ON A REPRESENTATION OF THE SOLUTION OF THE LINEAR ELASTICITY EQUATIONS

N. M. Borodachev

UDC 539.3

A new representation of the stress tensor in the linear theory of elasticity is proposed. The representation satisfies the equilibrium equations and the compatibility conditions for strains. In this representation, the stress tensor is expressed in terms of a harmonic vector. The second boundary-value problem for an elastic half-space and elastic layer is considered as an example.

Keywords: three-dimensional theory of elasticity, stress formulation, elastic half-space, elastic layer, Fourier transform

A representation of the stress tensor in terms of an asymmetric harmonic tensor was proposed in [1, 2]. A three-dimensional elastic problem in stress formulation was solved in [3–7, 9, 10, 12, 13].

This paper sets forth a new representation of the stress tensor in terms of a harmonic vector in the linear theory of elasticity. The representation satisfies the equilibrium equations and the compatibility conditions for strains.

1. Problem Formulation. In the three-dimensional linear elasticity problem in stress formulation, the stress tensor $\hat{T}$ has to satisfy the static equation

$$\operatorname{div}\hat{T}=0 \tag{1.1}$$

and the compatibility condition for strains [5]

$$\operatorname{Ink}\hat{\varepsilon}=\frac{1}{2\mu}\operatorname{Ink}\left(\hat{T}-\frac{\nu}{1+\nu}\sigma\hat{E}\right)=0, \tag{1.2}$$

where $\operatorname{Ink}\hat{Q}=\operatorname{rot}(\operatorname{rot}\hat{Q})^{*}$, $\hat{E}$ is a unit tensor, ν is Poisson's ratio, μ is the shear modulus, $\sigma = I_1(\hat{T})$ is the first invariant of the stress tensor, and $\hat{\varepsilon}$ is the linear strain tensor. Let body forces be absent.

It is sometimes more convenient to solve the three-dimensional elastic problem in the stress formulation than in the displacement formulation, especially if stresses are specified at the surface of the body (second problem).

The compatibility condition for strains is usually written in Beltrami's form

$$\nabla^2\hat{T}+\frac{\nabla\nabla\sigma}{1+\nu}=0, \tag{1.3}$$

where ∇ is the inverted delta and ∇^2 is the Laplacian.

Equation (1.3) is derived from (1.2) using some transformations and the equilibrium equation (1.1).

It is more convenient, however, to use the compatibility condition (1.2).

There are two methods for deriving a stress tensor $\hat{T}$ that would satisfy Eqs. (1.1) and (1.2).

National Aviation University, Kiev, Ukraine. Translated from Prikladnaya Mekhanika, Vol. 40, No. 11, pp. 85–91, November 2004. Original article submitted June 25, 2003.

1063-7095/04/4011-1263

The first method was proposed by Krutkov [4]. According to this method, the static equation (1.1) is satisfied first and then goes the compatibility equation (1.2). Using an invariant representation of the stress tensor, Krutkov reduced system (1.1), (1.2) to a rather awkward differential equation satisfied by the tensor of stress functions. Krutkov's method, however, appears very inconvenient when used to solve specific boundary-value problems.

We will use the second method. First, we will find an expression for $\hat{T}$ that would satisfy the compatibility condition (1.2) and only after that will we employ the equilibrium equation (1.1).

2. Solution Technique. Let us consider the compatibility condition (1.2). If the result of the operation Ink over a symmetric tensor is zero, then this tensor is the deformation of some vector. Hence,

$$\hat{T}-\frac{\nu}{1+\nu}\sigma\hat{E}=\text{def}\,\boldsymbol{c}, \tag{2.1}$$

where $\boldsymbol{c}$ is some vector, def $\boldsymbol{c}=\frac{1}{2}\left[(\nabla\boldsymbol{c})^{*}+\nabla\boldsymbol{c}\right]$.

Equating the traces of the tensors on the left- and right-hand sides of (2.1), we get

$$\frac{1-2\nu}{1+\nu}\sigma=\text{div}\,\boldsymbol{c}, \tag{2.2}$$

where $I_1(\text{def}\,\boldsymbol{c})=\text{div}\,\boldsymbol{c}$ and $I_1(\hat{E})=3$.

Substituting (2.2) into (2.1) yields

$$\hat{T}=\frac{\nu}{1-2\nu}\hat{E}\,\text{div}\,\boldsymbol{c}+\text{def}\,\boldsymbol{c}. \tag{2.3}$$

The stress tensor $\hat{T}$ in the form (2.3) identically satisfies the compatibility condition (1.2).

Let $\boldsymbol{c}=2\mu\boldsymbol{c}_0$. Then (2.3) becomes

$$\hat{T}=2\mu\left(\frac{\nu}{1-2\nu}\hat{E}\,\text{div}\,\boldsymbol{c}_0+\text{def}\,\boldsymbol{c}_0\right). \tag{2.4}$$

Let us now express the vector $\boldsymbol{c}_0$ in terms of a harmonic vector $\boldsymbol{B}$ ($\nabla^2\boldsymbol{B}=0$) and a scalar φ :

$$\boldsymbol{c}_0=\boldsymbol{B}-x_3\nabla\varphi\,, \tag{2.5}$$

where x_1, x_2, and x_3 are rectangular coordinates. We will relate φ and $\boldsymbol{B}$ below.

Substituting (2.5) into (2.4), we obtain

$$\hat{T}=2\mu\left\{\frac{\nu}{1-2\nu}\hat{E}\,\text{div}\left(\boldsymbol{B}-x_3\nabla\varphi\right)+\text{def}\left(\boldsymbol{B}-x_3\nabla\varphi\right)\right\}. \tag{2.6}$$

The stress tensor (2.6) satisfies the compatibility condition (1.2). Let us now make expression (2.6) satisfy the equilibrium equation (1.1) too. To this end, we substitute (2.6) into (1.1):

$$\text{div}\left[\frac{\nu}{1-2\nu}\hat{E}\,\text{div}\left(\boldsymbol{B}-x_3\nabla\varphi\right)+\text{def}\left(\boldsymbol{B}-x_3\nabla\varphi\right)\right]=0.$$

Since div $\left(\hat{E}\,\text{div}\,\boldsymbol{a}\right)=\nabla\,\text{div}\,\boldsymbol{a}$ and div def $\boldsymbol{a}=\frac{1}{2}\left(\nabla^2\boldsymbol{a}+\nabla\text{div}\,\boldsymbol{a}\right)$, we have

$$\nabla^2\left(\boldsymbol{B}-x_3\nabla\varphi\right)+\frac{\nu}{1-2\nu}\text{grad div}\left(\boldsymbol{B}-x_3\nabla\varphi\right)=0, \tag{2.7}$$

and

$$\nabla^2 \boldsymbol{B} = 0, \quad \nabla^2 \left(x_3 \nabla \varphi \right) = x_3 \nabla \left(\nabla^2 \varphi \right) + 2\nabla \frac{\partial \varphi}{\partial x_3},$$

$$\nabla \operatorname{div} \left(\boldsymbol{B} - x_3 \nabla \varphi \right) = \nabla \nabla \cdot \boldsymbol{B} - \nabla \frac{\partial \varphi}{\partial x_3} - \nabla \left(x_3 \nabla^2 \varphi \right).$$

Substituting these relations into (2.7), we obtain

$$\nabla \nabla \cdot \boldsymbol{B} - (3 - 4\nu) \nabla \frac{\partial \varphi}{\partial x_3} - (1 - 2\nu) x_3 \nabla \left(\nabla^2 \varphi \right) - \nabla \left(x_3 \nabla^2 \varphi \right) = 0. \tag{2.8}$$

If

$$\nabla^2 \varphi = 0, \quad \frac{\partial \varphi}{\partial x_3} = \frac{1}{3 - 4\nu} \nabla \cdot \boldsymbol{B}, \tag{2.9}$$

then (2.8) becomes an identity.

Thus, the stress tensor $\hat{T}$ in the form (2.6) also satisfies the equilibrium equation (1.1) if the function φ obeys the conditions (2.9).

In (2.6), the stress tensor $\hat{T}$ is expressed in terms of the harmonic vector $\boldsymbol{B}$ and the harmonic scalar φ that are related by (2.9). If the conditions (2.9) and the condition $\nabla^2 \boldsymbol{B} = 0$ are met, then (2.6) will satisfy Eqs. (1.1) and (1.2).

Expression (2.6) can be simplified somewhat. Since

$$\operatorname{div} \left(x_3 \nabla \varphi \right) = \frac{\partial \varphi}{\partial x_3} + x_3 \nabla^2 \varphi,$$

considering (2.9) we have

$$\operatorname{div} \left(\boldsymbol{B} - x_3 \nabla \varphi \right) = 2(1 - 2\nu) \frac{\partial \varphi}{\partial x_3}.$$

Hence, representation (2.6) takes the following final form:

$$\hat{T} = 2\mu \left[2\nu \hat{E} \frac{\partial \varphi}{\partial x_3} + \operatorname{def} \left(\boldsymbol{B} - x_3 \nabla \varphi \right) \right]. \tag{2.10}$$

The components of (2.10) can be written as

$$\sigma_{st} = \mu \left[4\nu \delta_{st} \frac{\partial \varphi}{\partial x_3} + \left(\frac{\partial B_s}{\partial x_t} + \frac{\partial B_t}{\partial x_s} \right) - \frac{\partial}{\partial x_t} \left(x_3 \frac{\partial \varphi}{\partial x_s} \right) - \frac{\partial}{\partial x_s} \left(x_3 \frac{\partial \varphi}{\partial x_t} \right) \right] \quad (s, t = 1, 2, 3), \tag{2.11}$$

where δ_{st} is the Kronecker delta.

The stress tensor $\hat{T}$ in the form (2.10) satisfies the static equation (1.1) and the compatibility condition (1.2). Using expression (2.11), we can first determine the components B_1, B_2, and B_3 of the vector $\boldsymbol{B}$ from the known boundary stresses and then the scalar φ from (2.9).

Thus, we can derive the final expression of $\hat{T}$ for a specific boundary-value problem.

An expression of $\hat{T}$ can also be derived from the Papkovich–Neuber solution

$$\boldsymbol{u} = \boldsymbol{B} - \frac{1}{4(1 - \nu)} \nabla (\boldsymbol{R} \cdot \boldsymbol{B} + B_0), \tag{2.12}$$

where $\boldsymbol{u}$ is the displacement vector and $\boldsymbol{R}$ is the position vector, $\nabla^2 \boldsymbol{B} = 0$, $\nabla^2 B_0 = 0$.

However, the structure of (2.12) does not allow us to establish the boundary conditions that have to be imposed on $\boldsymbol{B}$ and B_0 at the body's surface to solve boundary-value problems. Papkovich emphasized this circumstance in [8].

The structure of (2.10) is such that we can easily define boundary conditions.

3. Application of the Representation (2.10). By way of example let us apply expression (2.10) to solve the second boundary-value problem for an elastic half-space. It is a static problem with stresses specified at the boundary of the half-space. For $x_3 = 0$, we have

$$\sigma_{3t} = \begin{cases} -f_t(x_1, x_2) & \text{for } (x_1, x_2) \in \Omega_t, \\ 0 & \text{for } (x_1, x_2) \notin \Omega_t, \end{cases} \tag{3.1}$$

where Ω_t are the domains of loading in the plane $x_3 = 0$ ($t = 1, 2, 3$).

The expressions for the stresses σ_{31}, σ_{32}, and σ_{33} can be derived from (2.11) when $x_3 = 0$:

$$\sigma_{31} = \mu\left(\frac{\partial B_3}{\partial x_1} + \frac{\partial B_1}{\partial x_3} - \frac{\partial \varphi}{\partial x_1}\right), \quad \sigma_{32} = \mu\left(\frac{\partial B_3}{\partial x_2} + \frac{\partial B_2}{\partial x_3} - \frac{\partial \varphi}{\partial x_2}\right), \quad \sigma_{33} = 2\mu\left(\frac{\partial B_3}{\partial x_3} - (1-2\nu)\frac{\partial \varphi}{\partial x_3}\right). \tag{3.2}$$

Let us introduce functions N_t harmonic in the half-space $x_3 > 0$,

$$N_t(x_1, x_2, x_3) = \frac{1}{2\pi}\iint\limits_{\Omega_t} f_t(y_1, y_2) \ln(x_3 + r)\, dy_1 dy_2 \qquad (t = 1, 2, 3). \tag{3.3}$$

Then we have

$$\lim \frac{\partial^2 N_t}{\partial x_3^2} = \begin{cases} -f_t(x_1, x_2) & \text{for } (x_1, x_2) \in \Omega_t, \\ 0 & \text{for } (x_1, x_2) \notin \Omega_t. \end{cases} \tag{3.4}$$

Using (3.1)–(3.4), we arrive at the following system of equations:

$$\mu\left(\frac{\partial B_3}{\partial x_1} + \frac{\partial B_1}{\partial x_3} - \frac{\partial \varphi}{\partial x_1}\right) = \frac{\partial^2 N_1}{\partial x_3^2}, \quad \mu\left(\frac{\partial B_3}{\partial x_2} + \frac{\partial B_2}{\partial x_3} - \frac{\partial \varphi}{\partial x_2}\right) = \frac{\partial^2 N_2}{\partial x_3^2},$$

$$2\mu\left(\frac{\partial B_3}{\partial x_3} - (1-2\nu)\frac{\partial \varphi}{\partial x_3}\right) = \frac{\partial^2 N_3}{\partial x_3^2}.$$

Solving these equations, we obtain

$$B_1 = \frac{1}{2\mu}\left(2\frac{\partial N_1}{\partial x_3} - \frac{\partial N_3}{\partial x_1}\right) + 2\nu\frac{\partial \Psi}{\partial x_1}, \quad B_2 = \frac{1}{2\mu}\left(2\frac{\partial N_2}{\partial x_3} - \frac{\partial N_3}{\partial x_2}\right) + 2\nu\frac{\partial \Psi}{\partial x_2},$$

$$B_3 = \frac{1}{2\mu}\frac{\partial N_3}{\partial x_3} + (1-2\nu)\varphi, \quad \varphi = \frac{\partial \Psi}{\partial x_3}. \tag{3.5}$$

From (3.5) it follows that

$$\nabla \cdot \boldsymbol{B} = \frac{1}{\mu}\frac{\partial}{\partial x_3}(\nabla \cdot \boldsymbol{N}) + (1-4\nu)\frac{\partial \varphi}{\partial x_3}, \qquad \boldsymbol{N} = (N_1, N_2, N_3). \tag{3.6}$$

From (2.9) and (3.6) we get

$$\varphi = \frac{1}{2\mu}(\nabla \cdot \boldsymbol{N}). \tag{3.7}$$

Expressions (3.5) and (3.7) define the vector $\boldsymbol{B} = (B_1, B_2, B_3)$ and the scalar φ, respectively, appearing in (2.10), in the second boundary-value problem for an elastic half-space. Substituting (3.5) and (3.7) into (2.11), we obtain the final formulas for the stresses σ_{st}.

4. Elastic Layer. Let us demonstrate how expression (2.10) can be used to determine stresses in an isotropic elastic layer ($0 \le x_3 \le h$). The stresses σ_{31}, σ_{32}, and σ_{33} are specified at the boundary of the layer:

$$\sigma_{3t} = f_t^0(x_1, x_2) \quad \text{for} \quad x_3 = 0,$$

$$\sigma_{3t} = f_t^h(x_1, x_2) \quad \text{for} \quad x_3 = h \quad (t = 1, 2, 3). \tag{4.1}$$

From (2.11) we obtain

$$\sigma_{31} = \mu\left(\frac{\partial B_3}{\partial x_1} + \frac{\partial B_1}{\partial x_3} - \frac{\partial \varphi}{\partial x_1} - 2x_3 \frac{\partial^2 \varphi}{\partial x_1 \partial x_3}\right),$$

$$\sigma_{32} = \mu\left(\frac{\partial B_3}{\partial x_2} + \frac{\partial B_2}{\partial x_3} - \frac{\partial \varphi}{\partial x_2} - 2x_3 \frac{\partial^2 \varphi}{\partial x_2 \partial x_3}\right),$$

$$\sigma_{33} = 2\mu\left[\frac{\partial B_3}{\partial x_3} - (1-2\nu)\frac{\partial \varphi}{\partial x_3} - x_3 \frac{\partial^2 \varphi}{\partial x_3^2}\right]. \tag{4.2}$$

Each component B_m of the harmonic vector $\boldsymbol{B}$ satisfies the Laplace equation

$$\nabla^2 B_m = 0, \quad m = 1, 2, 3. \tag{4.3}$$

The two-dimensional Fourier transform of some function $f(x_1, x_2, x_3)$ is [11]:

$$\bar{f}(\xi_1, \xi_2, x_3) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, e^{i(\xi_1 x_1 + \xi_2 x_2)}\, dx_1 dx_2. \tag{4.4}$$

Applying the transform (4.4) to Eq. (4.3) and solving the resultant ordinary differential equation, we get

$$\bar{B}_m(\xi_1, \xi_2, x_3) = A_m(\xi_1, \xi_2)\sinh(\kappa x_3) + C_m(\xi_1, \xi_2)\cosh(\kappa x_3), \tag{4.5}$$

where $\kappa^2 = \xi_1^2 + \xi_2^2$ and $m = 1, 2, 3$.

A similar expression can be derived for the function φ satisfying the equation $\nabla^2 \varphi = 0$:

$$\bar{\varphi}(\xi_1, \xi_2, x_3) = A_0(\xi_1, \xi_2)\sinh(\kappa x_3) + C_0(\xi_1, \xi_2)\cosh(\kappa x_3). \tag{4.6}$$

Applying the transform (4.4) to (4.2) and considering that the stresses tend to zero at infinity, we obtain

$$\bar{\sigma}_{31} = \mu\left(-i\xi_1 \bar{B}_3 + \frac{\partial \bar{B}_1}{\partial x_3} + i\xi_1 \bar{\varphi} + 2i\xi_1 x_3 \frac{\partial \bar{\varphi}}{\partial x_3}\right),$$

$$\bar{\sigma}_{32} = \mu\left(-i\xi_2 \bar{B}_3 + \frac{\partial \bar{B}_2}{\partial x_3} + i\xi_2 \bar{\varphi} + 2i\xi_2 x_3 \frac{\partial \bar{\varphi}}{\partial x_3}\right),$$

$$\bar{\sigma}_{33} = 2\mu\left(\frac{\partial \bar{B}_3}{\partial x_3} - (1-2\nu)\frac{\partial \bar{\varphi}}{\partial x_3} - x_3 \frac{\partial^2 \bar{\varphi}}{\partial x_3^2}\right). \tag{4.7}$$

The Fourier transform of the boundary conditions (4.1) is

$$\bar{\sigma}_{3t} = \bar{f}_t^0(\xi_1, \xi_2) \quad \text{for} \quad x_3 = 0,$$

$$\bar{\sigma}_{3t} = \bar{f}_t^h(\xi_1, \xi_2) \quad \text{for} \quad x_3 = h \quad (t = 1, 2, 3). \tag{4.8}$$

Satisfying conditions (4.8) and using formulas (4.5)–(4.7), we obtain the system of equations

$$-i\xi_1 C_3 + \kappa A_1 + \xi_1 C_0 = \bar{f}_1^{0} / \mu, \quad -i\xi_2 C_3 + \kappa A_2 + \xi_2 C_0 = \bar{f}_2^{0} / \mu, \quad \kappa A_3 - (1-2\nu)\kappa A_0 = \bar{f}_3^{0} / 2\mu,$$

$$-i\xi_1 \left[A_3 \sinh(\kappa h) + C_3 \cosh(\kappa h)\right] + \kappa\left[A_1 \cosh(\kappa h) + C_1 \sinh(\kappa h)\right]$$

$$+ i\xi_1 \left[A_0 \sinh(\kappa h) + C_0 \cosh(\kappa h)\right] + 2i\xi_1 \kappa h \left[A_0 \cosh(\kappa h) + C_0 \sinh(\kappa h)\right] = \bar{f}_1^{h} / \mu,$$

$$-i\xi_2 \left[A_3 \sinh(\kappa h) + C_3 \cosh(\kappa h)\right] + \kappa\left[A_2 \cosh(\kappa h) + C_2 \sinh(\kappa h)\right]$$

$$+ i\xi_2 \left[A_0 \sinh(\kappa h) + C_0 \cosh(\kappa h)\right] + 2i\xi_2 \kappa h \left[A_0 \cosh(\kappa h) + C_0 \sinh(\kappa h)\right] = \bar{f}_2^{h} / \mu,$$

$$\kappa\left[A_3 \cosh(\kappa h) + C_3 \sinh(\kappa h)\right] - (1-2\nu)\,\kappa\left[A_0 \cosh(\kappa h) + C_0 \sinh(\kappa h)\right] - \kappa^2 h \left[A_0 \sinh(\kappa h) + C_0 \cosh(\kappa h)\right] = \bar{f}_3^{h} / 2\mu,$$

$$(3-4\nu)\,\kappa A_0 + i\xi_1 C_1 + \xi_2 C_2 - \kappa A_3 = 0, \quad (3-4\nu)\,\kappa C_0 + i\xi_1 A_1 + \xi_2 A_2 - \kappa C_3 = 0. \tag{4.9}$$

The last two equations in (4.9) have been derived using the Fourier transform of (2.9).

Solving the system of equations (4.9) in a symbolical form using Cramer's rule, we determine $A_0, C_0, A_1, C_1, A_2, C_2, A_3$, and C_3. Then, using expressions (4.5) and (4.6), we take the Fourier transforms of $\overline{B}_1, \overline{B}_2, \overline{B}_3$, and $\overline{\varphi}$. After that, we find $\overline{\sigma}_{31}, \overline{\sigma}_{32}$, and $\overline{\sigma}_{33}$ by formulas (4.7). And, finally, applying the inverse two-dimensional Fourier transform, we recover the stresses σ_{31}, σ_{32}, and σ_{33} as functions of the coordinates x_1, x_2, and x_3.

To derive the formulas for the other stresses (σ_{11}, σ_{22}, and σ_{12}), it is necessary to use expression (2.11). After that, we can take the Fourier transforms of $\overline{\sigma}_{11}, \overline{\sigma}_{22}$, and $\overline{\sigma}_{12}$ (i.e., formulas similar to (4.7)). These formulas include $\overline{B}_1, \overline{B}_2, \overline{B}_3$, and $\overline{\varphi}$. The original functions can easily be recovered numerically using the inverse Fourier transform.

REFERENCES

1. N. M. Borodachev, "Construction of exact solutions to three-dimensional elastic problems in stresses," *Int. Appl. Mech.*, **37**, No. 6, 762-768 (2001).
2. N. M. Borodachev, "Solution of a spatial stress problem for an elastic layer," *Int. Appl. Mech.*, **38**, No. 5, 556–561 (2002).
3. V. T. Grinchenko and A. F. Ulitko, *Equilibrium of Canonical Elastic Bodies* [in Russian], Naukova Dumka, Kiev (1985).
4. Yu. A. Krutkov, *Tensor of Stress Functions and General Solutions in Static Elasticity* [in Russian], Izd. AN SSSR, Moscow–Leningrad (1949).
5. A. I. Lur'e, *Theory of Elasticity* [in Russian], Nauka, Moscow (1970).
6. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Cambridge Univ. Press (1959).
7. W. Nowacki, *Theory of Elasticity* [in Polish], PWN, Warsaw (1970).
8. P. F. Papkovich, *Theory of Elasticity*, Oborongiz, Moscow–Leningrad (1939).
9. E. N. Borisov, "Approximate analytical solution of a physically nonlinear spatial problem on the elastic equilibrium of a multilayer rectangular plate," *Int. Appl. Mech.*, **39**, No. 8, 961–968 (2003).
10. N. M. Borodachev, "Solution of the three-dimensional thermoelastic problem in stresses," *Int. Appl. Mech.*, **39**, No. 4, 438–444 (2003).
11. B. Davies, *Integral Transforms and Their Applications*, Springer-Verlag, New York–Heidelberg–Berlin (1978).
12. Yu. N. Podil'chuk and Yu. K. Rubtsov, "Development of the boundary-element method for three-dimensional problems of static and nonstationary elasticity," *Int. Appl. Mech.*, **40**, No. 2, 160–168 (2004).
13. J. J. Rushchitsky, "Joint effect of microstructure and load on the static stress state of elastic bodies," *Int. Appl. Mech.*, **40**, No. 4, 370–387 (2004).