_1}'\right\}\right|_{w=\pm W_*}, \tag{9}$$

where W_* is the nonlinear mode amplitude.

The equation for W_* follows from the energy integral $\Pi|_{w=\pm W_*}=h$. This equation reads

$$\bar{h}=\frac{\bar{\gamma}\beta^2}{8}W_*^4-\frac{\bar{\gamma}\alpha^2}{2}W_*^2, \qquad h=\varepsilon\bar{h}. \tag{10}$$

Represented as a power series, the solution of Eq. (8) is

$$u_1=\varepsilon\bar{u}_1(w), \qquad \bar{u}_1(w)=a_0+a_1w+a_2w^2+\ldots, \tag{11}$$

where a_0, a_1, … are unknown coefficients.

Let us substitute the series (11) into Eq. (8) and equate the coefficients of like powers of w. We will restrict the analysis to the following powers of w: w^0, w^1, and w^2. Then we obtain three linear algebraic equations for five unknowns a_0, …, a_4. The boundary conditions (9) provide two more linear algebraic equations. The solution of the resulting system of linear algebraic equations has the form

$$a_1=a_3=0, \qquad a_0=-\frac{4\bar{h}}{\bar{\mu}}a_2,$$

$$\widetilde{a}_2=\left\{\left(2-\frac{(\bar{\mu}+4\bar{\gamma}\alpha^2)W_*^2}{12\bar{h}}\right)\left(4\bar{\gamma}\alpha^2-2\bar{\gamma}\beta^2W_*^2+\bar{\mu}\right)+\bar{\gamma}\beta^2W_*^2\right\}^{-1}\left\{2\bar{\mu}-\frac{\bar{\mu}W_*^2}{12\bar{h}}\left(4\bar{\gamma}\alpha^2-2\bar{\gamma}\beta^2W_*^2+\bar{\mu}\right)\right\},$$

$$\widetilde{a}_4=\frac{\bar{\mu}}{24\bar{h}}-\frac{\bar{\mu}+4\bar{\gamma}\alpha^2}{24\bar{h}}\widetilde{a}_2, \qquad a_j=\widetilde{a}_j\frac{\bar{\gamma}}{2\rho^2}, \qquad j=2,4. \tag{12}$$

Given $\mu=\gamma=\varepsilon=0.01$ and $\varphi=0.15$, system (2), (3) has been integrated numerically using the fourth-order Runge–Kutta method for initial conditions corresponding to the analytical nonlinear normal mode (11):

$$u_2(0)=\varepsilon\bar{u}_2(W_*), \qquad w(0)=W_*, \qquad \dot{u}_2(0)=\dot{w}(0)=0.$$

Figure 2*a* shows the nonlinear normal mode in the configuration space plotted from formulas (11) and (12), and Fig. 2*b* shows the nonlinear normal mode computed numerically. As is seen, the vibration amplitudes of the Mises truss in this mode are large, and the vibration amplitudes of the linear system are small. If this mode is stable, then such a motion is favorable for vibration damping.

3. Stability Analysis of Motions. Let us analyze the nonlinear normal mode (11) for stability. Since its curvature in the configuration space is rather small, we may restrict the analysis to the rectilinear approximation. We will use new variables (ξ, η) such that the ξ-axis is parallel to the rectilinear approximation of the nonlinear normal mode, and the η-axis is perpendicular to this direction. Then the orthogonal variation of the variable $\eta(t)$ defines the orbital stability of the nonlinear normal mode. Using the asymptotic formulas $u_1=O(\varepsilon)$ and $w=w_0+O(\varepsilon)$, we rearrange the equations of motion (2), (3) as [9, 10]

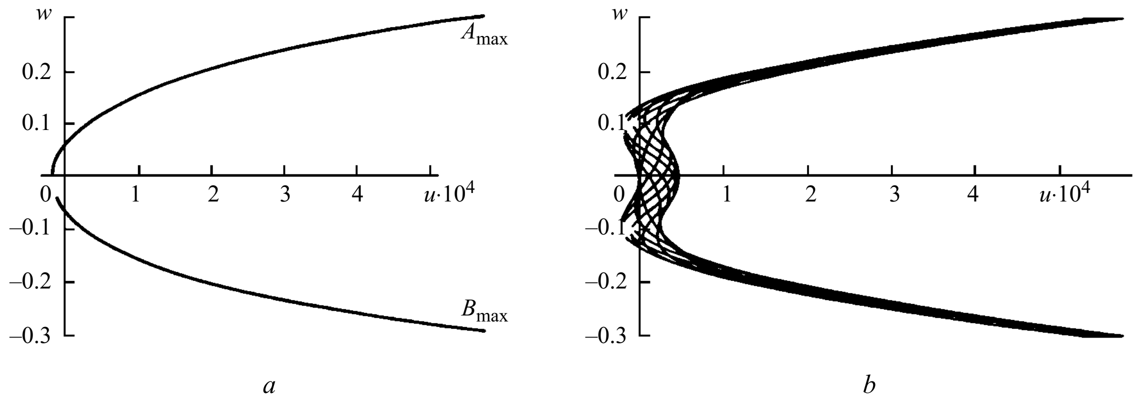

Fig. 2

$$\ddot{w}_0 - p^2\alpha^2 w_0 + \frac{p^2\beta^2}{2} w_0^3 = 0, \qquad \ddot{u}_1 + (1+\varepsilon\bar{\gamma})u_1 - \frac{\varepsilon\bar{\gamma}u_1 w_0^2}{\rho^3} - \frac{\varepsilon\bar{\gamma}w_0^2}{2\rho^2} = 0. \tag{13}$$

The first equation in system (13) does not depend on the second one and has the analytical solution

$$w_0 = \sqrt{2}\frac{\alpha}{\beta}\sqrt{1+\sqrt{1+4H}}\, cn(p\alpha\tau\sqrt[4]{1+4H}; k), \tag{14}$$

$$4H = \frac{\beta^4 W_*^4}{4\alpha^4} - \frac{\beta^2}{\alpha^2} W_*^2, \tag{15}$$

$$2k^2 = \left(1 + \frac{\beta^4 W_*^4}{4\alpha^4} - \frac{\beta^2}{\alpha^2} W_*^2\right)^{-1/2} + 1, \tag{16}$$

where k is the elliptic modulus, cn is an elliptic function, and H is the total energy of the independent oscillator of system (13).

Solution (14) describes the snap-through of the Mises truss. Let us introduce small perturbations $\eta(t)$ of the periodic solution $\bar{u}_1$: $u_1 = \bar{u}_1 + \eta$, which are described by the following variational equation:

$$\ddot{\eta} + (1+\varepsilon\bar{\gamma} - \frac{\varepsilon\bar{\gamma}}{\rho^3} w_0^2)\eta = 0. \tag{17}$$

From (16) we obtain the asymptotic relation

$$k^2 = k_0^2 - \varepsilon k_1 + O(\varepsilon^2),$$

$$k_1 = \frac{\bar{\gamma}\left(1.5 - c^{-1}\right)\left[W_*^4 - 2c^2(1-c)W_*^2\right] c^2(1-c)}{\left[4c^4(1-c)^2 + W_*^4 - 4c^2(1-c)W_*^2\right]^{3/2}},$$

$$k_0^2 = 0.5 + \frac{c^2(1-c)}{\sqrt{4c^4(1-c)^2 + W_*^4 - 4c^2(1-c)W_*^2}}. \tag{18}$$

Expanding the function $cn^2(t, k_0)$ into Fourier series, we write Eq. (17) as

$$\ddot{\eta} + \left[\Omega_0^2 - \varepsilon h \sum_{s=1}^{\infty} \frac{s q_0^s}{(1-q_0^{2s})} \cos(\Omega_* st)\right]\eta = 0,$$

$$h=\bar{\mu}\frac{4\pi^2}{K^2(k_0)},\qquad \Omega_*=\frac{\pi}{K(k_0)},\qquad q_0=\exp\left[-\frac{\pi K'(k_0)}{K(k_0)}\right],$$

$$\Omega_0^2=\frac{c(2k_0^2-1)}{2(1-c)p^2}-\frac{\varepsilon k_1 c}{(1-c)p^2}+\frac{\varepsilon\bar{\mu}(2k_0^2-1)(1+2c)}{4(1-c)}-\varepsilon 2\bar{\mu}\left[\frac{E(k_0)}{K(k_0)}-1+k_0^2\right], \tag{19}$$

where $K(k_0)$ and $E(k_0)$ are complete elliptic integrals of the first and second kinds.

Equation (19) is analyzed using the multiple-scale method [4, 6]. The solution of this equation can be represented as

$$\eta=\eta_0(T_0,T_1)+\varepsilon\eta_1(T_0,T_1)+\ldots,\qquad T_0=t,\qquad T_1=\varepsilon t,\qquad \Omega_0^2=\omega_0^2+\varepsilon\omega_1^2+\ldots. \tag{20}$$

After some transformations, according to the multiple-scale method, we obtain the equations

$$\frac{\partial^2\eta_0}{\partial T_0^2}+\omega_0^2\eta_0=0,\qquad \eta_0=A(T_1)\exp(i\omega_0T_0)+\bar{A}(T_1)\exp(-i\omega_0T_0), \tag{21}$$

$$\frac{\partial^2\eta_1}{\partial T_0^2}+\omega_0^2\eta_1+\omega_1^2\eta_0+2\frac{\partial^2\eta_0}{\partial T_0\partial T_1}-h\eta_0\sum_{s=1}^{\infty}\frac{sq_0^s}{(1-q_0^{2s})}\cos(\Omega_* st)=0. \tag{22}$$

Let us analyze parametric resonances of order s = 1, 2, 3, … described by the equations

$$\Omega_* s=2\omega_0+\varepsilon\sigma_s, \tag{23}$$

where σ_s is the amount off-resonance.

Changing variables, $A=\frac{a}{2}\exp(i\beta)$, $\gamma=\sigma_sT_1-2\beta$, we obtain the following system of modulation equations:

$$a'\omega_0=a\chi\sin(\sigma_sT_1-2\beta),\qquad \beta'\omega_0=\frac{\omega_1^2}{2}-\chi\cos(\sigma_sT_1-2\beta), \tag{24}$$

where $\chi=\dfrac{hsq_0^s}{4(1-q_0^{2s})}$.

Written in terms of the variables $(x, y) = [a\cos(\beta - 0.5\sigma_sT_1), \sin(\beta - 0.5\sigma_sT_1)]$, Eqs. (24) become

$$x'=y\left(\frac{\sigma}{2}-\frac{\omega_1^2}{2\omega_0}-\frac{\chi}{\omega_0}\right),\qquad y'=x\left(-\frac{\chi}{\omega_0}+\frac{\omega_1^2}{2\omega_0}-\frac{\sigma}{2}\right). \tag{25}$$

The solutions of Eqs. (25) are related to the dynamic system (19) as

$$\eta=x\cos\left(\frac{\Omega_* s}{2}t\right)-y\sin\left(\frac{\Omega_* s}{2}t\right)+O(\varepsilon). \tag{26}$$

The stability analysis of the dynamic system (19) reduces to the analysis of the trivial solutions of system (25). The characteristic numbers of this system are

$$\lambda_{1,2}=\pm\sqrt{\frac{\chi^2}{\omega_0^2}-\left(\frac{\sigma}{2}-\frac{\omega_1^2}{2\omega_0}\right)^2}. \tag{27}$$

This yields the boundary between stable and unstable modes:

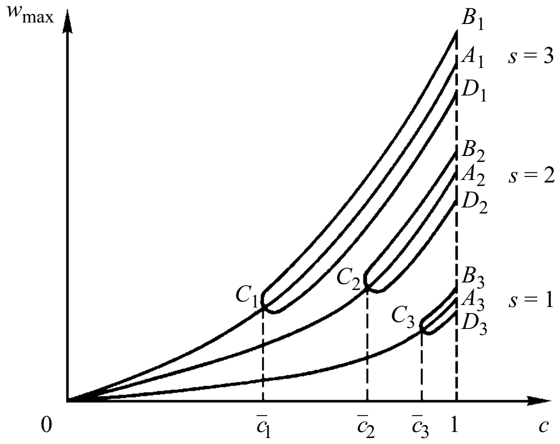

Fig. 3

$$\Omega_* s = 2\omega_0 + \varepsilon\left(\frac{\omega_1^2}{\omega_0} \pm \frac{2\chi}{\omega_0}\right) + O(\varepsilon^2). \tag{28}$$

Let us examine the orientation of this boundary on the parameter plane (c, W_*). To this end, we will discard the terms of order ε in (28). Using (19), we obtain the equation

$$\frac{\sqrt{2}cK(k_0)}{p\pi s} = \sqrt[4]{c^2(1-c)^2 + \frac{W_*^4}{4c^2} - (1-c)W_*^2} + O(\varepsilon). \tag{29}$$

Let us consider a flat Mises truss for which $1-c = \varepsilon_* c_1$, $\varepsilon_* \ll 1$. In this case, Eq. (29) becomes

$$\frac{\sqrt{2}cK(k_0)}{p\pi s} = \frac{W_*}{\sqrt{2c}}\left(1 - \varepsilon_* \frac{c^2 c_1}{W_*^2}\right) + O(\varepsilon_*^2) + O(\varepsilon), \tag{30}$$

where

$$k_0^2 = \frac{1}{2} + \varepsilon_* \frac{c^2 c_1}{W_*^2} + O(\varepsilon_*^2), \tag{31}$$

$$K(k_0) = K\left(\frac{1}{\sqrt{2}}\right) + \varepsilon_* K'\left(\frac{1}{\sqrt{2}}\right)\frac{c^2 c_1}{\sqrt{2}W_*^2} + O(\varepsilon_*^2). \tag{32}$$

For further analysis, we represent relation (32) as

$$W_* = \frac{2c^{3/2}}{p\pi s}\left[K\left(\frac{1}{\sqrt{2}}\right) + \varepsilon_* \frac{c^2 c_1^2}{W_*^2} E\left(\frac{1}{\sqrt{2}}\right)\right] + O(\varepsilon_*^2) + O(\varepsilon). \tag{33}$$

Figure 3 shows curves (OA_1), (OA_2), and (OA_3) plotted from Eq. (33). Since Eq. (28) includes $O(\varepsilon)$-terms, the boundary between stable and unstable modes is different from the curves (OA_1), (OA_2), and (OA_3). This boundary is represented in Fig. 3 by the curves $(B_1C_1D_1)$, $(B_2C_2D_2)$, and $(B_3C_3D_3)$. The quantities $\tilde{c}_s$, $s = 1, 2, \ldots$ (Fig. 3), are determined from Eq. (28) by the formulas

$$\widetilde{c}_s = \left[1 + \frac{4K^2\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{2}-1\right)}{p^2\pi^2 s^2} \right]^{-1} . \tag{34}$$

If $\mu = \gamma = \varepsilon = 0.01$ and $\varphi = 0.15$ in (1), then $\widetilde{c}_1 = 0.63$, $\widetilde{c}_2 = 0.88$ and $\widetilde{c}_3 = 0.94$.

In this case, the points A_i $(i = 1, 2, 3)$ (Fig. 3) have the following coordinates: $A_1(1, 1.18)$, $A_2(1, 0.59)$, and $A_3(1, 0.39)$.

From Fig. 3 it follows that if the angle φ is small, then the domain of unstable vibrations is of order ε. As the angle φ increases, the width and the number of instability domains decrease. Therefore, it is easy to select a value of φ such that the periodic motions are stable.

4. Conclusions. We propose to use the Mises truss as a free-vibration absorber. The elastic system whose vibrations are to be damped is represented by a linear oscillator with one degree of freedom. The free nonlinear vibrations of a system with two powers are studied using the nonlinear normal mode method. Periodic motions are analyzed for stability.

Based on the analysis performed, we draw the following conclusions. The local nonlinear normal mode studied in the paper is favorable for vibration damping. In this case, the Mises truss vibrates with large amplitudes and the elastic system, with small amplitudes. Such a motion is stable over a wide range of parameters.

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