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RESEARCH

Relativistic Nature of Wave-particle Duality

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Abstract

A new wave of relativistic nature is derived from the classical Hamilton-Jacobi equation on a curved space-time. The gravitational time dilation in the neighbourhood of an almost point-like mass is responsible for its existence. In order to obtain such type of wave (interpretable as a matter field), one has to resort to an old idea due to de Broglie according to which the physical **3**-dimensional space behaves as if it were covered with an infinity of clocks. The resulting particle field, that propagates in the physical **3**-dimensional space and is due to the interaction with the (classic) gravitational field of the mass, is shown to be associated with the usual scalar particle wave function of quantum mechanics. Therefore, the model here described, by linking Einstein's general relativity to the wave-like behaviour of particles via the viewpoint of de Broglie's Double Solution Theory rather than via the standard mechanisms of field quantisation, provides a new approach to quantum gravity. Finally, it is shown that this model provides a new interpretation of the single-particle interference and explains non-locality in terms of a novel quantum communication channel.

Keywords Quantum gravity · Gravitational time dilation · Hamilton-Jacobi equation on curved space-time · Matter waves · de Broglie's Double Solution Theory

1 Introduction

The seminal de Broglie idea of associating a relativistic point-like material particle with a wave propagating in physical 3-dimensional space is based on regarding the particle as a clock (see [1]), i.e., a purely time-dependent phenomenon $\Psi(t_0) = F\left[\varphi_0^{clock}(t_0)\right]$ relative to the particle proper frame with F being a real-valued function periodic in its argument and $\varphi_0^{clock}(t_0) := \nu_0^{clock} t_0$ with ν_0^{clock} being the clock proper frequency. But, although the clock sits in the particle, the motion of the latter discloses the possibility to imagine that the whole space is covered with an infinity of synchronised clocks in the following way. If the particle moves freely at speed v along the x-axis relative to an observer, substituting $\Psi(t_0)$ with Lorentz' transformation yields

$$\Psi(x, t) = F\left[\frac{\nu_0^{clock}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\left(t + \frac{v}{c^2}x\right)\right] \tag{1}$$

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with c being the speed-of-light in the empty space, that is, a monochromatic plane wave of frequency $\nu := \frac{\nu_0^{clock}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ and phase velocity $\mathcal{V} := \frac{c^2}{v}$ or, equivalently, an infinite distribution of synchronised clocks pointing, in general, to a different time that depends on the x coordinate of the clock position. Thus, to an observer relative to which a relativistic point-like particle equipped with its own clock moves freely the phase distribution of the infinite clocks appears globally as the phase of a travelling plane monochromatic wave which, in turn, may be interpreted as the phase of each clock of the infinite distribution relative to the observer frame. In particular, the presence of a point-like particle can be associated with an infinity of ideal clocks on the physical 3-dimensional space regardless of the point-like particle dynamics. This latter aspect will be hereafter expressed via the so-called *Principle of Clock Distribution: the empty physical* 3-*dimensional space may be imagined as covered with an infinity of point-like clocks, one for each point of space, all synchronised relative to a relativistic point-like particle proper frame*. If the clocks proper frequency is given in terms of the particle proper mass m_0 as $\nu_0^{clock} = \frac{m_0 c^2}{h}$ with h being the Planck constant, the plane monochromatic wave (1) is called *de Broglie's phase-wave* of the freely moving relativistic point-like particle. It is easy to show that such a de Broglie phase-wave keeps in phase with the particle internal clock (see [1, 2]) and, by the Principle of Clock Distribution, with each of the clocks sitting throughout the space. Then, the law of this phase accordance for free particle dynamics reads: for any inertial observer the phase of each ideal clock belonging to the infinite distribution is equal at each instant to the phase of the particle's de Broglie wave calculated at the position of the considered clock. As the total mechanical energy of the particle is

$$W = \frac{m_0 c^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{h\nu_0^{clock}}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = h\nu \tag{2}$$

relative to the observer (i.e., the well-known Planck relation), the wavelength of its de Broglie phase-wave, defined as $\lambda := \frac{\mathcal{V}}{\nu}$, is given by

$$\lambda = \frac{c^2}{v}\frac{h}{W} = \frac{m_0 c^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{m_0 v}\frac{h}{W} = \frac{h}{|\vec{p}|} \tag{3}$$

with $\vec{p} := \frac{m_0 \vec{v}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ being the kinetic 3-momentum of the particle relative to the observer. Thus, imagining an infinite number of tiny clocks placed at every point of space and pertaining to a massive point-like particle as established by the Principle of Clock Distribution leads to the celebrated de Broglie relation (3) by deploying the classical notions of relativistic mechanics as long as the particle moves freely. Substituting (1) with (2) and (3) the de Broglie phase-wave of the freely moving relativistic point-like particle takes on the well-known form in terms of particle 4-momentum usually referred to as the matter wave. However, as (3) cannot be derived from the Lorentz transformation for a general particle dynamics, the de Broglie relation is obtained in this case assuming that the phase velocity of the de Broglie phase-wave is given by $\mathcal{V}(\vec{r}, t) = \frac{W(\vec{r},t)}{|\vec{p}(\vec{r},t)|}$ with W and $\vec{p}$ denoting, respectively, the total mechanical energy and the kinetic 3-momentum that the relativistic point-like particle would have if it were placed at point $\vec{r}$ at time t (see [3]). For a particle's de Broglie phase-wave with phase φ such a form of $\mathcal{V}$ turns out to coincide with the usual phase velocity definition $\mathcal{V} := \frac{\frac{\partial \varphi}{\partial t}}{|\nabla \varphi|}$ provided that an appropriate relationship between φ and the Hamilton principal function S of Jacobi's theory be established. Notice that the form of $\mathcal{V}$ involving the particle kinetic 4-momentum as required here in order to extend (3) to a particle having a general dynamics deviates from the one given by de Broglie which was derived from the postulated proportionality between the 4-wave vector of the matter wave and the canonical 4-momentum of the particle (with proportionality constant h) and from which (3) was obtained in the absence of the electromagnetic field (see [3]). Despite this difference, the approach followed here permits to retrieve the matter wave postulated by de Broglie as a factorisation of the aforementioned particle's de Broglie phase-wave of relativistic nature and a phase factor dependent on the 4-potential. For instance, if the relativistic point-like particle with electric charge q moves in an electromagnetic field with 4-potential (A_0, A_1, A_2, A_3), the particle's de Broglie phase-wave with phase $\varphi := \frac{S}{h} + \frac{q}{hc}\int A_\alpha dx^\alpha$ (S, referred to as the Hamilton principal function, is a complete integral of the system Hamilton-Jacobi equation whose 4-gradient gives the particle canonical 4-momentum) has certainly the frequency and the wavelength given by, respectively, Planck's relation and de Broglie's relation for (2) and (3) can ensue also from the usual general definition of phase velocity provided that the phase have this form. In addition to this, also the law of phase accordance between the ideal clocks of the infinite distribution and the particle's de Broglie phase-wave can be assumed to hold when the particle moves in a general field of force.[1] Then, if in line with the case of free particle dynamics also in this case φ is obtained from the system Hamilton principal function such that both the Planck and the de Broglie relations keep holding good, the law of phase accordance between the clocks of the infinite distribution and the de Broglie phase-wave follows on. Since this amounts to saying that the point-like particle's de Broglie phase-wave at a given point has 4-wave vector proportional to the kinetic 4-momentum as if the particle were placed at that point (see [3]), one could summarise the foregoing arguments in the so-called *Principle of Phase Harmony*[2]*: the ideal clock of the infinite distribution sitting at point $\vec{r}$ and associated with a massive relativistic point-like particle keeps in phase with the particle's de Broglie phase-wave whose phase at point $\vec{r}$ and instant t is proportional (with proportionality constant h) to the system Hamilton principal function plus a minimal coupling term evaluated at $\vec{r}$ and t relative to the observer frame*. For the sake of clarity, the two terms forming the phase of the particle's de Broglie phase-wave as stated in the Principle of Phase Harmony must be interpreted as follows:

1. at any given time the term proportional to the system Hamilton principal function takes on different values depending on the observation point of space and its 4-gradient at $\vec{r}$ equals the canonical 4-momentum of the particle itself as if this at the instant considered were placed at point $\vec{r}$;
2. the minimal coupling term is non-zero only if the particle is electrically charged and moves in an electromagnetic field.

Against the backdrop of the foregoing arguments, the present work shows that it is possible to link Einstein's general relativity with de Broglie's ideas concerning the wave-particle

[1] For instance, this is possible if the proper phase of each ideal clock belonging to the infinite distribution reads $\nu_0^{clock} t_0 + k_0$ (the proper frequency ν_0^{clock} and the initial proper phase k_0 being the same for all clocks) and, consequently, the phase of the clock at $\vec{r}$ relative to the observer frame is given by $\varphi(\vec{r}, t) = \nu_0^{clock} t_0(\vec{r}, t) + k_0 = \nu_0^{clock} t_0 + k_0$ where t_0 denotes the clock proper time and t the time measured by it relative to the observer frame. In order that this clock have a definite frequency (whence it represents a purely periodic phenomenon typical of an actual clock) also relative to the observer frame, φ must be linear in t.

[2] This principle is related to a so-called 'phase harmony' condition postulated by de Broglie, whence the choice of its name. Lately, a specification of de Broglie's original 'phase harmony' condition has been deployed to lock a solitonic wave and a guiding field (see [4])

dualism which, overall, have been collected in the so-called Double Solution program, hereon referred to as de Broglie's Double Solution Theory. The model presented in the following represents an extension of this latter theory due to the introduction of gravity (from a classical viewpoint) at the particle micro-scale. In particular, such original approach describes the particle wave function by relating it to the temporal structure of space-time while in most interpretations of quantum mechanics the particle field is only a formal means for calculating physically relevant quantities with no interest for its very origin. Thanks to the relationship between the wave-like aspect of the particle and the clock distribution the model here proposed reconciles a fundamental quantum feature with a relativistic one and might thus be relevant to develop a theory of quantum gravity from a new perspective. In addition, it provides with a new description of the single-particle interference and of non-locality. The paper is organised as follows. In Section 2 the construction of the de Broglie phase-wave of the relativistic point-like particle in a curved space-time with spherically symmetric metric tensor is approached deploying the notion of gravitational time dilation inherited by Einstein's general relativity. In Section 3 the variational method for the dynamics of a relativistic point-like particle in a curved space-time is recalled to establish the evolution equation for the particle de Broglie phase-wave given at all times by the classical Hamilton-Jacobi equation on a curved space-time (see [5]). In accord with Einstein's field equations, any field present in the system (included the ones associated with the potential measurement devices) contributes to the space-time geometry at the particle position and on its vicinity that, in turn, affects the Hamilton-Jacobi equation. In Section 4 the phase of the particle de Broglie phase-wave in a curved space-time is expressed in terms of 4-momentum of the relativistic point-like particle whereas in the following section it is shown that, under certain conditions, this phase becomes complex-valued with non-zero imaginary part. At variance with the usual methods of field quantisation (e.g., canonical quantisation, path integral quantisation, geometric quantisation), the model here described deploys the point of view of de Broglie's Double Solution Theory sketched in Section 6.1 according to which the particle is regarded as a lump of energy (a so-called soliton or, perhaps, a solitary wave) smoothly nestled in an extended wave phenomenon that guides the former along the propagation of the latter in the physical 3-dimensional space (see [6]). Notice, for the sake of argument, that Bohmian Mechanics shares this guiding feature with de Broglie's Double Solution Theory albeit the well-known differences between these two theories in the interpretation of the wave fields (see [7]). The formal scheme of the model developed in Sections 3-5 for the point-like particle in a curved space-time is used in Section 7 to construct the de Broglie phase-wave of the particle conceived as a tiny object, though not strictly point-like. Moreover, an explicit form for the global wave is provided and it is shown that the model is a possible realisation of de Broglie's Double Solution Theory (see [6]). Recently, a global wave representing the particle in this context has been shown to exist in the form of a peaked soliton due to the self-focusing nature of self-gravitation under the assumption that the evolutionary wave equation is the so-called Schrödinger-Newton equation (see [8]) whose non-linear term might be responsible for the wave function collapse (see [9–11]). The relationship of proportionality, as postulated by de Broglie, between the particle de Broglie phase-wave and the classical Klein-Gordon field is discussed in Section 8 and leads to justify in a natural way the Principle of Interference (also called Born's rule): 'the probability that an observation will permit the localisation of a particle is proportional to the square of the particle wave function amplitude' (see [6]). Within the framework of the model here presented, the claimed proportionality between the two types of wave is explained appealing to the notion of curved space-time in the neighbourhood of matter, so underscoring how Einstein's general relativity can be made compatible with quantum mechanics. Likewise, one can carry over the connection between the particle statistics for the position observable and the u-wave to the case of a many-particle system, provided that the system u-wave be constructed from the u-wave of each single particle in the system. As for the statistics relative to any observable other than the position, this will result ultimately from measuring the position (see [2]), in accord with the priority given in de Broglie's theory of measurement to the 'Principle of Interference' and the position in the physical 3-dimensional space over the other observables. In Section 9 the single-particle interference and the possible existence of a superluminal quantum communication channel are described in terms of the introduced model. Like in the Bohmian approach, in these phenomena the measurement devices or observers are considered to be part of the physical system that bring on no collapse of the wave function, whence the interpretation here adopted turns out to be free from the so-called problem of time arising in quantum mechanics and present in all the quantum gravity theories based on it (see [12]).

2 Clock Phase Distribution in a Curved Space-time

By Einstein's general relativity, the pace of a clock is affected by the gravitational field due to the presence of mass in its neighbourhood, no matter how small the mass. In particular, each clock postulated by the Principle of Clock Distribution follows suit. Furthermore, if a point-like particle has spherical symmetry (meaning that the space-time in the neighbourhood of the particle is spherically symmetric), one can always assume that the components g_{01}, g_{02}, g_{03} of the space-time metric tensor $\left(g_{\alpha\beta}\right)_{\alpha\beta=0,1,2,3}$ be zero, whence the space-time interval between the events $\mathbf{x}_{in} = (ct_{in}, \vec{r}_{in})$ and $\mathbf{x} = (ct, \vec{r}) = \mathbf{x}_{in} + d\mathbf{x}$ with $d\mathbf{x} = \left(dx^0, dx^1, dx^2, dx^3\right) = (cdt, d\vec{r})$ can be cast in the form

$$ds^2 = g_{\alpha\beta}\left(\mathbf{x}\right) dx^\alpha dx^\beta := g_{00}\left(\mathbf{x}\right) dx^0 dx^0 + \sum_{h,k=1}^{3} g_{hk}\left(\mathbf{x}\right) dx^h dx^k \tag{4}$$

(see [13]). On the other hand, since ds^2 is a scalar, Einstein's Equivalence Principle ensures that (4) takes on the form

$$ds^2 = \eta_{\alpha\beta} dx_0^\alpha dx_0^\beta = c^2 \left(dt_0\right)^2 - \sum_{h=1}^{3} \left(dx_0^h\right)^2$$

with $\left(\eta_{\alpha\beta}\right)_{\alpha,\beta=0,1,2,3}$ being the Minkowski metric tensor relative to a locally inertial frame with respect to which the event $\mathbf{x}$ is denoted by $\mathbf{x}_0$. Hence,

$$\left(\frac{dt_0}{dt}\right)^2 - \frac{1}{c^2}\sum_{h=1}^{3}\left(\frac{dx_0^h}{dt}\right)^2 = g_{00}\left(ct, \vec{r}\right)\left\{1 - \left[\frac{\dot{l}\left(\vec{r}, \dot{\vec{r}}, t\right)}{\gamma\left(ct, \vec{r}\right)}\right]^2\right\} \tag{5}$$

with $\dot{l}\left(\vec{r}, \dot{\vec{r}}, t\right) := \sqrt{-\sum_{h,k=1}^{3} g_{hk}\left(ct, \vec{r}\right) \dot{x}^h\left(ct, \vec{r}\right) \dot{x}^k\left(ct, \vec{r}\right)}$ being the so-called *curvilinear 3-speed* of a clock at $\vec{r}$ with 3-velocity $\dot{\vec{r}}\left(\mathbf{x}\right) = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right)$ relative to the observer frame and $\gamma\left(ct, \vec{r}\right) := c\sqrt{g_{00}\left(ct, \vec{r}\right)}$ being the so-called *curvilinear speed-of-light* (see [3]). Then, as $\frac{dx_0^h}{dt} = \frac{dx_0^h}{dt_0}\frac{dt_0}{dt}$ for $h = 1, 2, 3$ and the considered clock is at rest relative to its own locally inertial frame it turns out from (5) that

$$dt_0 = \sqrt{g_{00}(ct,\vec{r})}\sqrt{1-\left[\frac{\dot{l}\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2}\,dt,$$

whence, if any clock sitting in the vicinity of a material point-like particle associated with a gravitational potential $\left(g_{\alpha\beta}\right)_{\alpha,\beta=0,1,2,3}$ has proper frequency ν_0^{clock}, one can define the quantity

$$\nu^{clock}(ct,\vec{r}) := \nu_0^{clock}\sqrt{g_{00}(ct,\vec{r})}\sqrt{1-\left[\frac{\dot{l}\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2} \tag{6}$$

at any event $\mathbf{x} = (ct,\vec{r})$ of the space-time $\mathcal{M}$ relative to the observer frame. For, in general, ν^{clock} depends on t through the metric tensor and the clock 3-velocity, it cannot be interpreted as a true frequency and it is thus called the *pseudo-frequency* of the clock at $\mathbf{x}$ relative to the observer frame. However, if $\left(g_{\alpha\beta}\right)_{\alpha,\beta=0,1,2,3}$ is such that the right-hand side of (6) is explicitly independent of time, ν^{clock} represents the true frequency of the clock at any point of the physical 3-dimensional space relative to the observer frame. The existence of such a space-time metric tensor is possible up to a suitable choice of a reference frame, at least in a neighbourhood of the considered event (see [14]). In particular, in the present case the four components of $\left(g_{\alpha\beta}\right)_{\alpha,\beta=0,1,2,3}$, though constrained to satisfy the above condition for the existence of a true clock frequency,[3] can still be chosen arbitrarily. Then, it turns out from (6) that the clock frequency $\nu_0^{clock}\sqrt{g_{00}}$ relative to the point-like particle proper frame[4] is the same for all the clocks in the flat portion of space-time whereas it differs smoothly for the clocks sitting in the curved portion of the space-time. In other words, the infinity of ideal clocks strewn in the physical 3-dimensional space according to the Principle of Clock Distribution, in phase with one another if sitting far from the gravitational field, run smoothly out of phase if lying in a region where a gravitational field is present. In particular, if $\nu_0^{clock} := \frac{m_0 c^2}{h}$, the frequency of the ideal clock at $\vec{r}$ relative to the observer frame reads

$$\nu^{clock}(\vec{r}) \overset{(6)}{=} \frac{m_0^{(g)}(ct,\vec{r})\left[\gamma(ct,\vec{r})\right]^2}{h}\sqrt{1-\left[\frac{\dot{l}\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2} \tag{7}$$

with $m_0^{(g)}(ct,\vec{r}) := \frac{m_0}{\sqrt{g_{00}(ct,\vec{r})}}$ being the so-called *curvilinear rest-mass* of the point-like particle relative to the observer frame (see [3]). Finally, by the Principle of Phase Harmony, the particle de Broglie phase-wave in a generally curved space-time is obtained from the phase distribution of the clocks hypothesised by the Principle of Clock Distribution whose frequency is given by (7).

[3] For instance, such condition is verified if both the space-time metric tensor and the 4-potential are static because in this case it follows from the geodesic equations that the 4-velocity at the event $(ct,\vec{r})$ is explicitly independent of time, whence the 3-velocity of the clock at point $\vec{r}$ is constant.

[4] In order that $\nu_0^{clock}\sqrt{g_{00}}$ be a true frequency the space-time metric tensor must be static relative to the point-like proper frame.

3 Relativistic Point-like Particle Dynamics in a Curved Space-time

Let consider the system made up of a relativistic point-like particle with proper mass m_0 and electric charge q moving in a curved space-time with metric tensor $\left(g_{\alpha\beta}\right)_{\alpha,\beta=0,1,2,3}$ as in (4) and acted upon by an electromagnetic field with 4-potential $\left(A^0, A^1, A^2, A^3\right) = \left(\phi, \vec{A}\right)$. Given the Lagrangian

$$\bar{L}\left[\vec{r}(t), \dot{\vec{r}}(t), t\right] := -m_0^{(g)}\left[ct, \vec{r}(t)\right]\left\{\gamma\left[ct, \vec{r}(t)\right]\right\}^2 \sqrt{1 - \left\{\frac{i\left[\vec{r}(t), \dot{\vec{r}}(t), t\right]}{\gamma\left[ct, \vec{r}(t)\right]}\right\}^2} - q g_{00}\left[ct, \vec{r}(t)\right]\phi\left[ct, \vec{r}(t)\right] - \frac{q}{c}\sum_{h,k=1}^{3} g_{hk}\left[ct, \vec{r}(t)\right] A^h\left[ct, \vec{r}(t)\right]\dot{x}^k(t) \tag{8}$$

for such a system, the conjugate momenta are

$$p_h(t) := \frac{\partial \bar{L}}{\partial \dot{x}^h}\left[\vec{r}(t), \dot{\vec{r}}, t\right]_{|\dot{\vec{r}}=\dot{\vec{r}}(t)} = -\frac{m_0^{(g)}\left[ct, \vec{r}(t)\right]\sum_{k=1}^{3} g_{hk}\left[ct, \vec{r}(t)\right]\dot{x}^k(t)}{\sqrt{1 - \left\{\frac{i\left[\vec{r}(t), \dot{\vec{r}}(t), t\right]}{\gamma\left[ct, \vec{r}(t)\right]}\right\}^2}} - \frac{q}{c}\sum_{k=1}^{3} g_{hk}\left[ct, \vec{r}(t)\right] A^k\left[ct, \vec{r}(t)\right]$$

for $h = 1, 2, 3$ and the Hamiltonian of the system reads

$$\bar{H}\left[\vec{r}(t), p^1(t), p^2(t), p^3(t), t\right] = q g_{00}\left[ct, \vec{r}(t)\right]\phi\left[ct, \vec{r}(t)\right] + \gamma\left[ct, \vec{r}(t)\right] \times \sqrt{(m_0 c)^2 - \sum_{h,k=1}^{3} g^{hk}\left[ct, \vec{r}(t)\right]\left\{p_h(t) + \frac{q}{c}A_h\left[ct, \vec{r}(t)\right]\right\}\left\{p_k(t) + \frac{q}{c}A_k\left[ct, \vec{r}(t)\right]\right\}}$$

with $p_h := g_{hk}p^k$ and $A_h := g_{hk}A^k$ for $h = 1, 2, 3$. Then, applying Jacobi's Theory and writing the 4-gradient in tensor notation $(\partial_0, \partial_1, \partial_2, \partial_3) := \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ one derives the system Hamilton-Jacobi equation

$$g^{\alpha\beta}(\mathbf{x})\left[\partial_\alpha \psi(\mathbf{x}) + \frac{q}{c}A_\alpha(\mathbf{x})\right]\left[\partial_\beta \psi(\mathbf{x}) + \frac{q}{c}A_\beta(\mathbf{x})\right] = (m_0 c)^2, \tag{9}$$

which is manifestly form-invariant for change of reference frame. If S is a complete integral of (9), the conjugate momenta of the system are, by Jacobi's theory, the components of the canonical 3-momentum of the point-like particle given by

$$\frac{\partial S}{\partial x^h}(\vec{r},t) = p_h(t) = -\frac{m_0^{(g)}(ct,\vec{r})\sum_{k=1}^{3} g_{hk}(ct,\vec{r})\,\dot{x}^k(ct,\vec{r})}{\sqrt{1-\left[\frac{\dot{l}\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2}} - \frac{q}{c}\sum_{k=1}^{3} g_{hk}(ct,\vec{r})\,A^k(ct,\vec{r}) \tag{10}$$

for $h = 1, 2, 3$. Then, substituting (10) for (9) yields

$$\frac{\partial S}{\partial t}(\vec{r},t) + q g_{00}(ct,\vec{r})\,\phi(ct,\vec{r}) = -\frac{m_0^{(g)}(ct,\vec{r})\left[\gamma(ct,\vec{r})\right]^2}{\sqrt{1-\left[\frac{\dot{l}\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2}} \tag{11}$$

whose right-hand side is the so-called *curvilinear total mechanical energy* of the relativistic point-like particle, i.e., the total mechanical energy[5] of the relativistic point-like particle with curvilinear rest-mass $m_0^{(g)}$ moving with curvilinear 3-speed $\dot{l}$ in a curved space-time where the light propagates at curvilinear speed γ relative to the observer frame (see [3]).

4 The Point-like Particle de Broglie Phase-wave

By Einstein's general relativity, any mass, no matter how small, causes the space-time to curve around it. This might be interpreted saying that the mass experiences the space-time curvature produced by its very presence. Consequently, the lagrangian of a system made up of a relativistic point-like particle with spherical symmetry, proper mass m_0 and electric charge q moving in an electromagnetic field with 4-potential (A_0, A_1, A_2, A_3) will be given by (8) with $\left(g_{\alpha\beta}\right)_{\alpha,\beta=0,1,2,3}$ being the metric tensor associated with m_0 (e.g., the Schwarzschild metric). Then, if one succeeds in finding a complete integral of (9), the phase of the de Broglie phase-wave, which, by the Principle of Phase Harmony, reads

$$\varphi(\vec{r},t) := \frac{S(\vec{r},t)}{h} + \frac{1}{hc}\int_{\Lambda(ct_{in},\vec{r}_{in};ct,\vec{r})} q[c g_{00}\left(ct',\vec{r}'\right)\phi\left(ct',\vec{r}'\right)dt' + \sum_{h,k=1}^{3} g_{hk}\left(ct',\vec{r}'\right)A^h\left(ct',\vec{r}'\right)dx'^k], \tag{12}$$

with $\Lambda(ct_{in},\vec{r}_{in};ct,\vec{r})$ denoting a world-line between $(ct_{in},\vec{r}_{in})$ and $(ct,\vec{r})$, turns out to be defined on the whole of the space-time because it can be written in terms of 4-gradient of S with components given by (10) and (11), that is, the canonical 4-momentum of the relativistic point-like particle all the way from where the space-time is flat to the immediate proximity of the point-like particle where it is curved[6]. Given such a complete integral, the phase variation

[5] Up to the sign.

[6] To be precise, (12) is defined throughout the space-time if the metric tensor has no singularities.

of the point-like particle's de Broglie phase-wave reads

$$\begin{aligned} d\varphi\left(\vec{r},t\right) &\overset{(12)}{=} \frac{dS\left(\vec{r},t\right)}{h}+\frac{q}{hc}A_{\alpha}\left(\mathbf{x}\right)dx^{\alpha} \\ &\overset{(10),(11)}{=} -\frac{m_0^{(g)}\left(ct,\vec{r}\right)\left[\gamma\left(ct,\vec{r}\right)\right]^2}{h}\sqrt{1-\left[\frac{l\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma\left(ct,\vec{r}\right)}\right]^2}\,dt, \end{aligned} \tag{13}$$

that is, the phase variation of the ideal clock from the infinite distribution sitting at $\vec{r}$ with frequency (7) relative to the observer frame[7]. Thus, the complete integral of (9) satisfying (10) and (11) generalises to the general dynamics of the particle the proof of phase agreement between the phase-wave associated with a point-like particle and the internal clock of the particle obtained by de Broglie for the freely moving particle, so justifying the Principle of Phase Harmony.

5 Explicit Form of the Point-like Particle de Broglie Phase-wave

The de Broglie phase-wave of the point-like particle has been dealt with so far only for formal purposes to help represent the physical particle, which instead will be conceived as an entity extended over space (see Section 7.1). In particular, in order to describe the internal structure of the particle, let express a complete integral of (9) in a more explicit form. To this scope, one starts out re-writing the space-time interval (4) in the form

$$\begin{aligned} ds^2 &= \gamma\left(ct,\vec{r}\right)dt^2-\left[l\left(\vec{r},\dot{\vec{r}},t\right)\right]^2dt^2 \\ &= \left[\gamma\left(ct,\vec{r}\right)\right]^2\left\{1-\left[\frac{l\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma\left(ct,\vec{r}\right)}\right]^2\right\}dt^2, \end{aligned} \tag{14}$$

where the inverse square root of the quantity within curly brackets generalises to a curved space-time the notion of Lorentz contraction factor usually defined on a flat space-time and is thus called *curvilinear Lorentz contraction factor*. This factor can be re-written as

$$\left\{1-\left[\frac{l\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma\left(ct,\vec{r}\right)}\right]^2\right\}^{-\frac{1}{2}} = \pm\frac{\gamma\left(ct,\vec{r}\right)dt}{\sqrt{ds^2}} = \pm\frac{\gamma\left(ct,\vec{r}\right)dt}{\sqrt{\operatorname{sgn}\left(ds^2\right)d\xi^2}} \tag{15}$$

with ξ being, by definition, the real arc-length parameter of $\Lambda\left(ct_{in},\vec{r}_{in};ct,\vec{r}\right)$ such that $ds^2=\operatorname{sgn}\left(ds^2\right)d\xi^2$. Let at first consider just the portion of the spherically symmetric curved space-time

$$\mathcal{M}^{int} := \left\{\mathbf{x}\in\mathcal{M}|ds^2 := g_{\alpha\beta}\left(\mathbf{x}\right)dx^{\alpha}dx^{\beta}<0\right\}$$

[7] At any point of space $\vec{r}$ which is kept fixed one has $d\vec{r}=0$, whence $d\nu^{clock}\left(\vec{r}\right)=\nabla\nu^{clock}\left(\vec{r}\right)\cdot d\vec{r}=0$.

with the metric tensor as in (4). Since $d\xi^2 > 0$, a complete integral of (9) takes on the form

$$S^{\pm}(\vec{r}, t) = \begin{cases} S(\vec{r}_{in}, t_{in}) \pm c \int_{\xi_{in}}^{\bar{\xi}} m_0 d\xi' \pm ic \int_{\bar{\xi}}^{\xi} m_0 d\xi' \\ -\frac{1}{c} \int_{\Lambda(\mathbf{x}_{in};\mathbf{x})} q g_{\alpha\beta} \left(\mathbf{x}'\right) A^{\alpha} \left(\mathbf{x}'\right) dx'^{\beta}, & \mathbf{x} \in \mathcal{M}^{int} \\ S(\vec{r}_{in}, t_{in}) \pm c \int_{\xi_{in}}^{\xi} m_0 d\xi' \\ -\frac{1}{c} \int_{\Lambda(\mathbf{x}_{in};\mathbf{x})} q g_{\alpha\beta} \left(\mathbf{x}'\right) A^{\alpha} \left(\mathbf{x}'\right) dx'^{\beta}, & \mathbf{x} \in \mathcal{M} \setminus \mathcal{M}^{int} \end{cases} \tag{16}$$

where $\mathbf{x}_{in} = \mathbf{x}'(\xi_{in})$ is supposed to be in $\mathcal{M} \setminus \mathcal{M}^{int}$, $\bar{\mathbf{x}} = \mathbf{x}'\left(\bar{\xi}\right)$ is the point where $\Lambda(\mathbf{x}_{in}; \mathbf{x})$ intersects $\partial\mathcal{M}^{int}$, $\mathbf{x} = \mathbf{x}'(\xi)$[8] and $S(\vec{r}_{in}, t_{in})$ is real-valued.[9] In other words, the gravitational field is responsible for $S^{\pm}$ having a non-zero imaginary part on $\mathcal{M}^{int}$ whereas $S^{\pm}$ is purely real-valued only on $\mathcal{M} \setminus \mathcal{M}^{int}$.

6 Main Features of de Broglie's Double Solution Theory

Before applying the foregoing result concerning a complete integral of (9) to a particle's de Broglie phase-wave, it is worthwhile to recall some fundamental aspects of de Broglie's Double Solution Theory, an unfinished wave-monistic program aimed at clarifying the wave-particle dualism that, conceived to extend the usual quantum mechanics, must agree with this latter theory predictions within a certain domain of application.

6.1 Single Particle Case

In accordance with de Broglie's Double Solution Theory, for any given particle there exist two different waves:

1. the usual complex-valued particle wave function whose squared amplitude at $\mathbf{x}$ represents the probability that the point-like particle be found at $\vec{r}$ at time t and which has only a subjective character;[10]
2. the complex-valued so-called 'u-wave' (denoted with u) of objective character, sometimes also referred to as matter wave, which, unlike the wave function, depends only on the intrinsic features of the particle and propagates in the physical 3-dimensional space.

While the first wave evolves according to a suitable linear wave equation (e.g., the linear Schrödinger equation for a non-relativistic particle, the linear Klein-Gordon equation for a relativistic zero-spin particle, the linear Dirac equation for a relativistic half-spin particle, etc.), the second one is supposed to obey this same equation with, in addition, a non-linear term whose contribution is paramount within the so-called 'singular region' where the wave amplitude is very large as well as at the edges of the wave front. As the 'singular region'

[8] By inverting $\mathbf{x} = \mathbf{x}'(\xi)$, it turns out that $\xi = \xi'(\mathbf{x})$ and $\bar{\xi} = \xi'(\bar{\mathbf{x}}) = \bar{\xi}(\mathbf{x})$ as $\bar{\mathbf{x}}$ depends on $\mathbf{x}$ through $\Lambda(\mathbf{x}_{in}; \mathbf{x})$.

[9] In accordance with the Principle of Phase Harmony, the dependence of $S^{\pm}$ on $\vec{r}$ and t has the following interpretation: at any given time t the ideal clock of the infinite distribution at point $\vec{r}$ keeps in phase with the de Broglie phase-wave of a (potentially electrically charged) point-like particle of proper mass m_0 that follows the world-line $\Lambda(ct_{in}, \vec{r}_{in}; ct, \vec{r})$ for which $\int_{\Lambda(ct_{in},\vec{r}_{in};ct,\vec{r})} m_0 ds$ is stationary.

[10] Within the framework of the Double Solution Theory the range of possible values rather than a single value associated with a measurement is ascribed to the random perturbation of the system occurring when the measurement is performed and represents only the incapability of the experimenter to reveal prior the measurement a feature that remains, after all, immanent in the particle.

is tiny, the particle can be considered point-like as far as its dynamics is concerned, this latter being a feature that represents only an approximation in the Double Solution Theory (the so-called Pilot-Wave Theory) whereas it holds with exactness in the usual quantum mechanics theory. Beside the 'singular region' the Double Solution Theory postulates the existence of a so-called 'intermediate region' (see [6]), where the amplitude of the u-wave begins to change very rapidly over space and time and the wave equation for the u-wave is only approximately equal to the linear one commonly considered in the standard theory,[11] and, finally, the existence of a so-called 'external region' (see [6]), where the u-wave is proportional to the particle wave function to a good approximation and the wave equation for the u-wave is the same as the usual linear one. If Ψ denotes the point-like particle wave function referring to the particle localisation probability after actual detection, the de Broglie Double Solution Theory maintains that the particle u-wave can be decomposed as follows:

1. $u = u_0 + v$ with $|u_0| >> |v|$, whence $u \approx u_0$, on the 'singular region';
2. $u \approx u_0 + v$ with $v \approx K\Psi$ by reason of the already rapid increase of u_0 on the 'intermediate region';
3. $u, u_0 \approx 0$ and $v = K\Psi$, whence $u \approx K\Psi$, with K being a complex constant dependent on the particle on the 'external region'.

Notice that in more recent approaches to the Double Solution Theory the original decomposition of the u-wave as superposition of u_0 and v has been replaced by a 'factorisation ansatz' involving two waves of different nature (see [15, 16]). As for the proportionality between Ψ and v on the 'external region' mentioned before, arguments in favour of this hypothesis based upon the notion of ergodicity date back to de Broglie (see [6, 17]) whereas a more detailed analysis about the attendant origin of Born's rule was carried out by Bohm and collaborators via the introduction of a so-called 'sub-quantum medium' responsible for randomly forcing the system to reach an equilibrium state (see [7, 18, 19]). Finally, in recent times several other researchers have faced up to the emergence of Born's rule deploying different strategies, especially within the framework of Bohmian Mechanics, (see [20]) either with no appeal to statistical arguments (see [21]) or, to the contrary, calling for statistical assumptions based on the notion of equivariance (see [22–25]), calling on Gleason's theorem (see [26]) or using the H-theorem without the postulate of sub-quantum fluctuations (see [26, 27]). To conclude this brisk account of the de Broglie Double Solution Theory, one should pay heed to how the proportionality between Ψ and v contributes to eliminate the following paradox that was underscored by Pauli to dismiss the Pilot-Wave Theory: the particle guidance formula, obtained from the continuity equation which is derived, in turn, from the linear wave equation for the mere particle wave function without introducing the u-wave, shows that the dynamics of a physical entity such as the particle is determined by the phase which, in turn, depends on the amplitude of the fictitious probability wave. However, Pauli's objection is removed, if there exists a u-wave supposed to solve, to a good approximation, the linear wave equation on a closed 3-surface surrounding the 'singular region'. In order to guide the particle along its motion, the particle u-wave must be a solution of the linear wave equation, to a good approximation, on such a 3-surface which must then lie in the 'intermediate region'.[12] The quasi-proportionality between the u-wave and the wave function on the 'intermediate region' is a sufficient condition to ensure these features and, consequently, that the particle velocity depend ultimately on the phase and the amplitude of the physical u-wave (see [6]). Once again, it is worth stressing that de Broglie's Double Solution Theory

[11] Though, still slightly non-linear.

[12] In addition, the size of the 'singular region' is very small compared to the local wavelength.

is expected to provide the same results as the standard quantum theory as far as the ‘external region’ of the particle is concerned but that, on the contrary, differences between the two theories appear at the level of the region occupied by the particle which, being regarded as strictly point-like, is beside the mark for the latter theory whereas, being supposed to occupy a tiny, though spatially extended, portion of the physical 3-dimensional space, represents a major aspect of investigation for the former theory.

6.2 Many-particle Case

In the case of a many-particle system, de Broglie’s Double Solution Theory establishes a relationship between the wave function of each single particle in the system associated with the particle u-wave defined on the physical 3-dimensional space and the system wave function Ψ defined on the configuration space usually considered in the standard quantum theory (see [6, 28–31]). For the sake of simplicity, let consider the case of a two-particle system, the step from the two-particle case to that of N particles with $N > 2$ not entailing any difficulty of principle. Furthermore, let assume that the particles move at non-relativistic speed relative to an observer, whence their wave functions Ψ_1 and Ψ_2 associated with the two single particle u-waves obey the linear Schrödinger equation, as far as each particle’s ’external region’ is concerned. If every point in the physical 3-dimensional space is determined by the radius 3-vector $\vec{r}$ from the origin to the given point, $\vec{R}^{(h)}$ denotes the position 3-vector of particle h, m_h is the mass of particle h (for $h = 1, 2$) and the single particle wave functions have the form

$$\Psi_1\left(\vec{r}, \vec{R}^{(2)}, t\right) = a_1\left(\vec{r}, \vec{R}^{(2)}, t\right) e^{\frac{i}{\hbar} S_1\left(\vec{r}, \vec{R}^{(2)}, t\right)},$$

$$\Psi_2\left(\vec{r}, \vec{R}^{(1)}, t\right) = a_2\left(\vec{r}, \vec{R}^{(1)}, t\right) e^{\frac{i}{\hbar} S_2\left(\vec{r}, \vec{R}^{(1)}, t\right)},$$

the equations constraining the motion of the particles are the Hamilton-Jacobi equations obtained as the real part of the linear Schrödinger equation for the corresponding particle after substituting it with Ψ_h for $h = 1, 2$, i.e.,

$$\frac{\partial S_1}{\partial t}\left(\vec{R}^{(1)}, \vec{R}^{(1)} - \vec{R}^{(2)}, t\right) = \frac{1}{2m_1}\left|\nabla S_1\left(\vec{R}^{(1)}, \vec{R}^{(1)} - \vec{R}^{(2)}, t\right)\right|^2 + F_1\left(\vec{R}^{(1)}, t\right) + F_{12}\left(\vec{R}^{(1)} - \vec{R}^{(2)}\right) + \frac{\hbar^2}{2m_1}\frac{\Box a_1\left(\vec{R}^{(1)}, \vec{R}^{(1)} - \vec{R}^{(2)}, t\right)}{a_1\left(\vec{R}^{(1)}, \vec{R}^{(1)} - \vec{R}^{(2)}, t\right)}$$

at the particle 1 position $\vec{R}^{(1)}$ and

$$\frac{\partial S_2}{\partial t}\left(\vec{R}^{(2)}, \vec{R}^{(1)} - \vec{R}^{(2)}, t\right) = \frac{1}{2m_2}\left|\nabla S_2\left(\vec{R}^{(2)}, \vec{R}^{(1)} - \vec{R}^{(2)}, t\right)\right|^2 + F_2\left(\vec{R}^{(2)}, t\right) + F_{21}\left(\vec{R}^{(1)} - \vec{R}^{(2)}\right) + \frac{\hbar^2}{2m_2}\frac{\Box a_2\left(\vec{R}^{(2)}, \vec{R}^{(1)} - \vec{R}^{(2)}, t\right)}{a_2\left(\vec{R}^{(2)}, \vec{R}^{(1)} - \vec{R}^{(2)}, t\right)}$$

at the particle 2 position $\vec{R}^{(2)}$, where F_h represents an external potential that may possibly act upon particle h (for $h = 1, 2$) and F_{12} (resp., F_{21}) represents the action of particle 2 (resp., particle 1) on particle 1 (resp., particle 2). Moreover, if the system wave function has the form

$$\Psi\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right) = a\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right) e^{\frac{i}{\hbar} S\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right)},$$

it can be shown that

$$\begin{cases} S\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right) = S_1\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right) + S_2\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right) - S_{12}\left(\vec{R}^{(1)} - \vec{R}^{(2)}, t\right) \\ \frac{\hbar^2}{2}\left[\frac{1}{m_1}\frac{\Box_1 a\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right)}{a\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right)} + \frac{1}{m_2}\frac{\Box_2 a\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right)}{a\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right)}\right] = \frac{\hbar^2}{2m_1}\frac{\Box a_1\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right)}{a_1\left(\vec{R}^{(1)}, \vec{R}^{(2)}, t\right)} \\ + \frac{\hbar^2}{2m_2}\frac{\Box a_2\left(\vec{R}^{(2)}, \vec{R}^{(1)}, t\right)}{a_2\left(\vec{R}^{(2)}, \vec{R}^{(1)}, t\right)} - Q_{12}\left(\vec{R}^{(1)} - \vec{R}^{(2)}, t\right) \end{cases}$$

where $\Box_h$ is the d'Alembertian relative to the particle h position coordinates.[13] Here, the particles are treated as point-like singularities and these relationships between the phases and the amplitudes of the wave functions must hold at the points where the particles are positioned, the underlying wave equation being the linear Schrödinger equation with an interaction potential defined at every instant at every point of the physical 3-dimensional space and dependent on the simultaneous position of both particles.

7 The "Extended Particle" Model

The notion of de Broglie's phase-wave presented above can be deployed to specify the nature of the particle u-wave postulated by de Broglie's Double Solution Theory. But, to achieve this goal, one must first stress the role played by the space-time geometry in the de Broglie phase-wave and, in particular, rule out any potential singularity of the metric in order that the de Broglie phase-wave be defined everywhere on $\mathcal{M}$. Since the Schwarzschild metric presents a singularity at the point where the mass (supposed point-like) is sitting, one should not expect such geometry to underlie the space-time structure.[14] Conversely, the representation of such a particle as a spherical mass distribution is free from this shortcoming because the metric tensor associated with a ball has no singularities whatsoever (see [32]). Therefore, it seems preferable to consider the particle as a massive extended object and refer to it as the "*extended particle*". However, since the metric tensor associated with a ball has the same form as the Schwarzschild metric outside the surface of the ball (see [32]), the Schwarzschild geometry may still be used as an approximation in the region surrounding the mass distribution.

7.1 The "Extended Particle" de Broglie Phase-wave

As mentioned in Section 5 the formal results concerning the relativistic point-like particle de Broglie phase-wave can be applied to a mass distribution. To this scope, first of all let recall from Section 1 that the ideal clock of the infinite distribution associated with a relativistic point-like particle of proper mass m_0 sitting at point $\vec{r}$ can be represented by the periodic function $C(\vec{r})\, e^{i 2\pi \nu^{clock}(\vec{r}) t}$ where $C(\vec{r})$ is a complex scalar and t and $\nu^{clock}(\vec{r})$ denote, respectively, the time measured by the clock and the clock frequency given by (7) relative

13 In the last system of equations it suffices that S_{12} and Q_{12} be functions of the $\vec{R}^{(1)} - \vec{R}^{(2)}$ components rather than the 3-vector $\vec{R}^{(1)} - \vec{R}^{(2)}$.

14 Even in the most simple scenario of a single spherically symmetric particle not interacting with any electromagnetic field.

to the observer frame. Then, if the point-like particle has electric charge q and moves in an electromagnetic field with 4-potential $\left(A^0, A^1, A^2, A^3\right)$, its de Broglie phase-wave reads, by the Principle of Phase Harmony,

$$F\left(\vec{r}, t\right) = C\left(\vec{r}\right) e^{i2\pi\varphi(\vec{r},t)} \tag{17}$$

with φ given by (12). Now, like for the case of free particle dynamics described in Section 1, S in (12) must have a real part linear in the time variable and a constant imaginary part in order that the de Broglie phase-wave at a given point of space keep in phase with the clock at that point as required by the Principle of Phase Harmony, otherwise (17) would lose its property of periodicity typical of a clock. Then, substituting (12) with (16) it follows from (17) that, in general, the events of $\mathcal{M}^{int}$ cannot be associated with clocks and, consequently, the Principle of Phase Harmony cannot apply to those events.[15] Conversely, as φ is real-valued on $\mathcal{M} \setminus \mathcal{M}^{int}$, the Principle of Phase Harmony does apply to the events of this portion of space-time, which appear as clocks of definite frequency also relative to the observer frame if φ is linear in the time variable. Let then assume that the particle has a mass and an electric charge distributed upon a tiny region V_0 of the physical 3-dimensional space, whose 3-volume $\|V_0\| := dx_0^1 dx_0^2 dx_0^3$ is independent of time, with time-dependent and finite mass density and electric charge density relative to the mass distribution centre-of-mass frame.[16] By the 4-volume transformation formula from Einstein's general relativity, that is,

$$\sqrt{\left|\det\left[g_{\alpha\beta}\left(\mathbf{x}\right)\right]_{\alpha,\beta=0,1,2,3}\right|} dx^0 dx^1 dx^2 dx^3$$
$$= \sqrt{\left|\det\left[g_{0\alpha\beta}\left(\mathbf{x}_0\right)\right]_{\alpha,\beta=0,1,2,3}\right|} dx_0^0 dx_0^1 dx_0^2 dx_0^3,$$

and supposing that the time coordinate transforms from the observer frame to the mass distribution centre-of-mass frame as

$$\frac{dt_0}{dt}\sqrt{\left|\frac{\det\left[g_{0\alpha\beta}\left(\mathbf{x}_0\right)\right]_{\alpha,\beta=0,1,2,3}}{\det\left[g_{\alpha\beta}\left(\mathbf{x}\right)\right]_{\alpha,\beta=0,1,2,3}}\right|} = K^{'}$$

with $K^{'} \in \mathbb{R}$ being a constant, the 3-volume of the mass distribution $\|V\| := dx^1 dx^2 dx^3$ relative to the observer frame is independent of the coordinates. Now, by (17), the de Broglie phase-wave associated with the point-like mass at $\vec{r}_0^{\,'}$ belonging to the mass distribution relative to the mass distribution centre-of-mass frame is given by

$$F_0^{\pm}\left(\vec{r}_0, t_0; \vec{r}_0^{\,'}\right) = C\left(\vec{r}_0^{\,'}\right) e^{i2\pi\varphi_0^{\pm}\left(\vec{r}_0, t_0; \vec{r}_0^{\,'}\right)}$$

with $C\left(\vec{r}_0^{\,'}\right) \in \mathbb{C}$ being a scalar and $\varphi_0^{\pm}$ as in (12) where $S_0^{\pm}$ has the form (16) but with the role of the proper mass and the electric charge played, respectively, by the mass density and the electric charge density relative to the mass distribution centre-of-mass frame. Thus, the superposition of the de Broglie phase-waves associated with the point-like masses forming

15 Adapting the image given in footnote 9 to this case, $S^{\pm}$ at a given event of $\mathcal{M}^{int}$ can be interpreted as the Hamilton principal function of a (potentially electrically charged) point-like particle of proper mass $m_0^{(g)}$ that moves at speed $\dot{l}$ (in general, greater than the curvilinear speed-of-light but, however, smaller than c, whence its total energy keeps real and finite) along the world-line $\Lambda\left(ct_{in}, \vec{r}_{in}; ct, \vec{r}\right)$ for which $\int_{\Lambda\left(ct_{in}, \vec{r}_{in}; ct, \vec{r}\right)} m_0 ds$ is stationary.

16 Hereon, the subscript 0 denotes the quantities relative to the mass distribution centre-of-mass frame.

the entire mass distribution seems a natural choice for the so-called *de Broglie phase-wave of the "extended particle"*, that is, by definition,

$$\int_{V_0} F_0^{\pm}\left(\vec{r}_0, t_0; \vec{r}_0^{\,'}\right) d^3 r_0^{'} = \int_{V_0} C\left(\vec{r}_0^{\,'}\right) e^{i2\pi\varphi_0^{\pm}\left(\vec{r}_0, t_0; \vec{r}_0^{\,'}\right)} d^3 r_0^{'}$$
$$= \|V_0\|\, C\left(\vec{R}_0\right) e^{i2\pi\varphi_0^{\pm}\left(\vec{r}_0, t_0; \vec{R}_0\right)} \tag{18}$$

relative to the mass distribution centre-of-mass frame with $\vec{R}_0$ being the 3-position of a suitable point P of V_0 at rest relative to the centre-of-mass frame whose existence is ensured by the mean value theorem for integrals.[17] Then, when transformed into the observer frame (18) reads

$$\|V_0\|\, C\left(\vec{R}_0\right) e^{i2\pi\varphi_0^{\pm}\left(\vec{r}_0, t_0; \vec{R}_0\right)} = \|V\|\, C_0 e^{i2\pi\varphi^{\pm}\left[\vec{r}, t; \vec{R}(t)\right]} \tag{19}$$

where $\|V\|$ is a real constant in virtue of the time-coordinate transformation between the observer frame and the centre-of-mass frame, $C_0 := \frac{C\left(\vec{R}_0\right)}{K'}$ is a scalar independent of the coordinates and $\vec{R}\,(t)$ is the 3-position of P relative to the observer frame. The de Broglie phase-wave of the "extended particle" given by (18) and (19) is then claimed to represent the u-wave of de Broglie's Double Solution Theory up to a complex exponential factor dependent on the 4-potential and, consequently, will be hereon denoted by $u^{\pm}$. For later purposes, both the 'singular region' and the 'intermediate region' postulated by de Broglie's Double Solution Theory will be here assumed to lie at time t within the region of the physical 3-dimensional space corresponding to the hypersurface with fixed t of the curved portion of space-time given by

$$\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right] := \{\vec{r}^{\,'} \in \mathbb{R}^3 | \sum_{h,k=1}^{3} g_{hk}\left[ct, \vec{r}^{\,'} - \vec{R}_{cm}(t)\right] dx^{'h} dx^{'k}$$
$$= g_{\alpha\beta}\left[ct, \vec{r}^{\,'} - \vec{R}_{cm}(t)\right] dx^{'\alpha} dx^{'\beta} \neq \eta_{\alpha\beta} dx^{'\alpha} dx^{'\beta}$$
$$= \sum_{h,k=1}^{3} \eta_{hk} dx^{'h} dx^{'k}\}$$

where $\vec{R}_{cm}(t)$ is the 3-position of the mass distribution centre-of-mass relative to the observer frame.[18] In addition, if V_0 contains at any time $t \in [t_{in}, +\infty[$ the hypersurface

$$\mathcal{Q}^3_{<}\left[\vec{R}_{cm}(t)\right] := \{\vec{r}^{\,'} \in \mathcal{Q}^3\left[\vec{R}_{cm}(t)\right] | g_{\alpha\beta}\left[ct, \vec{r}^{\,'} - \vec{R}_{cm}(t)\right] dx^{'\alpha} dx^{'\beta}$$
$$= \sum_{h,k=1}^{3} g_{hk}\left[ct, \vec{r}^{\,'} - \vec{R}_{cm}(t)\right] dx^{'h} dx^{'k} < 0\}$$

[17] If V_0 is independent of time, $\vec{R}_0$ is certainly constant relative to the centre-of-mass frame. But V_0 could also vary over time if the mass were suitably distributed to have, for instance, centre-of-mass at $\vec{R}_0$, because P is obviously at rest relative to itself.

[18] In the definition of $\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right]$ the space-time interval equals the space interval for a fixed t as $dx^{'0} = cdt^{'} = 0$ on the hypersurface where $t^{'} \equiv t$.

such that $\mathcal{M}^{int} = \bigcup_{t\in[t_{in},+\infty[} \{ct\} \times \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ and if the 'singular region' and the 'intermediate region' form a partition of $\bar{V}(t) := V_0 \bigcap \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right] = \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$,[19] it follows from (16) that (19) is complex-valued with a space-time coordinate-dependent modulus on the 'singular region' and on the 'intermediate region'. Finally, the 'external region' will be here assumed to lie at time t in $\mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$. Now, by construction of the de Broglie phase-wave for the "extended particle", the phase of the u-wave given by (19) depends on a complete integral of (9) as in (12) where the role of m_0 and q is played, respectively, by $\mu\left[\vec{R}(t), t\right]$ and $\rho\left[\vec{R}(t), t\right]$. As the left-hand side of (9) is the contraction of a 2-rank tensor with two 1-rank tensors, $\mu\left[\vec{R}(t), t\right]$ must be invariant. But this is compatible with the relativistic transformation of a mass density if there exists a frame relative to which the point P is at rest.[20] The mass must then be distributed on V_0 so that this condition for the mass density at P be satisfied at any time. The corresponding frame would represent the rest-frame of the particle as if this were assumed to be point-like with rest-mass equal to $\mu\left[\vec{R}(t), t\right]$ and electric charge relative to the observer frame[21] equal to $\rho\left[\vec{R}(t), t\right]$.

7.2 Application of the "Extended Particle" Model to de Broglie's Double Solution Theory

Deploying (12) and (16) in the de Broglie phase-wave of the "extended particle" at time t and supposing henceforth that the gauge has been fixed according to the Fock-Schwinger condition (i.e., $A_\alpha dx^\alpha = 0$) it turns out from (19) that

$$\begin{aligned} u^{\pm}(\vec{r}, t) &= \|V\|\, C_0 e^{\mp\frac{c}{\hbar}\int_{\bar{\xi}(ct,\vec{r})}^{\xi'(ct,\vec{r})} \mu\left[\vec{R}\left(t'\left(\xi'\right)\right), t'\left(\xi'\right)\right] d\xi'} \\ &\quad\times e^{\frac{i}{\hbar}\left\{S^{\pm}\left[\vec{r}_{in}, t_{in}; \vec{R}(t_{in})\right] \pm c\int_{\xi'(ct_{in},\vec{r}_{in})}^{\bar{\xi}(ct,\vec{r})} \mu\left[\vec{R}\left(t'\left(\xi'\right)\right), t'\left(\xi'\right)\right] d\xi'\right\}} \\ &= \|V\|\, C_0 e^{\mp\frac{G_1(\vec{r},t)}{\hbar}} e^{\frac{i}{\hbar}\left\{S^{\pm}\left[\vec{r}_{in}, t_{in}; \vec{R}(t_{in})\right] \pm G_2(\vec{r},t)\right\}} \end{aligned} \tag{20}$$

on $\mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ because the phase of the de Broglie phase-wave is complex-valued on that portion of physical 3-dimensional space with

$$G_1(\vec{r}, t) := c\int_{\bar{\xi}(ct,\vec{r})}^{\xi'(ct,\vec{r})} \mu\left[\vec{R}\left(t'\left(\xi'\right)\right), t'\left(\xi'\right)\right] d\xi'$$
$$G_2(\vec{r}, t) := c\int_{\xi'(ct_{in},\vec{r}_{in})}^{\bar{\xi}(ct,\vec{r})} \mu\left[\vec{R}\left(t'\left(\xi'\right)\right), t'\left(\xi'\right)\right] d\xi'$$

being strictly positive as $\xi'(ct, \vec{r}) > \bar{\xi}(ct, \vec{r})$ and $\bar{\xi}(ct, \vec{r}) > \xi'(ct_{in}, \vec{r}_{in})$. If the mass density takes on large value around $\vec{R}(t)$ at any time t, (20) shows that u^+ has a space-time coordinate-dependent amplitude that can attain at any time t small values on $\mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$.

19 Notice that V_0 might also have at any time t a non-zero intersection with $\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right] \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right] \subseteq \mathbb{R}^3 \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$.

20 Such a frame is certainly given by the one co-moving with P and could coincide with the "extended particle" centre-of-mass frame (see footnote 17).

21 In general, the rest-mass and the electric charge of the particle could be time-dependent.

Therefore, in order that the u-wave amplitude reach at any time very large values on the 'singular region' in accordance with de Broglie's Double Solution Theory, the 'singular region' and the 'intermediate region'[22] can form a partition of $\bar{V}(t)$ only if one chooses u^- as the "extended particle" u-wave (see Fig. 1). Finally, it turns out from (19) that the u-wave at time t is given by

$$u^-(\vec{r},t) = \|V\| C_0 e^{\frac{i}{\hbar}\left\{S^-\left[\vec{r}_{in},t_{in};\vec{R}(t_{in})\right]-G_3(\vec{r},t)\right\}} \tag{21}$$

with

$$G_3(\vec{r},t) := c\int_{\xi'(ct_{in},\vec{r}_{in})}^{\xi'(ct,\vec{r})} \mu\left[\vec{R}\left(t'\left(\xi'\right)\right),t'\left(\xi'\right)\right]d\xi'$$

on $\mathbb{R}^3 \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ because the phase of the de Broglie phase-wave is real-valued on that portion of physical 3-dimensional space. Thus, (21) has a constant and uniform amplitude which, being $\mathcal{O}(\|V\|)$ with $\|V\|$ small,[23] can be considered zero, to a good approximation, in accordance with the requirement of de Broglie's Double Solution Theory for the u-wave on the 'external region'. As a result of the foregoing construction, the u-wave becomes a highly peaked soliton for a suitable choice of the gravitational potentials in line with the spirit of de Broglie's Double Solution Theory and, as previously mentioned, with the quest of meaningful solutions to the Schrödinger-Newton equation (see [8]). Notice that, although the Double Solution Theory is not a collapse-theory, from the point of view of this latter kind of theory the wave just constructed should be considered as representing an already collapsed particle (see, e.g., [9, 10, 33, 34]). In order that this model of the "extended particle" be valid whenever the particle, considered as a stable entity, is not absorbed nor emitted by an atom, there must exist[24] a conserved quantity associated with the u-wave on $\bar{V}(t)$ that can be interpreted as a sort of particle "internal energy" with density dependent on $\left|u^-\right|$. But, since G_1 is determined by the values taken on by the mass density at the point P during the time interval $\left[t'\left(\bar{\xi}\right),t'(\xi)\right]$, a state of constant "internal energy" represents a constraint for the internal dynamics of the "extended particle" which, compatibly with the lack of a relativistic notion of rigidity, does not necessarily have to undergo an undeformability condition as required in previous approaches (see [4, 35]). However, which type of internal dynamics would provide with a u-wave having, in general, the form (20) on $\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ but with a constant "internal energy" as long as the regime corresponding to a particle stationary state is maintained? As a possible example, one might devise the following mechanism. If the system evolves unperturbed over a large enough time interval until reaching a stationary state (e.g., when no absorption nor emission from an atom take place), the mass distribution at P evolves so that

$$\lim_{t\to+\infty} G_1(\vec{r},t) = \bar{G}(\vec{r}) \tag{22}$$

22 By 'intermediate region' it is here meant the portion of 'intermediate region' lying at any time t in $\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$. In addition to this, the 'intermediate region' might include also a portion lying at any time t, e.g., in $\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right] \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$.

23 By the chosen relativistic transformation of 4-volume, $\|V\| = K'\|V_0\|$ with $\|V_0\| << 1$.

24 At least under certain circumstances such as, e.g., when an electron is bound to an atom nucleus.

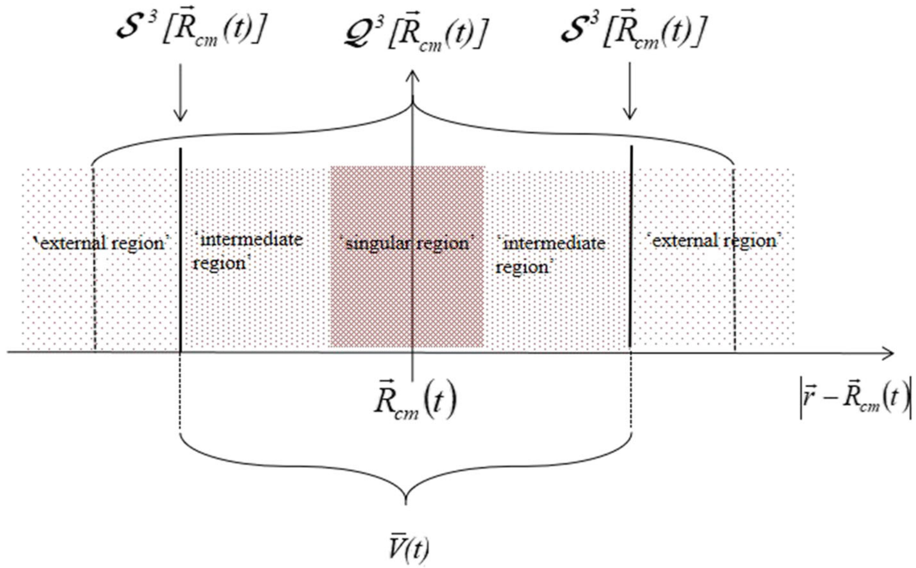

Fig. 1 The three regions of de Broglie's Double Solution Theory within the framework of the "extended particle" model. The 3-surface enclosing the 'singular region' and lying in the 'intermediate region' is chosen to coincide at any time t with the boundary of $\bar{V}(t)$

on $\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ and at $t >> t_{in}$

$$G_2(\vec{r}, t) = S^-\left[\vec{r}_{in}, t_{in}; \vec{R}(t_{in})\right] + \frac{\hbar}{2\pi} W t - \frac{\hbar}{2\pi}\psi(\vec{r}) \tag{23}$$

on $\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ and

$$G_3(\vec{r}, t) = S^-\left[\vec{r}_{in}, t_{in}; \vec{R}(t_{in})\right] + \frac{\hbar}{2\pi} W t - \frac{\hbar}{2\pi}\psi(\vec{r}) \tag{24}$$

on $\mathbb{R}^3 \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ with W being a real constant and ψ being a real function dependent only on the space coordinates.[25] Then, if t_{in} refers to an instant after which the conditions for a stationary state are fulfilled, it turns out that at $t >> t_{in}$

$$\begin{aligned} u^-(\vec{r}, t) &= \lim_{t' \to t} u^-\left(\vec{r}, t'\right) \\ &\overset{(20)}{=} \lim_{t' \to t}\left(\|V\| C_0 e^{\frac{G_1\left(\vec{r}, t'\right)}{\hbar}} e^{\frac{i}{\hbar}\left\{S^-\left[\vec{r}_{in}, t_{in}; \vec{R}(t_{in})\right] - G_2\left(\vec{r}, t'\right)\right\}}\right) \\ &\approx \|V\| C_0 e^{\frac{i}{\hbar} S^-\left[\vec{r}_{in}, t_{in}; \vec{R}(t_{in})\right]} \end{aligned} \tag{25}$$

[25] The behaviour of G_1, G_2 and G_3 at each $\vec{r}$ depends, by definition, on the 4-velocity that the mass at P should have to reach $\vec{r}$ at the considered time. In turn, this 4-velocity derives from a solution of the geodesic equations on $\mathcal{M}^{int}$ for a suitable value of its initial 4-velocity (see also footnote [15]).

$$\times \lim_{t' \to +\infty} \left(e^{\frac{G_1\left(\vec{r},t'\right)}{\hbar}} e^{\frac{i}{\hbar}\left\{S^-\left[\vec{r}_{in},t_{in};\vec{R}(t_{in})\right]-G_2\left(\vec{r},t'\right)\right\}} \right)$$

$$\overset{(22),(23)}{=} \|V\| \, C_0 e^{\frac{\bar{G}(\vec{r})}{\hbar}} e^{-\frac{i}{2\pi}[Wt-\psi(\vec{r})]}$$

on $\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ to a good approximation, and

$$u^-(\vec{r},t) \overset{(21)}{=} \|V\| \, C_0 e^{\frac{i}{\hbar}\left\{S^-\left[\vec{r}_{in},t_{in};\vec{R}(t_{in})\right]-G_3(\vec{r},t)\right\}}$$
$$\overset{(24)}{=} \|V\| \, C_0 e^{-\frac{i}{2\pi}[Wt-\psi(\vec{r})]} \tag{26}$$

on $\mathbb{R}^3 \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$. Thus, the u-wave given by (20) and (21) with G_1, G_2 and G_3 satisfying, respectively, (22), (23) and (24) becomes, to a good approximation, a stationary wave defined on the whole of $\mathbb{R}^3$ with constant amplitude and with phase linear in the time variable during the considered time interval, whence one obtains the stationary states of a particle (see next section). Therefore, in principle, it seems possible to derive the particle stationary states from the particle de Broglie phase-wave. Similarly to the interpretation of solitonic solutions for the Schrödinger-Newton equation (see [33]), also the u-wave stationary state obtained in this case might be viewed as originating from a previous general state through an excess energy radiation process. Yet, the example just described to derive a stationary state seems artificial as the asymptotic conditions (22), (23) and (24) were chosen ad hoc. Instead, in order to organically explain the origin of the stationary states one should show, for instance, that these asymptotic conditions can be ultimately obtained from the much sought-after non-linear wave equation postulated by the de Broglie Double Solution Theory. However, bar this latter limitation, it is worth stressing the following properties of the u-wave constructed above:

1. it provides forthrightly the global u-wave postulated by de Broglie's Double Solution Theory resorting to the relativistic behaviour of clocks in a curved space-time and avoiding any arbitrary choice of different decomposition types (see Section 6.1);
2. it entails the guidance relation at each point of the physical 3-dimensional space and, in particular, on the closed 3-surface surrounding the 'singular region', a fundamental feature of the Pilot-Wave Theory for particles that has been also extended to quantum field theories (see [36]);
3. though from a distinct perspective, it shares with the approach in [15] the idea that gravitation at small scale is responsible for the cohesion of electrically charged particles in the form of a stable soliton which otherwise would disintegrate under the action of the purely repulsive internal electromagnetic field;
4. it eschews to associate the particle u-wave with two waves of different nature as hypothesized by the 'factorisation ansatz' of [15] and [16].

8 Relationship Between the "Extended Particle" de Broglie Phase-wave and the Scalar Particle Wave Function

Let consider a single particle[26] of rest-mass $\bar{m}_0$, electric charge $\bar{q}_0$ and zero spin in interaction with an electromagnetic field having 4-potential (A_0, A_1, A_2, A_3). The wave function of such

[26] The particle is here assumed point-like in accordance with the usual interpretation.

a particle solves the Klein-Gordon equation

$$\frac{1}{\sqrt{-\eta}}\partial_\alpha\left[\sqrt{-\eta}\eta^{\alpha\beta}\partial_\beta\Psi(\mathbf{x})\right]+\left(\frac{\bar{m}_0 c}{\hbar}\right)^2\Psi(\mathbf{x})$$
$$-\left(\frac{\bar{q}_0}{\hbar c}\right)^2\eta^{\alpha\beta}A_\alpha(\mathbf{x})A_\beta(\mathbf{x})\Psi(\mathbf{x})-i\frac{2\bar{q}_0}{\hbar c}\eta^{\alpha\beta}A_\alpha(\mathbf{x})\partial_\beta\Psi(\mathbf{x})=0 \quad (27)$$

where $\eta := \det\left(\eta^{\alpha\beta}\right)_{\alpha,\beta=0,1,2,3}=-1$. Substituting (27) with the wave function in the form $\Psi(\mathbf{x})=|\Psi(\mathbf{x})|\,e^{-\frac{i}{\hbar}S(\mathbf{x})}$, its real part reads

$$\eta^{\alpha\beta}\left[\partial_\alpha S(\mathbf{x})+\frac{\bar{q}_0}{c}A_\alpha(\mathbf{x})\right]\left[\partial_\beta S(\mathbf{x})+\frac{\bar{q}_0}{c}A_\beta(\mathbf{x})\right]=\left[M_0(\mathbf{x})\,c\right]^2 \quad (28)$$

where $M_0(\mathbf{x}) := \sqrt{\bar{m}_0^2+\frac{\hbar^2}{c^2}\frac{\eta^{\alpha\beta}\partial_\alpha\partial_\beta[|\Psi(\mathbf{x})|]}{|\Psi(\mathbf{x})|}}$ is the so-called *variable rest-mass* of the particle. Now, in accordance with de Broglie's Double Solution Theory, although the particle wave function has a subjective nature as far as its amplitude is concerned, its phase must bear at any time t an element of objectivity inherited by the particle u-wave, at least on a 3-surface, say, $\mathcal{S}^3\left[\vec{R}_{cm}(t)\right]$, that encloses the 'singular region' and lies in the 'intermediate region' (see [6]). Now, within the framework of the "extended particle" model the phase of the particle u-wave (19) is given by (12) where the role of m_0 and q is played, respectively, by $\mu\left[\vec{R}(t),t\right]$ and $\rho\left[\vec{R}(t),t\right]$ and, by the discussion at the end of Section 7.1, with $\mu\left[\vec{R}(t),t\right]=\bar{m}_0$. However, although at variance with de Broglie's Double Solution Theory,[27] the phase of Ψ carries an objective information concerning the particle dynamics if one supposes that

[27] In de Broglie's Double Solution Theory the phase of the wave function and the phase of the u-wave coincide (at least) upon a 3-surface enclosing the particle 'singular region' and lying in the particle 'intermediate region'. On the other hand, in the "extended particle" model the phase coincidence at time t on $\mathbb{R}^3\setminus\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ between the particle wave function and the particle u-wave given by (21) requires the gauge to be fixed according to the Fock-Schwinger condition on that portion of the physical 3-dimensional space. But this latter choice of gauge is at loggerheads with Lorenz gauge deployed by de Broglie to derive the guidance formula from the linear Klein-Gordon equation on the particle 'external region'. Yet, one could arrange as a workaround that at any time t the particle 'intermediate region' has a portion lying, e.g., in $\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right]\setminus\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ where $\mathcal{S}^3\left[\vec{R}_{cm}(t)\right]$ is placed. In this case, the Lorenz gauge condition might be imposed exactly on $\mathcal{S}^3\left[\vec{R}_{cm}(t)\right]$, whence the guidance formula would be retrieved, and smoothly changed into the Fock-Schwinger gauge condition in order to make the phase of the wave function equal to the phase of the u-wave given by (21) almost everywhere on $\mathbb{R}^3\setminus\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$. This would then consent to tightly seal the phase of the u-wave given by the "extended particle" model with the phase of the wave function calculated with the usual method of quantum mechanics. However, such a sleight of hand seems artificial and suggests to replace altogether the Fock-Schwinger condition with the Lorenz gauge, whence the u-wave is the factorisation of a wave as in (21) with a purely complex exponential term dependent on the 4-potential. Therefore, one concludes that, while the particle's de Broglie phase-wave still satisfies (13), the electromagnetic field is responsible, in general, for the loss of phase harmony between the u-wave and the clocks from the infinite distribution on $\mathbb{R}^3\setminus\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$. Now, as the electromagnetic produced by the electrically charged mass distribution of the "extended particle" is never exactly zero on $\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right]$, it could be argued that the notion of particle's de Broglie phase-wave is effectively useless. Nevertheless, such argument would be wrong because the electromagnetic field is decomposable into a part due to the "extended particle" mass distribution and into another one due to external sources. Then, if the latter were zero, far afield from the mass distribution on $\mathbb{R}^3\setminus\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ the 4-potential could be neglectable, whence the particle's de Broglie phase-wave from the Principle of Phase-Harmony could in principle exist along with the u-wave.

$S(ct,\vec{r})$ is given by $S^{\pm}\left[\vec{r},t;\vec{R}(t)\right]$ at any time t on $\mathbb{R}^3 \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ where $S^{\pm}$ is real-valued. Then, assuming $\rho\left[\vec{R}(t),t\right]=\bar{q}_0$, at any time t on $\mathbb{R}^3 \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ one obtains

$$\begin{aligned}
&\left\{\mu\left[\vec{R}(t),t\right]c\right\}^2 \overset{(9)}{=} g^{\alpha\beta}(ct,\vec{r})\left\{\partial_\alpha S^{\pm}\left[\vec{r},t;\vec{R}(t)\right]+\frac{\rho\left[\vec{R}(t),t\right]}{c}A_\alpha(ct,\vec{r})\right\}\\
&\times\left\{\partial_\beta S^{\pm}\left[\vec{r},t;\vec{R}(t)\right]+\frac{\rho\left[\vec{R}(t),t\right]}{c}A_\beta(ct,\vec{r})\right\}\\
&=\eta^{\alpha\beta}\left\{\partial_\alpha S^{\pm}\left[\vec{r},t;\vec{R}(t)\right]+\frac{\bar{q}_0}{c}A_\alpha(ct,\vec{r})\right\}\left\{\partial_\beta S^{\pm}\left[\vec{r},t;\vec{R}(t)\right]+\frac{\bar{q}_0}{c}A_\beta(ct,\vec{r})\right\}\\
&-\left[\eta^{\alpha\beta}-g^{\alpha\beta}(ct,\vec{r})\right]\left\{\partial_\alpha S^{\pm}\left[\vec{r},t;\vec{R}(t)\right]+\frac{\bar{q}_0}{c}A_\alpha(ct,\vec{r})\right\}\left\{\partial_\beta S^{\pm}\left[\vec{r},t;\vec{R}(t)\right]+\frac{\bar{q}_0}{c}A_\beta(ct,\vec{r})\right\}\\
&\overset{(28)}{=}(\bar{m}_0c)^2+\hbar^2\frac{\eta^{\alpha\beta}\partial_\alpha\partial_\beta\left[|\Psi(ct,\vec{r})|\right]}{|\Psi(ct,\vec{r})|}\\
&-\left[\eta^{\alpha\beta}-g^{\alpha\beta}(ct,\vec{r})\right]\left\{\partial_\alpha S^{\pm}\left[\vec{r},t;\vec{R}(t)\right]+\frac{\bar{q}_0}{c}A_\alpha(ct,\vec{r})\right\}\left\{\partial_\beta S^{\pm}\left[\vec{r},t;\vec{R}(t)\right]+\frac{\bar{q}_0}{c}A_\beta(ct,\vec{r})\right\},\\
&\overset{(10),(11)}{=}(\bar{m}_0c)^2+\hbar^2\frac{\eta^{\alpha\beta}\partial_\alpha\partial_\beta\left[|\Psi(ct,\vec{r})|\right]}{|\Psi(ct,\vec{r})|}-\left[\eta^{00}-g^{00}(ct,\vec{r})\right]\frac{\left[\mu^{(g)}(ct,\vec{r})\right]^2\left[\gamma(ct,\vec{r})\right]^4}{1-\left[\frac{i\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2}\\
&-\frac{\left[\mu^{(g)}(ct,\vec{r})\right]^2}{1-\left[\frac{i\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2}\sum_{h,k,l,m=1}^{3}\left[\eta^{hk}-g^{hk}(ct,\vec{r})\right]g_{hl}(ct,\vec{r})\,\dot{x}^l(ct,\vec{r})\,g_{km}(ct,\vec{r})\,\dot{x}^m(ct,\vec{r})
\end{aligned}$$

with $\mu^{(g)}(ct,\vec{r}):=\frac{\mu\left[\vec{R}(t),t\right]}{\sqrt{g_{00}(ct,\vec{r})}}$, whence, substituting $\mu\left[\vec{R}(t),t\right]$ with $\bar{m}_0$,

$$\begin{aligned}
&\hbar^2\eta^{\alpha\beta}\partial_\alpha\partial_\beta\left[|\Psi(ct,\vec{r})|\right]=\left(\left\{\mu\left[\vec{R}(t),t\right]\right\}^2-m_0^2\right)c^2\,|\Psi(ct,\vec{r})|\\
&+\{\left[\eta^{00}-g^{00}(ct,\vec{r})\right]\frac{\left[\mu^{(g)}(ct,\vec{r})\right]^2\left[\gamma(ct,\vec{r})\right]^4}{1-\left[\frac{i\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2}\\
&+\frac{\left[\mu^{(g)}(ct,\vec{r})\right]^2}{1-\left[\frac{i\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2}\sum_{h,k,l,m=1}^{3}\left[\eta^{hk}-g^{hk}(ct,\vec{r})\right]g_{hl}(ct,\vec{r})\,\dot{x}^l(ct,\vec{r})\,g_{km}(ct,\vec{r})\,\dot{x}^m(ct,\vec{r})\}\,|\Psi(ct,\vec{r})|\\
&=\frac{\left[\mu^{(g)}(ct,\vec{r})\right]^2}{1-\left[\frac{i\left(\vec{r},\dot{\vec{r}},t\right)}{\gamma(ct,\vec{r})}\right]^2}\{\left[\eta^{00}-g^{00}(ct,\vec{r})\right]\left[\gamma(ct,\vec{r})\right]^4\\
&-\sum_{l,m=1}^{3}\left[\sum_{k=1}^{3}g_{lk}(ct,\vec{r})\,g_{km}(ct,\vec{r})+g_{lm}(ct,\vec{r})\right]\dot{x}^l(ct,\vec{r})\,\dot{x}^m(ct,\vec{r})\}\,|\Psi(ct,\vec{r})| \qquad (29)
\end{aligned}$$

at any time t on $\mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ and, in particular, on $\mathcal{S}^3 \left[\vec{R}_{cm}(t)\right]$, if this 3-surface is assumed to lie in $\mathcal{Q}^3 \left[\vec{R}_{cm}(t)\right] \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$. Thus, (29) shows that $|\Psi|$ at any time t and any point $\vec{r} \in \mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ depends, in general, on how much the space-time metric tensor at the event $(ct, \vec{r})$ deviates from the Minkowski metric. But, since the metric tensor of the space-time is obtained from Einstein's field equations which, in general, contain an energy-stress tensor term, the above relation between the phase of Ψ and the phase of the de Broglie phase-wave at any time t and any point $\vec{r} \in \mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ makes $|\Psi|$ depend on the actual presence of some energy in the proximity of the event $(ct, \vec{r})$. Thus, in agreement with the Principle of Interference according to which $|\Psi|^2$ represents the particle localisation probability density, one is naturally lead to consider the deviation from zero of $|\Psi|$, solution of (29) at a point $\vec{r} \in \mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$, as an indication that some form of energy is present in a neighbourhood of $\vec{r}$. Furthermore, if one assumes that the particle wave function and the particle u-wave at any time t are proportional on $\mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$, with the "extended particle" model it is possible to justify the interpretation of the wave function amplitude given by the Principle of Interference at any time t not only on $\mathcal{S}^3 \left[\vec{R}_{cm}(t)\right] \subseteq \mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ but also on the rest of $\mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$. In fact, by (21), the relation $u^- = K\Psi$ for a suitable constant $K = |K| e^{i \arg K} \in \mathbb{C}$ yields $|\Psi| = \frac{\|V\| C_0}{|K|}$ that, as $\|V\| \approx 0$, is approximately zero and can thus be interpreted, to a good approximation, as the absence of energy in the form of material particle on the 'external region' in accordance with the meaning attributed by the Principle of Interference to $|\Psi|$. On the other hand, within the framework of the "extended particle" model the interpretation of Ψ according to the Principle of Interference entails the quasi-proportionality between the particle wave function and the particle u-wave on the 'external region' as claimed by de Broglie's Double Solution Theory. In effect, if a wave function of non-zero amplitude at any time t only on a limited region $V(t)$ of the physical 3-dimensional space contained in the hypersurface $\mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ of the "extended particle" model, the particle u-wave is given outside $V(t)$ by (21) with constant and uniform amplitude[28] (see Fig. 2). Then, deploying the quasi-proportionality between the subjective probability wave for the position observable and the objective u-wave one can determine, in principle, the measurement probabilities for any other observable because, in general, the measurement of a physical quantity is ultimately based upon the observation of position (see [2]). To keep in line with the de Broglie's Double Solution Theory requirements, let identify $\mathcal{S}^3 \left[\vec{R}_{cm}(t)\right]$ at any time t with the closest 3-surface to $\vec{R}_{cm}(t)$ on which the particle u-wave and the particle wave function are exactly proportional.[29] Now, as the u-wave and the wave function are smooth functions, their exact proportionality at any time t on $\mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ can be extended to a quasi-proportionality, to a good approximation, at any time t on the portion of $\mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ close to its boundary with $\mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ where $\mathcal{S}^3 \left[\vec{R}_{cm}(t)\right]$ is supposed to lie. Then, substituting $u^- \approx K\Psi$ with (20) and separating the real part from

[28] Hence, in the "extended particle" model there exists exact proportionality, rather than quasi-proportionality, between the particle wave function and its u-wave on the 'external region'.

[29] The values of the u-wave and the wave function at any time t on $\mathcal{S}^3 \left[\vec{R}_{cm}(t)\right]$ are given by, respectively, (20) and a solution of the usual linear wave equation of quantum mechanics.

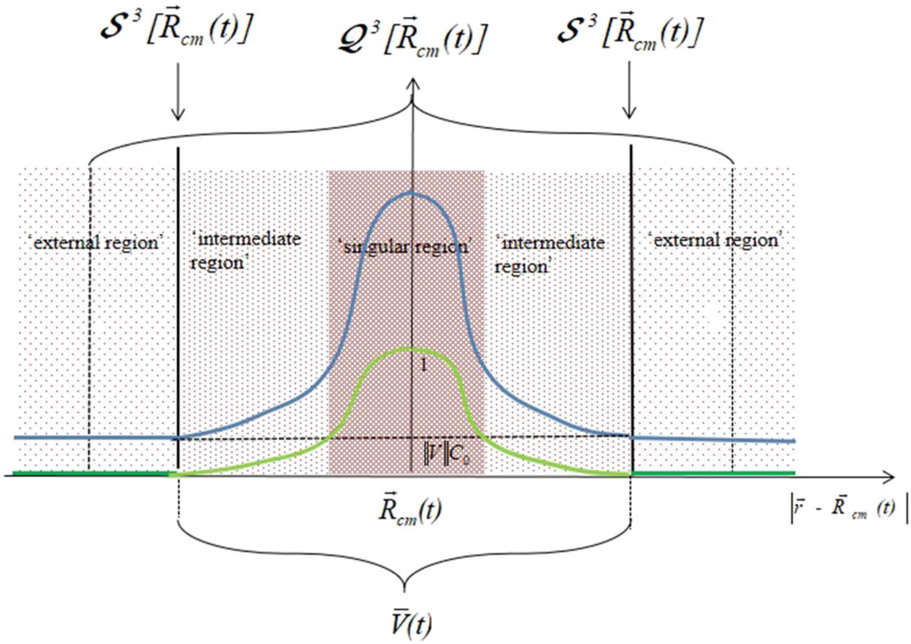

Fig. 2 According to the "extended particle" model, at any time t the particle de Broglie phase-wave u^-, that represents the u-wave of de Broglie's Double Solution Theory, has amplitude (in blue) proportional to the amplitude of the particle wave function (in green) on $\mathbb{R}^3 \setminus \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$

the imaginary part, it turns out that

$$\begin{cases} G_1(\vec{r},t) \approx \hbar \ln\left[|\Psi(\vec{r},t)|\right] + \hbar \ln\left(\frac{|K|}{\|V\|C_0}\right) \\ G_2(\vec{r},t) \approx S^-\left[\vec{r}_{in}, t_{in}; \vec{R}(t_{in})\right] - \hbar \arg K + S(ct,\vec{r}), \end{cases} \tag{30}$$

to a good approximation, at any time t on the portion of $\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ close to $\partial\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$. Then, given the features of $\mathcal{S}^3\left[\vec{R}_{cm}(t)\right]$ required by both de Broglie's Double Solution Theory[30] and the "extended particle" model[31], $\partial\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ represents a good choice for $\mathcal{S}^3\left[\vec{R}_{cm}(t)\right]$, whence (30) holds exactly with $G_1 \equiv 0$ on $\mathcal{S}^3\left[\vec{R}_{cm}(t)\right]$[32]. Since at any time t the values of G_1 and G_2 at the point $\vec{r}$ depend, by definition, on a suitable world-line $\Lambda(ct_{in}, \vec{r}_{in}; ct, \vec{r})$ described by the mass density at point P at all $t' \in [t_{in}, t]$ as if P were a point-like mass that reached the point $\vec{r}$ at time t under the action of a gravitational field, it follows from (30) that the localisation probability of the particle at each time depends,

30 In particular, the phase relationship between the u-wave and the wave particle which justifies the guidance formula (see also Section 6.1).

31 In particular, the assumption that $\mathcal{S}^3\left[\vec{R}_{cm}(t)\right] \subseteq \mathbb{R}^3 \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ which justifies the Principle of Interference in terms of the "extended particle" model.

32 In alternative, if the 'intermediate region' is extended at any time t up to $\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right] \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$, $\mathcal{S}^3\left[\vec{R}_{cm}(t)\right]$ can be placed within this portion of physical 3-dimensional space.

to a good approximation, on the gravitational potential associated with the mass distribution and, correspondingly, with the mass distribution internal dynamics. But, in accordance with Einstein's general relativity, the evolution of the point-like mass belonging to the mass distribution is obtained from

$$\begin{cases} \frac{d^2x^0}{d\tau^2}(\tau) + \Gamma^0_{\alpha\beta}(\tau)\frac{dx^\alpha}{d\tau}(\tau)\frac{dx^\beta}{d\tau}(\tau) = Q^0(\tau) \\ \frac{d^2x^1}{d\tau^2}(\tau) + \Gamma^1_{\alpha\beta}(\tau)\frac{dx^\alpha}{d\tau}(\tau)\frac{dx^\beta}{d\tau}(\tau) = Q^1(\tau) \\ \frac{d^2x^2}{d\tau^2}(\tau) + \Gamma^2_{\alpha\beta}(\tau)\frac{dx^\alpha}{d\tau}(\tau)\frac{dx^\beta}{d\tau}(\tau) = Q^2(\tau) \\ \frac{d^2x^3}{d\tau^2}(\tau) + \Gamma^3_{\alpha\beta}(\tau)\frac{dx^\alpha}{d\tau}(\tau)\frac{dx^\beta}{d\tau}(\tau) = Q^3(\tau) \end{cases} \tag{31}$$

where τ is given by $c^2(d\tau)^2 = ds^2$, $\Gamma^\sigma_{\alpha\beta}$ is the affine connection of $\mathcal{M}$ and $\left(Q^0, Q^1, Q^2, Q^3\right)$ is the 4-force dependent on $x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)$ as well as, in general, their derivatives of suitably high order (see [37]). Thus, if one manages to know the energy-stress tensor throughout the space-time, Einstein's field equations permit, in principle, to work out the space-time metric, to derive the geodesic equations (31) and, consequently, to obtain the explicit form of G_1 and G_2. In particular, as long as G_1 depends explicitly on time the form of the particle wave function Ψ follows on from (30) with $|\Psi|$ being explicitly time-dependent (as, e.g., during the transitions between stationary states). In such a regime the particle de Broglie phase-wave u^- and the particle wave function Ψ present the following features:

1. u^- is given by (20) and $u^- = u_0 + v \approx u_0$, to a good approximation, on the 'singular region' (i.e., where $|u_0| >> |v|$) represented by the core of $\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ closer to the mass distribution centre-of-mass;
2. u^- is given by (20) and $u^- = u_0 + v \approx v \approx K\Psi$, to a good approximation, on the 'intermediate region' (i.e., where $|u_0| << |v|$ due to the proximity to the 'external region') represented by the part of $\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ closer to $\partial\mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$ with $K \in \mathbb{C}$ being a suitable constant dependent on the particle[33];
3. u^- is given by (21) and $u^- = u_0 + v \approx v = K\Psi$, to a good approximation, on the 'external region' (i.e., where $|u_0| << |v|$) represented by $\mathbb{R}^3 \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$[34].

On the other hand, when the mass distribution reaches a dynamical regime in which the asymptotic conditions (22), (23) and (24) are satisfied (as, e.g., in the stationary states), the particle de Broglie phase-wave u^- and the particle wave function Ψ present the following features:

1. u^- is given by (25), to a good approximation, on the 'singular region';
2. u^- is given by (25) and $u^- \approx K\Psi$, to a good approximation, on the 'intermediate region'[35];
3. u^- is given by (26) and $u^- \approx K\Psi$, to a good approximation, on the 'external region'.

[33] Notice that, if the 'intermediate region' extends up to $\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right] \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$, u^- is also given by (21) on this portion of physical 3-dimensional space.

[34] Or, by an infinite portion of $\mathbb{R}^3 \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$, if the 'intermediate region' extends up to $\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right] \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$.

[35] Notice that, if the 'intermediate region' extends up to $\mathcal{Q}^3\left[\vec{R}_{cm}(t)\right] \setminus \mathcal{Q}^3_<\left[\vec{R}_{cm}(t)\right]$, u^- is also given by (26) on this portion of physical 3-dimensional space.

As a result of the quasi-proportionality between the particle u-wave and the particle wave function on the 'intermediate region' and the 'external region', (30) with (22) and (23) and (26) with (24) ensure that Ψ has a constant amplitude and, to a good approximation, a phase which is linear in time everywhere in the physical 3-dimensional space except within the 'singular region'. Thus, by deploying the proportionality relation between the particle wave function and the particle u-wave, considered valid at least at the so-called quantum equilibrium (see [7, 18, 22–24, 26, 27]), the distinctive feature of the probability wave as a function that "weighs" the amount of matter present in a certain region of the physical 3-dimensional space is motivated by the "extended particle" model deploying the interaction between the de Broglie field and the gravitational field, an aspect possible only because both these fields exist in the physical 3-dimensional space. To the author knowledge, this relationship between the de Broglie field and the gravitational field in finding the reasons for the Born rule represents a novelty introduced via the "extended particle" model that have been hitherto ignored by any other approach to quantum mechanics aimed at this scope, inclusive of the deterministic ones such as de Broglie's Double Solution Theory and, a fortiori, Bohmian Mechanics. Notice that the foregoing analysis carried out for the single particle de Broglie phase-wave can be applied to the case of many-particle systems, essentially following the original method developed within the framework of de Broglie's Double Solution Theory for particles that are represented as point-like singularities (see Section 6.2). However, it must be recalled that in the "extended particle" model every particle occupies a spatially extended region, i.e., the union of the particle 'singular region' and 'intermediate region', and that the wave equation is given by the linear one usually considered in standard quantum mechanics only on the particle 'external region'. Then, from the standpoint of the "extended particle" model, the relationships between the phases and the amplitudes of the wave functions obtained supposing the particles as point-like entities are to hold at the points of the 'external region' close to the 'intermediate region' of each particle, that is, where the wave equation can still be considered linear to a good approximation. But, since the construction of the u-wave provided in Sec. 7.2 ensures that each particle 'singular region' is locked to its 'external region' and each particle wave function is related to the corresponding u-wave as in the single particle case, the relationships between the phases and the amplitudes of the wave functions, given by (17) for the two-particle system, keep establishing, in principle, the connection between the system wave function defined in configuration space and the particle u-wave defined in the physical 3-dimensional space.

9 Phenomenology Interpretation Via the "Extended Particle" Model

In a system made up of a single particle a point can be reached by the particle only if, in accordance with the Principle of Phase Harmony, the clock at that point is in phase with the particle's de Broglie phase-wave evaluated at the clock 3-position. Then, thanks to the guidance law and to the continuity of the particle's de Broglie phase-wave on $\partial \mathcal{Q}^3_< \left[\vec{R}_{cm}(t)\right]$ at each time t, the particle itself will be driven by the external branch of the particle's de Broglie phase-wave given by (26) to the point where that clock sits maintaining the phase agreement between the particle's de Broglie phase-wave and the clocks disseminated in the physical 3-dimensional space along the propagation path. Conversely, in a system made up of two distinct particles a point may not be reached simultaneously by their respective de Broglie phase-waves because, in general, de Broglie's phase-waves of distinct particles have different phase[36] and, consequently, a clock that is reached by the de Broglie phase-wave of one particle (which is in phase with this particle's de Broglie phase-wave evaluated at the clock 3-position) cannot keep the phase agreement at the same time with the de Broglie phase-wave of the other particle. Thus, if a particle moves in a system where several other particles are simultaneously in a general motion, the external branch of its de Broglie phase-wave vanishes and, consequently, the particle's de Broglie phase-wave ends up having edges rather than being extended throughout the physical 3-dimensional space like in a system made up of a single particle. In other words, the de Broglie field of a particle is, in general, short-ranged due to the effects upon the clocks disseminated on the 'external region' of that particle produced by all the other particles in the system.

9.1 Two-slit Experiment

When a single particle is sent through a two-slit set-up and no disturbances take place at either slit (e.g., no photons are emitted by a particle detector in the neighbourhood of either slit), the phase agreement between the particle's de Broglie phase-wave and the clocks strewn throughout the physical 3-dimensional space is maintained all along the propagation. However, the presence of the obstacle (i.e., the walls beside the two slits) modifies the shape of the particle's de Broglie phase-wave in its proximity[37], whence the two branches of the particle's de Broglie phase-wave coming off the slits of the interference set-up keep coherent along some paths and go out of phase along other paths. Now, it is only along the paths of phase coherence leading to a possible point of non-zero intensity on the particle detection screen that the clocks postulated in the Principle of Clock Distribution are in phase with the branches of the particle's de Broglie phase-wave coming off both slits. Since these latter waves, which extend on the particle 'external region', guide the 'intermediate region' and the 'singular region' of the particle and their phase agreement with the clocks close to the boundary of the mass distribution holds during the whole particle dynamics, the particle must follow one of the paths of phase coherence. But, as there are more paths leading to some points (i.e., the maxima of the interference pattern) than to other points on the screen, there is higher probability that more particles follow the paths heading to the maxima than anywhere else, whence it is more likely to actually encounter a particle at a maximum than anywhere else. However, if some other particle enters the set-up (e.g., the photons emitted by a detector), the phase agreement between the branches coming off the two slits breaks down by the mechanism described above and, consequently, no interference pattern appears (see [38]).

9.2 Non-local Quantum Channel Communication

Although, two distinct particles can interact, in general, through the electromagnetic field and the gravitational field, but not via their de Broglie field due to the latter's short-range, under some appropriate conditions it is possible, in principle, to tune the phases of the two single particle's de Broglie phase-waves in such a way that they match smoothly on their

[36] Unless both particles undergo a dynamics for which (13) takes on the same value at the observation point.

[37] The phase of the particle's de Broglie phase-wave varying according to (13) is modified by the gravitational potential of the walls beside the two slits.

'external region'. Let then consider an isolated system made up of two far apart "extended particles" having de Broglie's phase-waves in phase with one another on their common 'external region'. The de Broglie phase-wave associated with such a system, obtained from a suitable preparation of the two particles, presents two bumps, each of which corresponding to the 'singular region' of the particles, smoothly connected by a plane and monochromatic wave of such a low amplitude that it could hardly be detected. Such a system de Broglie's phase-wave defined on the physical 3-dimensional space would be associated with the system wave function defined on the configuration space as sketched in Section 6.2 and could be viewed as the formal means to represent a novel superluminal communication link between the two particles. To see how this might work, let first point out that, according to the standpoint of the "extended particle" model, if a local disturbance (i.e., the action of an external field, the presence of an obstacle or the measurement of an observable) takes place in a limited region of the physical 3-dimensional space somewhere between the two bumps, the clocks lying in that region break off their phase agreement with the system de Broglie phase-wave because the disturbance (i.e., either an electromagnetic wave or the de Broglie phase-wave of the object representing the obstacle) interferes, in general, randomly with the de Broglie field. Thus, the de Broglie phase-wave of the original system breaks into two separate de Broglie's phase-wave, one for each particle, made up of two bumps that are not any more connected by the plane and monochromatic wave of low amplitude in the interloping 'external region'. In this case, one is confronted with a circumstance that, according to de Broglie's theory of measurement (for which any measurement operation, except the measurement of position, may be viewed as a spectral analysis), is essential to establish a bijective correspondence between the localisation of a corpuscule and the value of a physical quantity one wants to measure, i.e., the co-existence of two separate solitons. This situation is what one strives to reach in order to improve the separating power in optical or mass spectrography (see [17]). But conversely, if a local disturbance occurs within the 'singular region' of one of the two particles, say, particle 1, taking care not to break off its low-amplitude connection with the 'singular region' of particle 2, the energy-stress tensor is modified there and, consequently, the solutions of Einstein's field equations change alongside the solution of the geodesic equations, whence the 4-velocity field as well as the mass distribution within the 'singular region' of particle 1 differ from what they were before the disturbance. In order to build the system's de Broglie phase-wave that represents such a two-bump soliton-like state, one starts out recalling that, by (19), the single particle's de Broglie phase-wave for particle i ($i = 1, 2$) at any time t has phase given by

$$
\begin{aligned}
2\pi\varphi^{(i)}\left[\vec{r}, t; \vec{R}^{(i)}\left(t^{'}\right)\right] &\overset{(12)}{=} \frac{1}{\hbar}\int_{\Lambda(ct_{in},\vec{r}_{in};ct,\vec{r})}\{\frac{\partial S^{(i)}}{\partial t^{'}}\left[\vec{r}^{'}, t^{'}; \vec{R}^{(i)}\left(t^{'}\right)\right]dt^{'} \\
&+\sum_{h=1}^{3}\frac{\partial S^{(i)}}{\partial x^{'h}}\left[\vec{r}^{'}, t^{'}; \vec{R}^{(i)}(t)\right]dx^{'h}\} \\
&+\frac{1}{\hbar c}\int_{\Lambda(ct_{in},\vec{r}_{in};ct,\vec{r})}\rho_i\left[\vec{R}^{(i)}\left(t^{'}\right), t^{'}\right]\{cg_{00}\left(ct^{'}, \vec{r}^{'}\right)\phi\left(ct^{'}, \vec{r}^{'}\right)dt^{'} \\
&+\sum_{h,k=1}^{3}g_{hk}\left(ct^{'}, \vec{r}^{'}\right)A^{k}\left(ct^{'}, \vec{r}^{'}\right)dx^{'h}\}
\end{aligned}
\tag{32}
$$

$$\stackrel{(10),(11)}{=} -\frac{1}{\hbar}\int_{\Lambda(ct_{in},\vec{r}_{in};ct,\vec{r})}\{\frac{\mu_i^{(g)}\left[ct',\vec{r}';\vec{R}^{(i)}\left(t'\right)\right]\left[\gamma\left(ct',\vec{r}'\right)\right]^2}{\sqrt{1-\left[\frac{i\left(\vec{r}',\dot{\vec{r}}'^{(i)},t'\right)}{\gamma\left(ct',\vec{r}'\right)}\right]^2}}dt' + \sum_{h=1}^{3}\frac{\mu_i^{(g)}\left[ct',\vec{r}';\vec{R}^{(i)}\left(t'\right)\right]\sum_{k=1}^{3} g_{hk}\left(ct',\vec{r}'\right)\dot{x}'^{(i)k}\left(ct',\vec{r}'\right)}{\sqrt{1-\left[\frac{i\left(\vec{r}',\dot{\vec{r}}'^{(i)},t'\right)}{\gamma\left(ct',\vec{r}'\right)}\right]^2}}dx'^h\}$$

where the electric charge density $\rho_i\left(\vec{R}^{(i)},t\right)$ and the so-called *curvilinear rest-mass density* $\mu_i^{(g)}\left(ct,\vec{r};\vec{R}^{(i)}\right) := \frac{\mu^{(i)}\left(\vec{R}^{(i)},t\right)}{\sqrt{g_{00}(ct,\vec{r})}}$ denote, respectively, the electric charge and the curvilinear rest-mass of the point-like particle at the position 3-vector $\vec{R}^{(i)}$ of the average point obtained from the mean value theorem for integrals applied to the de Broglie phase-wave of the "extended particle" i. Since the de Broglie phase-wave of the system is smooth throughout the space-time,

$$\begin{cases} S^{(1)}\left[\vec{r}_s(t),t;\vec{R}^{(1)}(t)\right] = S^{(2)}\left[\vec{r}_s(t),t;\vec{R}^{(2)}(t)\right] \\ \frac{\partial S^{(1)}}{\partial t}\left[\vec{r}_s(t),t;\vec{R}^{(1)}(t)\right] = \frac{\partial S^{(2)}}{\partial t}\left[\vec{r}_s(t),t;\vec{R}^{(2)}(t)\right] \\ \frac{\partial S^{(1)}}{\partial x^1}\left[\vec{r}_s(t),t;\vec{R}^{(1)}(t)\right] = \frac{\partial S^{(2)}}{\partial x^1}\left[\vec{r}_s(t),t;\vec{R}^{(2)}(t)\right] \\ \frac{\partial S^{(1)}}{\partial x^2}\left[\vec{r}_s(t),t;\vec{R}^{(1)}(t)\right] = \frac{\partial S^{(2)}}{\partial x^2}\left[\vec{r}_s(t),t;\vec{R}^{(2)}(t)\right] \\ \frac{\partial S^{(1)}}{\partial x^3}\left[\vec{r}_s(t),t;\vec{R}^{(1)}(t)\right] = \frac{\partial S^{(2)}}{\partial x^3}\left[\vec{r}_s(t),t;\vec{R}^{(2)}(t)\right] \end{cases} \tag{33}$$

must be satisfied along a 3-curve $\mathcal{C} : t \longmapsto \vec{r}_s(t)$ at all times. Thus, the aforementioned two-bump soliton-like state may be defined as the u-wave having the behaviour of the single particle u-wave constructed in Section 7.2 close to the 'singular region' of either particle and satisfying the boundary condition (33) somewhere in the common 'external region' of the two particles. Since the variation of phase during the time-interval dt for the system's de Broglie phase-wave in the portion of the 'external region' close to the 3-surface delimiting the 'intermediate region' and the 'singular region' of particle 1 reads, by (32),

$$d\varphi^{(1)}\left[\vec{r},t;\vec{R}^{(1)}(t)\right] = -\frac{1}{h}\{\frac{\mu_1^{(g)}\left[ct,\vec{r};\vec{R}^{(1)}(t)\right]\left[\gamma(ct,\vec{r})\right]^2}{\sqrt{1-\left[\frac{i\left(\vec{r},\dot{\vec{r}}^{(1)},t\right)}{\gamma(ct,\vec{r})}\right]^2}}dt + \sum_{h=1}^{3}\frac{\mu_1^{(g)}\left[ct,\vec{r};\vec{R}^{(1)}(t)\right]\sum_{k=1}^{3} g_{hk}(ct,\vec{r})\dot{x}^{(1)k}(ct,\vec{r})}{\sqrt{1-\left[\frac{i\left(\vec{r},\dot{\vec{r}}^{(1)},t\right)}{\gamma(ct,\vec{r})}\right]^2}}\}dx^h,$$

the phase of a perturbation of $d\varphi^{(1)}$ due to a variation of the 3-velocity field $\delta\dot{\vec{r}}$ within the particle 1 mass distribution[38] is given by

$$\delta\varphi^{(1)}\left[\vec{r},t;\vec{R}^{(1)}(t)\right]=-\frac{1}{h}\sum_{l=1}^{3}\frac{\partial}{\partial\dot{x}^{(1)l}}\left\{\frac{\mu_1^{(g)}\left[ct,\vec{r};\vec{R}^{(1)}(t)\right]\left[\gamma(ct,\vec{r})\right]^2}{\sqrt{1-\left[\frac{l\left(\vec{r},\dot{\vec{r}}^{(1)},t\right)}{\gamma(ct,\vec{r})}\right]^2}}\right\}dt\delta\dot{x}^{(1)l}$$

$$-\frac{1}{h}\sum_{l,h=1}^{3}\frac{\partial}{\partial\dot{x}^{(1)l}}\left\{\frac{\mu_1^{(g)}\left[ct,\vec{r};\vec{R}^{(1)}(t)\right]\sum_{k=1}^{3}g_{hk}(ct,\vec{r})\,\dot{x}^{(1)k}(ct,\vec{r})}{\sqrt{1-\left[\frac{l\left(\vec{r},\dot{\vec{r}}^{(1)},t\right)}{\gamma(ct,\vec{r})}\right]^2}}\right\}dx^h\delta\dot{x}^{(1)l}$$

$$=-\frac{1}{h}\mu_1^{(g)}\left[ct,\vec{r};\vec{R}^{(1)}(t)\right]\left[\gamma(ct,\vec{r})\right]^2\sum_{l=1}^{3}\frac{\partial}{\partial\dot{x}^{(1)l}}\left\{\frac{1}{\sqrt{1-\left[\frac{l\left(\vec{r},\dot{\vec{r}}^{(1)},t\right)}{\gamma(ct,\vec{r})}\right]^2}}\right\}\delta\dot{x}^{(1)l}dt$$

$$-\frac{1}{h}\mu_1^{(g)}\left[ct,\vec{r};\vec{R}^{(1)}(t)\right]\sum_{h,l=1}^{3}\frac{\partial}{\partial\dot{x}^{(1)l}}\left\{\frac{\sum_{k=1}^{3}g_{hk}(ct,\vec{r})\,\dot{x}^{(1)k}(ct,\vec{r})}{\sqrt{1-\left[\frac{l\left(\vec{r},\dot{\vec{r}}^{(1)},t\right)}{\gamma(ct,\vec{r})}\right]^2}}\right\}\delta\dot{x}^{(1)l}dx^h,$$

whence the perturbation of the particle 1 de Broglie phase-wave has phase velocity

$$\mathcal{V}'(\vec{r},t)=\frac{c^2\sum_{l=1}^{3}\frac{\partial}{\partial\dot{x}^{(1)l}}\left\{\frac{1}{\sqrt{1-\frac{\sum_{k=1}^{3}\left[\dot{x}^{(1)k}(ct,\vec{r})\right]^2}{c^2}}}\right\}\delta\dot{x}^{(1)l}}{\sqrt{\sum_{h=1}^{3}\left(\sum_{l=1}^{3}\frac{\partial}{\partial\dot{x}^{(1)l}}\left\{\frac{\dot{x}^{(1)h}(ct,\vec{r})}{\sqrt{1-\frac{\sum_{k=1}^{3}\left[\dot{x}^{(1)k}(ct,\vec{r})\right]^2}{c^2}}}\right\}\delta\dot{x}^{(1)l}\right)^2}} \tag{34}$$

on the 'external region' of both particles corresponding to the flat portion of space-time. Then, it can be shown that, given any finite value of a potentially estimated lower bound for the superluminal speed $\mathcal{V}'$, say, $\overline{\mathcal{V}}'_{\min}>c$, there exists an internal dynamics of the particle 1 mass distribution such that $\mathcal{V}'>\overline{\mathcal{V}}'_{\min}$ on the 'external region' of both particles corresponding to the flat portion of space-time. Notice that this dynamics is compatible with the construction of the particle de Broglie phase-wave given in Section 7.2 (see, in particular, footnote [25]) as well as with the precepts of Einstein's relativity applied to the external local perturbation occurring on the portion of the particle 1 'external region' close to the particle 1 'intermediate region'

[38] As mentioned before, this 3-velocity field variation is brought on by a local disturbance taking place within the 'singular region' of particle 1.

(in particular, the validity of condition $ds^2 > 0$ on the 'external region'). Nevertheless, the introduced local perturbation induces a change in the mass dynamics within the boundary of the particle 1 'singular region' (where, by definition, $ds^2 < 0$) for which the values of the parameters $\dot{x}^{(1)1}$, $\dot{x}^{(1)2}$ and $\dot{x}^{(1)3}$ in (34) on the 'external region' of both particles corresponding to the flat portion of space-time become such that $\sum_{h=1}^{3} \left|\dot{x}^{(1)h}\right|^2 \leq c^2$ at all times. Summarising, if the phase agreement between the de Broglie phase-waves of particle 1 and particle 2 is ensured throughout the space during the motion of the particles and, consequently, the system de Broglie phase-wave keeps being smooth during the time interval while a perturbation propagates through the system, a variation of the 3-velocity field of the mass distribution corresponding to particle 1 causes a perturbation of the system de Broglie phase-wave throughout the 'external region' of particle 1 that, having phase velocity greater than c, may affect[39] the 3-velocity field of the mass distribution on the 'singular region' of particle 2. In other words, a local disturbance acting upon one particle[40] determines a non-local[41] effect upon a second particle, no matter how far from the first, provided that the weak interconnection between the two particles due to the system de Broglie phase-wave is not broken. Though by a different procedure, this conclusion seems to agree, for example, with the stochastic interpretation of quantum mechanics according to which the transmission of information is not restricted to the speed-of-light but undergoes a purely local theory that leads to a non-local theory (see [19]). The possibility for non-locality to arise from a classical context is already known by the physics community. For instance, a non-local behaviour has been noticed for the so-called resonant states, i.e., solitary wave solutions to certain classical non-linear wave equations (see [39–41]) and several experiments have shown that, for some modifications of the Schrödinger equation involving a non-linear term induced by gravity, communications between separate parts of the system cannot travel at finite speed during the collapse process of the system wave function (see [42]). From a similar perspective, before trying to experimentally test whether the "extended particle" model interpretation of non-locality is actually plausible, one should perhaps know the precise form of the non-linear term that must be added to the linear wave equation. For the time being, the question of whether the description of non-locality provided by the "extended particle" model is correct remains an open issue.

10 Discussion and Conclusion

Einstein's general relativity and de Broglie's guidance law exhibit the following remarkable consonance: the dynamics of the particle is obtained, according to the former, via the geodesic equations from the knowledge of the 4-force acting upon the particle and of the gravitational field[42] and, according to the latter, via the guidance formula from the phase of a wave with objective character that propagates in the physical 3-dimensional space. Thus, combining these two interpretations one expects that the gravitational field and the de Broglie field interact with each other through the particle by determining its motion. To better understand

[39] On the face of it, instantaneously but, in reality, at a finite superluminal speed.

[40] For instance, a measurement of an observable associated with this particle.

[41] 'Non-local' here means that there exist interactions between events that are too far apart in the physical 3-dimensional space and too close together in time for the events to be connected even by signals moving at the speed of light.

[42] In turn, these are derivable, respectively, from the energy-stress tensor and from solving Einstein's field equations.

the close relationship between these two fields one might appeal to a theory (or better said, a theoretical program) from which the guidance law can be derived. To be in agreement with quantum mechanics this theoretical program, called the Double Solution Theory, must satisfy some minimal requirements that have been thoroughly described by de Broglie, its inventor, long time ago (see [6]). But, in its original unfinished version this program addresses only partly its relationship with Einstein's general relativity, leaving room to deploy this latter theory in the attempt to disclose the very nature of the physical wave the existence of which it postulates. And this is actually the goal the model here described aims at by resorting to de Broglie's very seminal idea of wave-particle duality, that is, the wave associated with a point-like particle represents the distribution of phase of an infinity of clocks disseminated throughout the physical 3-dimensional space. At variance with the usual description of an elementary particle as a point-like entity, in this work the particle mass is assumed to be distributed over a finite 3-volume in order to avoid singularities for the space-time metric tensor and the evolution of any pair of points belonging to the mass distribution corresponds to two events separated by a space-like interval. Although arising from an utterly different viewpoint, this representation of the particle seems to agree with a result by Dirac:

> it is possible for a signal to be transmitted faster than light through the interior of an electron. The finite size of the electron now reappears in a new sense, the interior of the electron being a region of failure ... of some of the elementary properties of space-time

(see [43]). The wave associated with the "extended particle" built here resembles the wave of pure phase devised by de Broglie for a point-like particle (except for having an amplitude dependent on the space-time coordinates over a small portion of space) that can be related to the usual wave function of quantum mechanics as required by the Double Solution Theory. The connection between these two waves of different nature is realised recurring to a specific choice for the purely periodic phenomenon representing the clock that sits at each point of space outside the 3-volume occupied by the mass distribution, i.e., the complex exponential function considered by de Broglie (see, e.g., [6]). One might speculate as to why the relationship between the particle's de Broglie phase-wave and the particle wave function should depend on such a particular choice of periodic function among all those potentially valid to represent a purely periodic phenomenon. In this respect, a lack of generality might just be a limitation of the model presented in this work. Then, if the 3-volume occupied by the particle mass is made up of the 'singular region', the 'intermediate region' and, possibly, part of the 'external region' as conjectured by de Broglie's Double Solution Theory, the values of the amplitude and the phase of the particle wave function obtained following the usual methods of quantum mechanics provide, to a good approximation, with constraints for the internal dynamics of the mass distribution. In other words, the "extended particle" model arising in this study suggests that the measured values of the probability amplitudes and the frequencies associated with the transitions between energy stationary states are related deterministically to the space-time metric tensor and to the 4-velocity field of the point-like masses forming the "extended particle". In accordance with the classical framework of Einstein's general relativity, these two latter fields are obtained, respectively, from the Einstein field equations and the geodesic equations. But, for this to happen, the mass distribution must not disintegrate under the repulsion of the Coulomb field acting between the components of the "extended particle", whence one expects that there exists an attractive field at this scale such as, e.g., a Poincaré-like particle internal field (see [35, 44]) which contributes to the energy-stress tensor appearing in the Einstein field equations. Perhaps, such a gluing effect might originate from an appropriate non-linear term in the wave equation for the particle de Broglie phase-wave that, in accordance with the Double Solution Theory, should be sensibly non-zero only on a limited portion of the physical 3-dimensional space. The common approach to establish such a non-linear wave equation has so far rested upon guessing the non-linear term to be added to the usual linear wave equation considered in standard quantum mechanics and, even in the framework of the approximated version of this theory represented by the Pilot-Wave Theory, the existence of a possible solitonic solution to the wave equation has been shown choosing a priori the wave functional that guides the soliton. Instead, the strategy followed in the "extended particle" model is different from the common one to the extent that one deploys de Broglie's original idea (i.e., the existence of a periodic phenomenon associated with the particle) as well as general relativity arguments (i.e., the clock phase shift due to clock interaction with the gravitational field) to show that the particle wave function is associated with a time-evolving soliton defined in the physical 3-dimensional space and from this one can determine the exact form of the non-linear term in the complete wave equation (for conciseness sake, further details on this latter aspect will be provided in a subsequent work to be published). Notice that the importance of the non-linearity in the wave equations of quantum mechanics has been already highlighted in the past, especially in relation to the appearance of the discrete spectra, the stationary states and their stability as well as with the explanation of the quantum jumps, a feature that is strange to any genuinely linear theory (see [45–48]). However, so far, the supplementary tailor-made non-linear terms have been introduced to produce a desired effect (see [4, 49–57]). Also the non-linearity of the Schrödinger-Newton equation belongs to the class of theories based upon an a priori guess but it is of particular interest for comparison with the "extended particle" model due to its link with the gravitational effect on the particle wave function. In addition, recent speculations about the connection between the Schrödinger-Newton equation and de Broglie's Double Solution Theory ([15, 33, 58]) as well as a large literature on the existence, uniqueness and stability of global solutions for this equation and on approximated and asymptotic behaviour of its solutions (see [59–63]) might inspire further investigations on the "extended particle" model.

As shown in the present work, the elementary particle behaves like a fluid extended over the physical 3-dimensional space whose internal dynamics obeys the classical laws of motion but with some constraints that allow to retrieve the findings of quantum mechanics as if the particle were point-like. Although foreign to most approaches, this wave-monistic description of the particle has the advantage of being far from violating the No Singularity Principle (see [64]). Moreover, unlike the studies concerning the role of the non-linearity cited previously, this model for the relativistic scalar particle might help not only derive the non-linear wave equation envisaged by de Broglie's Double Solution Theory but also propose a suitable dynamical system for the internal structure of the "extended particle" where the atoms discrete energy spectrum could emerge, for instance, as a consequence of a limit cycle phenomenon (see [65] and references therein, [66]). Though, it would remain to check whether the "extended particle" model can be extended to the case of spinor particles albeit, at first sight, it seems that the main ideas behind it developed for the relativistic zero-spin particle can be also carried over to the relativistic half-spin particle. Finally, thanks to this model based on postulating that the physical 3-dimensional space is filled with, so to speak, an *aether* made up of clocks it has been possible to provide, consistently with the notion of de Broglie's phase-wave, an intelligible description of the single particle interference phenomenology as well as of non-locality in terms of a quantum channel, with its potentially novel application to superluminal communications. Possibly, one may also use the "extended particle" model to provide further insight into other quantum effects (e.g., the quantum tunneling).

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