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RESEARCH

On the Constructions of Quantum MDS Convolutional Codes

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Abstract
Quantum convolutional codes, which are the correct generalization to quantum domain of their classical analogs, were introduced to overcome decoherence during long distance quantum communications. In this paper, we construct some classes of quantum convolutional codes via classical constacyclic codes. These codes are maximum-distance-separable (MDS) codes in the sense that they achieve the Singleton bound for pure convolutional stabilizer codes. Furthermore, compared with some of the codes available in the literature, our codes have better parameters and are more general.

Keywords Quantum convolutional codes · Constacyclic codes · Negacyclic codes · MDS codes

1 Introduction

Quantum convolutional codes were first introduced by Chau in [3, 4], and then Ollivier and Tillich [21] gave the stabilizer framework for such codes, encoding and decoding methods of quantum convolutional codes were also described. Similar to quantum codes, the construction of quantum convolutional codes with good properties is also an interesting task. Many quantum convolutional codes had been constructed by various methods. Almeida and Palazzo [9] constructed a concatenated quantum convolutional code with parameters $[(4, 1, 3)]$. Quantum convolutional codes of rate $(n-2)/n$ had been constructed from self-orthogonal classical convolutional codes by Forney et al. in [10], which also have low decoding complexity. Grassl and R$\ddot{o}$tteler [11] constructed some quantum convolutional codes via product codes, and they also presented an algorithm for non-catastrophic encoders in [12].Tan and Li [22] constructed quantum convolutional codes through LDPC codes. Wilde and Brun [24] used two arbitrary binary convolutional codes to construct entanglement-assisted quantum convolutional codes. As quantum codes, the parameters of quantum convolutional codes are mutually restricted. Aly et al. [2] established the Singleton bound for pure convolutional stabilizer codes, and a class of quantum convolutional codes achieves such bound is derived

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from generalized Reed-Solomon codes. Later, La Guardia [17] constructed several classes of optimal quantum convolutional codes via BCH codes using an algebraic method. Due to the rich algebraic structure and efficient encoding and decoding circuits, constacyclic codes including cyclic codes and negacyclic codes are preferred objects on the construction of quantum convolutional codes. Lots of quantum convolutional codes with good parameters had been constructed from them (please see [1, 8, 14, 25] and the references therein). Particularly, many quantum MDS convolutional codes had been constructed. Among the obtained results, the lengths of these quantum MDS convolutional codes divide q^2-1 (please see [7, 8, 25, 28]) or q^2+1 (please see [5–7, 18–20, 25, 26, 28]).

Going on the line of the above studies, we construct eight classes of quantum MDS convolutional codes via classical constacyclic codes (including negacyclic codes). Precisely speaking, for any odd prime power q, the quantum MDS convolutional codes have the following parameters.

- $[(\frac{q^2+1}{\rho}, \frac{q^2+1}{\rho}-4\delta+2, 1; 2, 2\delta+2)]_q$, where $q=2\rho m+2a+1$, $\rho=a^2+(a+1)^2$, a and m are positive integers, and $2\le\delta\le\frac{(2a+1)q+1}{2\rho}-1$.
- $[(\frac{q^2+1}{\rho}, \frac{q^2+1}{\rho}-4\delta+2, 1; 2, 2\delta+2)]_q$, where $q=2\rho m-2a-1$, $\rho=a^2+(a+1)^2$, a and m are positive integers, and $2\le\delta\le\frac{(2a+1)q-1}{2\rho}-1$.
- $[(\frac{q^2-1}{\mu}, \frac{q^2-1}{\mu}-2\delta, 1; 1, \delta+2)]_q$, where $q=\mu m+\ell$, $\mu=\ell^2-1$, m is a positive integer, ℓ is a positive odd integer, and $2\le\delta\le\ell m-1$.
- $[(\frac{q^2-1}{\mu}, \frac{q^2-1}{\mu}-2\delta, 1; 1, \delta+2)]_q$, where $q=\mu m-\ell$, $\mu=\ell^2-1$, m is a positive integer, ℓ is a positive odd integer, and $2\le\delta\le(\ell-1)m-3$.
- $[(\frac{q^2-1}{\mu}, \frac{q^2-1}{\mu}-2\delta, 1; 1, \delta+2)]_q$, where $q=\mu m+\ell$, $\mu=\frac{\ell^2-1}{2}$, $\ell\equiv 1(\mathrm{mod}4)$, m is a positive integer, ℓ is a positive odd integer, and $2\le\delta\le\ell m$.
- $[(\frac{q^2-1}{\mu}, \frac{q^2-1}{\mu}-2\delta, 1; 1, \delta+2)]_q$, where $q=\mu m+\ell$, $\mu=\frac{\ell^2-1}{2}$, $\ell\equiv 3(\mathrm{mod}4)$, m is a positive integer, ℓ is a positive odd integer, and $2\le\delta\le\frac{\ell+1}{2}m-1$.
- $[(\frac{q^2-1}{\mu}, \frac{q^2-1}{\mu}-2\delta, 1; 1, \delta+2)]_q$, where $q=\mu m-\ell$, $\mu=\frac{\ell^2-1}{2}$, $\ell\equiv 1(\mathrm{mod}4)$, m is a positive integer, ℓ is a positive odd integer, and $2\le\delta\le\frac{\ell-1}{2}m-3$.
- $[(\frac{q^2-1}{\mu}, \frac{q^2-1}{\mu}-2\delta, 1; 1, \delta+2)]_q$, where $q=\mu m-\ell$, $\mu=\frac{\ell^2-1}{2}$, $\ell\equiv 3(\mathrm{mod}4)$, m is a positive integer, ℓ is a positive odd integer, and $2\le\delta\le\ell m-4$.

The paper is organized as follows. In Section 2, we recall some basic results of constacyclic codes. In Section 3, we state some notations and results about quantum convolutional codes. In Sections 4 and 5, we give the constructions of quantum MDS convolutional codes based on classical constacyclic codes of length $\frac{q^2+1}{\rho}$ and classical negacyclic codes of length $\frac{q^2-1}{\mu}$, respectively. A conclusion is given in Section 6.

2 Constacyclic Codes

In this paper, we will consider the codes under the Hermitian inner product. Let $\mathbb{F}_{q^2}$ be the finite field with q^2 elements, where p is an odd prime number and q is a power of p. A linear code $\mathscr{C}$ over $\mathbb{F}_{q^2}$ of length n with dimension k and minimum distance d, denoted as $[n,k,d]_{q^2}$, is a linear subspace of $\mathbb{F}_{q^2}^n$. The parameters of such codes satisfy the well-known Singleton bound: $d\le n-k+1$. If d achieves such bound, then $\mathscr{C}$ is called an MDS code. For any two vectors $\mathbf{x}=(x_0,x_1,\ldots,x_{n-1})$, and $\mathbf{y}=(y_0,y_1,\ldots,y_{n-1})\in\mathbb{F}_{q^2}^n$, the Hermitian inner product is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle_H := x_0 y_0^q + x_1 y_1^q + \cdots + x_{n-1} y_{n-1}^q.$$

If $\langle \mathbf{x}, \mathbf{y} \rangle_H = 0$, then the vectors $\mathbf{x}$ and $\mathbf{y}$ are called orthogonal with respect to the Hermitian inner product. The Hermitian dual code $\mathscr{C}^{\perp_H}$ of $\mathscr{C}$ is defined as

$$\mathscr{C}^{\perp_H} := \{\mathbf{x} \in \mathbb{F}_{q^2}^n | \langle \mathbf{x}, \mathbf{y} \rangle_H = 0 \text{ for all } \mathbf{y} \in \mathscr{C}\}.$$

It is easy to see that $\mathscr{C}^{\perp_H}$ is also a linear code, which has dimension $n - \dim(\mathscr{C})$.

For any nonzero element $\lambda \in \mathbb{F}_{q^2}$, defining a map

$$\varphi : \mathbb{F}_{q^2}^n \longrightarrow \mathbb{F}_{q^2}^n,$$

$$(c_0, c_1, \ldots, c_{n-1}) \longmapsto (\lambda c_{n-1}, c_0, \ldots, c_{n-2}).$$

If $\varphi(\mathscr{C}) = \mathscr{C}$, then $\mathscr{C}$ is called a λ-constacyclic code. In the case $\lambda = 1$, $\mathscr{C}$ is a cyclic code, while in the case $\lambda = -1$, $\mathscr{C}$ is the so-called negacyclic code. Defining the following map

$$\psi : \mathbb{F}_{q^2}^n \longrightarrow \mathscr{R} = \frac{\mathbb{F}_{q^2}[x]}{\langle x^n - \lambda \rangle},$$

$$(c_0, c_1, \ldots, c_{n-1}) \longmapsto c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}.$$

Then $\mathscr{C}$ is a λ-constacyclic code if and only if $\psi(\mathscr{C}) = \{\psi(\mathbf{c}) | \mathbf{c} \in \mathscr{C}\}$ is an ideal of the quotient ring $\mathscr{R}$. As we know, $\mathscr{R}$ is a principal ideal ring. Assume that $\mathscr{C} = \langle f(x) \rangle$ is a λ-constacyclic code of length n over $\mathbb{F}_{q^2}$, where $f(x)$ is a monic polynomial of minimal degree in $\mathscr{C}$. Then $f(x) | (x^n - \lambda)$ and $f(x)$ is called the generator polynomial of $\mathscr{C}$. The dimension of $\mathscr{C}$ is $n - deg(f(x))$.

Assume that $\gcd(n, q) = 1$, the order of λ is r, i.e., $\mathrm{ord}(\lambda) = r$, and the multiplicative order of q^2 modulo rn is u, i.e., $\mathrm{ord}_{rn}(q^2) = u$. Then there is a primitive rn-th root of unity $\xi \in \mathbb{F}_{q^{2u}}$ such that $\xi^n = \lambda$, which implies that $x^n - \lambda = \prod_{i=0}^{n-1}(x - \xi^{1+ri})$. Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ be the ring of integers modulo n and $\mathbb{Z}_{rn}$ be the ring of integers modulo rn. Suppose that $m_i(x)$ is the minimal polynomial of ξ^{1+ri} over $\mathbb{F}_{q^2}$. If $\mathscr{C} = \langle f(x) \rangle$ is a λ-constacyclic code of length n over $\mathbb{F}_{q^2}$, then there is a subset $\Omega \subseteq \mathbb{Z}_n$ such that $f(x) = \prod_{i \in \Omega} m_i(x)$. For every $i \in \mathbb{Z}_{rn}$, the q^2-cyclotomic coset of i modulo rn is defined by

$$C_i := \{iq^{2l} (\mathrm{mod} rn) | 0 \leq l \leq l_i - 1\},$$

where l_i is the smallest positive integer such that $i \equiv iq^{2l_i}$ (mod rn). Let $\mathscr{Z} = \{i \in \mathbb{Z}_{rn} | f(\xi^i) = 0\}$. The set $\mathscr{Z}$ is called the defining set of $\mathscr{C}$. It is clear that $\mathscr{Z}$ is a union of some q^2-cyclotomic cosets and $\dim(\mathscr{C}) = n - |\mathscr{Z}|$, where $|\mathscr{Z}|$ means the cardinality of the set $\mathscr{Z}$. As we know, it is hard to know the exact value of the distance of code $\mathscr{C}$, but it can be estimated by the following well-known bound.

Theorem 1 ***(BCH bound)*** [16] *Let δ be an integer in the range $2 \leq \delta \leq n$. Assume that $\mathscr{C}$ is a λ-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z}$. If $\mathscr{Z}$ consists of $\delta - 1$ consecutive elements, then $d(\mathscr{C}) \geq \delta$.*

The following lemma gives a parity-check matrix of $\mathscr{C}$, which will play an important role in our construction.

Lemma 1 [16, 19] *Assume that $\lambda \in \mathbb{F}_{q^2}^*$, $\mathrm{ord}(\lambda) = r$, and $\mathrm{ord}_{rn}(q^2) = u$. Taking a primitive rn-th root of unity $\beta \in \mathbb{F}_{q^{2u}}$ such that $\beta^n = \lambda$. Suppose that $\mathscr{C}$ is a λ-constacyclic code of* *length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z} = \bigcup_{i=b}^{\delta-2} C_{1+ri}$, where b is a nonnegative integer. Then a parity-check matrix of $\mathscr{C}$ can be obtained from the matrix*

$$H = \begin{pmatrix} 1 & \beta^{1+rb} & \beta^{2(1+rb)} & \dots & \beta^{(n-1)(1+rb)} \\ 1 & \beta^{1+r(b+1)} & \beta^{2[1+r(b+1)]} & \dots & \beta^{(n-1)[1+r(b+1)]} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta^{1+r(\delta-3)} & \beta^{2[1+r(\delta-3)]} & \dots & \beta^{(n-1)[1+r(\delta-3)]} \\ 1 & \beta^{1+r(\delta-2)} & \beta^{2[1+r(\delta-2)]} & \dots & \beta^{(n-1)[1+r(\delta-2)]} \end{pmatrix}$$

by expanding each entry as a column vector which contains u rows with respect to certain $\mathbb{F}_{q^2}$-basis of $\mathbb{F}_{q^{2u}}$ and then removing any linearly dependent rows.

3 Quantum Convolutional Codes

In this section, we recall some basic notations and results about classical convolutional codes and quantum convolutional codes over finite fields. Let $\mathbb{F}_{q^2}[D]$ be the polynomial ring in the indeterminate D with coefficients in $\mathbb{F}_{q^2}$ and $G(D)$ be a $k \times n$ matrix over $\mathbb{F}_{q^2}[D]$. $G(D)$ is called basic if $G(D)$ has a polynomial right inverse. A basic polynomial generator matrix $G(D)$ is called reduced if the overal constraint length $\gamma = \sum_{i=1}^{k} \gamma_i$ attains the minimal value among all basic generator matrices of the convolutional code $\mathscr{V}$.

Definition 1 [2] A rate k/n convolutional code $\mathscr{V}$ with parameters $(n, k, \gamma; \upsilon, d_f)_{q^2}$ is a submodule of $\mathbb{F}_{q^2}[D]^n$ generated by a reduced basic matrix $G(D) = (g_{ij}(D)) \in \mathbb{F}_{q^2}[D]^{k\times n}$, that is, $\mathscr{V} = \{\mathbf{u}(D)G(D)|\mathbf{u}(D) \in \mathbb{F}_{q^2}[D]^k\}$, where n is the length, k is the dimension, $\gamma_i = \max\limits_{1\le j\le n} \{\deg g_{ij}(D)\}$ is the i-th row degree, $\gamma = \sum_{i=1}^{k} \gamma_i$ is the degree, $\upsilon = \max\limits_{1\le i\le k} \{\gamma_i\}$ is the memory and $d_f = \min\{\mathrm{wt}(\mathbf{v}(D))|\mathbf{v}(D) \in \mathscr{V}, \mathbf{v}(D) \neq 0\}$ is the free distance of $\mathscr{V}$. Here, $\mathrm{wt}(\mathbf{v}(D)) = \sum_{i=1}^{n} \mathrm{wt}(v_i(D))$, where $\mathrm{wt}(v_i(D))$ is the number of nonzero coefficients of $v_i(D)$.

Given any two n-tuples $\mathbf{u}(D) = \sum_i \mathbf{u}_i D^i$ and $\mathbf{v}(D) = \sum_i \mathbf{v}_i D^i$ in $\mathbb{F}_{q^2}[D]^n$, their Hermitian inner product is defined as $\langle \mathbf{u}(D)|\mathbf{v}(D)\rangle_H = \sum_i \mathbf{u}_i \cdot \mathbf{v}_i^q$, where $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_{q^2}^n$ and $\mathbf{v}_i^q = (v_{i_1}^q, v_{i_2}^q, \dots, v_{i_n}^q)$. The Hermitian dual code $\mathscr{V}^{\perp_H}$ of a convolutional code $\mathscr{V}$ is defined as

$$\mathscr{V}^{\perp_H} = \{\mathbf{u}(D) \in \mathbb{F}_{q^2}[D]^n | \langle \mathbf{u}(D)|\mathbf{v}(D)\rangle_H = 0 \text{ for all } \mathbf{v}(D) \in \mathscr{V}\}.$$

The following lemma presents a relationship between $\mathscr{V}$ and $\mathscr{V}^{\perp_H}$.

Lemma 2 [28] *If $\mathscr{V}$ is an $(n, k, \gamma)_{q^2}$ convolutional code, then $\mathscr{V}^{\perp_H}$ is an $(n, n-k, \gamma)_{q^2}$ convolutional code.*

As we know, convolutional codes can be constructed from block codes. Let $\mathscr{C}$ be an $[n, k, d]_{q^2}$ linear code with parity-check matrix H. Splitting H into $\upsilon+1$ disjoint submatrices H_i such that

$$H = \begin{pmatrix} H_0 \\ H_1 \\ \vdots \\ H_\upsilon \end{pmatrix}, \tag{1}$$

where each H_i has n columns. Let κ be the largest number of rows among all the matrices H_i, where $0 \le i \le \upsilon$. Each matrix H_i is enlarged to a matrix $\widetilde{H}_i$ by adding zero-rows at the bottom such that $\widetilde{H}_i$ has κ rows in total. Let

$$G(D) = \widetilde{H}_0 + \widetilde{H}_1 D + \cdots + \widetilde{H}_\upsilon D^\upsilon. \tag{2}$$

Then $G(D)$ generates a convolutional code $\mathscr{V}$. The parameters of $\mathscr{V}$ can be derived from the following theorem.

Theorem 2 [1] *Let $\mathscr{C}$ be an $[n, k, d]_{q^2}$ linear code with parity-check matrix $H \in \mathbb{F}_{q^2}^{(n-k)\times n}$. Suppose that H is partitioned into submatrices $H_0, H_1, \ldots, H_\upsilon$ as in (1) such that $\kappa = rank(H_0)$ and $rank(H_i) \le \kappa$ for $1 \le i \le \upsilon$. Let $G(D)$ be defined as in (2), and the convolutional code $\mathscr{V} = \{\boldsymbol{u}(D)G(D) | \boldsymbol{u}(D) \in \mathbb{F}_{q^2}[D]^{n-k}\}$. Then we can get*

(i) The matrix $G(D)$ is a reduced basic generator matrix.
(ii) If $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$, then $\mathscr{V} \subseteq \mathscr{V}^{\perp_H}$.
(iii) Let d_f and $d_f^{\perp_H}$ be the free distance of $\mathscr{V}$ and $\mathscr{V}^{\perp_H}$, respectively. Let d_i be the minimum Hamming distance of the code $\mathscr{C}_i = \{\boldsymbol{c} \in \mathbb{F}_{q^2}^n | \boldsymbol{c}\widetilde{H}_i^T = 0\}$, and let $d^{\perp_H}$ be the minimum Hamming distance of $\mathscr{C}^{\perp_H}$. Then $\min\{d_0 + d_\upsilon, d\} \le d_f^{\perp_H} \le d$ and $d_f \ge d^{\perp_H}$.

Quantum convolutional codes are defined as infinite versions of quantum stabilizer codes. The stabilizer can be defined by a matrix with polynomial entries

$$S(D) = (X(D)|Z(D)) \in \mathbb{F}_q[D]^{(n-k)\times 2n},$$

which satisfies $X(D)Z(1/D)^T - Z(D)X(1/D)^T = 0$. If a quantum convolutional code $\mathscr{Q}$ is generated by the full-rank stabilizer matrix $S(D)$, then $\mathscr{Q}$ has parameters $[(n, k, \upsilon; \gamma, d_f)]_q$, where n is the frame size, k is the number of logical qudits per frame, γ is the degree, $\upsilon = \max_{1\le i\le n-k, 1\le j\le n}\{\max\{\deg X_{ij}(D), \deg Z_{ij}(D)\}\}$ is the memory, and d_f is the free distance of $\mathscr{Q}$. It had already been shown in [2] that the free distance of a quantum convolutional code $\mathscr{Q}$ must satisfy the following version of the Singleton bound.

Theorem 3 *The free distance of an $[(n, k, \upsilon; \gamma, d_f)]_q$ $\mathbb{F}_{q^2}$-linear pure convolutional stabilizer code is bounded by*

$$d_f \le \frac{n-k}{2}\left(\left\lfloor \frac{2\gamma}{n+k} \right\rfloor + 1\right) + \gamma + 1.$$

If the free distance d_f of the quantum convolutional code $\mathscr{Q}$ attains such bound, then $\mathscr{Q}$ is called a quantum MDS convolutional code. Actually, it is not an easy task to construct quantum MDS convolutional codes. The following theorem gives a connection between classical convolutional codes and quantum convolutional codes.

Theorem 4 [1] *Let $\mathscr{V}$ be an $(n, (n-k)/2, \gamma; \upsilon)_{q^2}$ convolutional code such that $\mathscr{V} \subseteq \mathscr{V}^{\perp_H}$. Then there exists an $[(n, k, \upsilon; \gamma, d_f)]_q$ convolutional stabilizer code, where $d_f = wt(\mathscr{V}^{\perp_H} \setminus \mathscr{V})$.*

From the aforementioned theorem, we can apply Hermitian self-orthogonal convolutional codes over $\mathbb{F}_{q^2}$ to construct quantum convolutional codes. In this paper, we first use constacyclic codes and negacyclic codes to construct some Hermitian self-orthogonal convolutional codes and then some quantum MDS convolutional codes are derived from such convolutional codes.

4 Quantum MDS Convolutional Codes of Length $\frac{q^2+1}{\rho}$

Let $\eta \in \mathbb{F}_{q^2}^*$ and $\mathrm{ord}(\eta) = q+1$. In this section, we will construct some quantum MDS convolutional codes of length $n = \frac{q^2+1}{\rho}$ via η-constacyclic codes, where $\rho = a^2 + (a+1)^2$, and a is a positive integer. As n should be an integer, it can be easily obtained that q is a prime power with the form $q = \rho m + 2a + 1$ or $q = \rho m - 2a - 1$, where m is a positive integer. Here we only consider q being odd with the form $q = 2\rho m \pm (2a+1)$. In order to proceed further, we first recall some relevant results shown in the literature.

Lemma 3 *[15] Let $n = \frac{q^2+1}{\rho}$, $s = \frac{q^2+1}{2}$, and ρ be odd. Then all cyclotomic cosets modulo $(q+1)n$ containing $1 + (q+1)i$ are as follows:*

(1) $C_s = \{s\}$ and $C_{s \pm \frac{q+1}{2}n} = \{s \pm \frac{q+1}{2}n\}$.

(2) $C_{s-(q+1)i} = \{s - (q+1)i, s + (q+1)i\}$ for $1 \le i \le n/2 - 1$.

Lemma 4 *Let q be an odd prime power with the form $q = 2\rho m + 2a + 1$, where $\rho = a^2 + (a+1)^2$, and a, m are positive integers. Let $n = \frac{q^2+1}{\rho}$, $s = \frac{q^2+1}{2}$. If $\mathscr{C}$ is an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z} = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \le \delta \le \frac{(2a+1)q+1}{2\rho} - 1$, then $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$.*

Proof The case $a \ge 2$ had been proved in [13], while the case $a = 1$ had been proved in [27]. Hence, the result is true. □

Lemma 5 *Let $n = \frac{q^2+1}{\rho}$ and q be an odd prime power with the form $q = 2\rho m + 2a + 1$, where $\rho = a^2 + (a+1)^2$, and a, m are positive integers. Then there exists an $(n, n-2\delta+1, 2; 1, 2\delta+2)_{q^2}$ convolutional code which contains its Hermitian dual, where $2 \le \delta \le \frac{(2a+1)q+1}{2\rho} - 1$.*

Proof According to Lemma 3, the order of q^2 modulo $(q+1)n$ is equal to 2. Suppose that $\beta \in \mathbb{F}_{q^4}$ is a primitive $(q+1)n$-th root of unity such that $\beta^n = \eta$. Let $\mathscr{C}$ be an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z} = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $2 \le \delta \le \frac{(2a+1)q+1}{2\rho} - 1$. By Lemma 1, the parity-check matrix of $\mathscr{C}$, denoted by H, can be obtained from the following matrix

$$H_{\mathscr{C}} = \begin{pmatrix} 1 & \beta^{s} & \beta^{2s} & \cdots & \beta^{(n-1)s} \\ 1 & \beta^{s+(q+1)} & \beta^{2[s+(q+1)]} & \cdots & \beta^{(n-1)[s+(q+1)]} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta^{s+(\delta-1)(q+1)} & \beta^{2[s+(\delta-1)(q+1)]} & \cdots & \beta^{(n-1)[s+(\delta-1)(q+1)]} \\ 1 & \beta^{s+\delta(q+1)} & \beta^{2[s+\delta(q+1)]} & \cdots & \beta^{(n-1)[s+\delta(q+1)]} \end{pmatrix},$$

by expanding each entry as a column vector which contains 2 rows with respect to certain $\mathbb{F}_{q^2}$-basis of $\mathbb{F}_{q^4}$ and then removing any linearly dependent rows. Hence, H has rank $2\delta+1$. Note that $\mathscr{C}$ is an MDS code with parameters $[n, n-2\delta-1, 2\delta+2]$. Moreover, $\mathscr{C}^{\perp_H}$ is also an MDS code with parameters $[n, 2\delta+1, n-2\delta]$.

Suppose that $\mathscr{C}_0$ is an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z}_0 = \bigcup_{j=0}^{\delta-1} C_{s-(q+1)j}$, where $2 \leq \delta \leq \frac{(2a+1)q+1}{2\rho} - 1$. Then $\mathscr{C}_0$ is an MDS code with parameters $[n, n-2\delta+1, 2\delta]$, and its Hermitian dual code $\mathscr{C}_0^{\perp_H}$ is also an MDS code with parameters $[n, 2\delta-1, n-2\delta+2]$. Furthermore, the parity-check matrix of $\mathscr{C}_0$, denoted by H_0, can be obtained from the following matrix

$$H_{\mathscr{C}_0} = \begin{pmatrix} 1 & \beta^{s} & \beta^{2s} & \cdots & \beta^{(n-1)s} \\ 1 & \beta^{s+(q+1)} & \beta^{2[s+(q+1)]} & \cdots & \beta^{(n-1)[s+(q+1)]} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta^{s+(\delta-1)(q+1)} & \beta^{2[s+(\delta-1)(q+1)]} & \cdots & \beta^{(n-1)[s+(\delta-1)(q+1)]} \end{pmatrix},$$

by expanding each entry as a column vector which contains 2 rows with respect to certain $\mathbb{F}_{q^2}$-basis of $\mathbb{F}_{q^4}$ and then removing any linearly dependent rows. In fact, $H_{\mathscr{C}_0}$ is a submatrix of $H_{\mathscr{C}}$, so H_0 can be directly derived from the parity-check matrix H. Particularly, the rank of H_0 is $2\delta-1$.

Suppose that $\mathscr{C}_1$ is an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z}_1 = C_{s-(q+1)\delta}$, where $2 \leq \delta \leq \frac{(2a+1)q+1}{2\rho} - 1$. Then $\mathscr{C}_1$ is an $[n, n-2]$ code with minimum Hamming distance ≥ 2. The parity-check matrix of $\mathscr{C}_1$, denoted by H_1, can be obtained by expanding the following matrix

$$H_{\mathscr{C}_1} = \left(1,\ \beta^{s+\delta(q+1)},\ \beta^{2[s+\delta(q+1)]},\ \cdots,\ \beta^{(n-1)[s+\delta(q+1)]}\right),$$

which also can be directly derived from the parity-check matrix H.

Due to the above construction, one can see that H has been partitioned into two submatrices H_0 and H_1 such that

$$H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}.$$

Assume that $G(D) = \widetilde{H}_0 + \widetilde{H}_1 D$, where $\widetilde{H}_0 = H_0$ and $\widetilde{H}_1$ is obtained from H_1 by adding zero-rows at its bottom such that $\widetilde{H}_1$ has the same number of rows as H_0. Due to Theorem 2(i), $G(D)$ is a reduced basic generator matrix, and the convolutional code $\mathscr{V}$ generated by such matrix has dimension $2\delta-1$ and degree 2. It follows from Lemma 2 that the Hermitian dual code $\mathscr{V}^{\perp_H}$ has dimension $n-2\delta+1$ and degree 2. According to Theorem 2(iii), the free distance $d_f^{\perp_H}$ of $\mathscr{V}^{\perp_H}$ satisfies that $\min\{d_0+d_1, d\} \leq d_f^{\perp_H} \leq d$, where d_0, d_1 and d are the minimum Hamming distances of the constacyclic codes $\mathscr{C}_0, \mathscr{C}_1$ and $\mathscr{C}$, respectively. Then we can get $d_f^{\perp_H} = 2\delta+2$. Hence, the convolutional code $\mathscr{V}^{\perp_H}$ has parameters $(n, n-2\delta+1, 2; 1, 2\delta+2)_{q^2}$. Finally, it follows from Theorem 2(ii) that $\mathscr{V}$ is Hermitian self-orthogonal due to the fact that $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$.

The construction of quantum MDS convolutional codes can be given by using the above lemma.

Theorem 5 *Let $n = \frac{q^2+1}{\rho}$ and q be an odd prime power with the form $q = 2\rho m + 2a + 1$, where $\rho = a^2 + (a+1)^2$, and a, m are positive integers. Then there exists a quantum MDS convolutional code with parameters $[(n, n-4\delta+2, 1; 2, 2\delta+2)]_q$, where $2 \leq \delta \leq \frac{(2a+1)q+1}{2\rho} - 1$.*

Proof Due to Lemma 5, a convolutional code $\mathscr{V}$ with parameters $(n, 2\delta-1, 2; 1, d_f)_{q^2}$ can be obtained, which also satisfies $\mathscr{V} \subseteq \mathscr{V}^{\perp_H}$ for any $2 \leq \delta \leq \frac{(2a+1)q+1}{2\rho} - 1$. According to Theorem 2(iii), $d_f \geq n - 2\delta$. It is easy to see that $d_f^{\perp_H} = 2\delta + 2 < n - 2\delta$. Therefore, a quantum convolutional code with parameters $[(n, n-4\delta+2, 1; 2, 2\delta+2)]_q$ can be derived from $\mathscr{V}$ due to Theorem 4. Such quantum convolutional code is a quantum MDS convolutional code due to the following fact:

$$\frac{n-k}{2}\left(\left\lfloor \frac{2\gamma}{n+k} \right\rfloor + 1\right) + \gamma + 1 = (2\delta-1)(0+1) + 2 + 1 = 2\delta + 2 = d_f.$$

Now we consider the case $q = 2\rho m - 2a - 1$, where $\rho = a^2 + (a+1)^2$, and a, m are positive integers. Similar to Lemma 4, we also have the following result from [13] and [27].

Lemma 6 *Let q be an odd prime power with the form $q = 2\rho m - 2a - 1$, where $\rho = a^2 + (a+1)^2$, and a, m are positive integers. Let $n = \frac{q^2+1}{\rho}$, $s = \frac{q^2+1}{2}$. If $\mathscr{C}$ is an η-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z} = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \leq \delta \leq \frac{(2a+1)q-1}{2\rho} - 1$, then $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$.*

Similar to the discussion of Lemma 5 and Theorem 5, we have the following results.

Lemma 7 *Let $n = \frac{q^2+1}{\rho}$ and q be an odd prime power with the form $q = 2\rho m - 2a - 1$, where $\rho = a^2+(a+1)^2$, and a, m are positive integers. Then there exists an $(n, n-2\delta+1, 2; 1, 2\delta+2)_{q^2}$ convolutional code which contains its Hermitian dual, where $2 \leq \delta \leq \frac{(2a+1)q-1}{2\rho} - 1$.*

Theorem 6 *Let $n = \frac{q^2+1}{\rho}$ and q be an odd prime power with the form $q = 2\rho m - 2a - 1$, where $\rho = a^2 + (a+1)^2$, and a, m are positive integers. Then there exists a quantum MDS convolutional code with parameters $[(n, n-4\delta+2, 1; 2, 2\delta+2)]_q$, where $2 \leq \delta \leq \frac{(2a+1)q-1}{2\rho} - 1$.*

Remark 1 Let $a = 1$, then $n = \frac{q^2+1}{5}$. Quantum MDS convolutional codes of length $\frac{q^2+1}{5}$ with q being an odd prime power had already been constructed in [28]. Comparing their results with ours, one can easily see from Table 1 that our codes have larger free distances. What's more, our construction is more general.

Remark 2 Let $a = 2$, then $n = \frac{q^2+1}{13}$. Actually, quantum MDS convolutional codes of length $\frac{q^2+1}{13}$ with q being an odd prime power had already been studied in [20]. Comparing their results with ours, one can easily see from Table 2 that our results coincide with theirs within such length. Hence, our results can be seen as a generalization of theirs.

Table 1 Quantum MDS convolutional codes of length $\frac{q^2+1}{5}$

Parameters $[(n, k, \upsilon; \gamma, d_f)]_q$	q	Our δ	δ in [28]
$[(\frac{q^2+1}{5}, \frac{q^2+1}{5} - 4\delta + 2, 1; 2, 2\delta + 2)]_q$	$20m+3$	$2 \leq \delta \leq 6m$	$2 \leq \delta \leq 5m+1$
	$20m+7$	$2 \leq \delta \leq 6m+1$	$2 \leq \delta \leq 5m+2$
	$20m-3$	$2 \leq \delta \leq 6m-2$	$2 \leq \delta \leq 5m-1$
	$20m-7$	$2 \leq \delta \leq 6m-3$	$2 \leq \delta \leq 5m-2$

Table 2 Quantum MDS convolutional codes of length $\frac{q^2+1}{13}$

Parameters $[(n, k, \upsilon; \gamma, d_f)]_q$	q	Our δ	δ in [20]
$[(\frac{q^2+1}{13}, \frac{q^2+1}{13} - 4\delta + 2, 1; 2, 2\delta + 2)]_q$	$26m + 5$	$2 \leq \delta \leq 5m$	$2 \leq \delta \leq 5m$
	$26m - 5$	$2 \leq \delta \leq 5m - 2$	$2 \leq \delta \leq 5m - 2$

Example 1 Let $a = 3$, then $n = \frac{q^2+1}{25}$. Quantum MDS convolutional codes of length $n = \frac{q^2+1}{25}$ are constructed. Some new quantum MDS convolutional codes obtained from Theorems 5 and 6 are listed in Table 3.

5 Quantum MDS Convolutional Codes of Length $\frac{q^2-1}{\mu}$

In this section, we will construct some classes of quantum MDS convolutional codes via classical negacyclic codes of length $n = \frac{q^2-1}{\mu}$ with $\mu = \ell^2 - 1$ and $\mu = \frac{\ell^2-1}{2}$, respectively. Since $q^2 \equiv 1(\mathrm{mod} 2n)$, the q^2-cyclotomic coset C_x modulo $2n$ is $C_x = \{x\}$ for each odd x in the range $1 \leq x \leq 2n - 1$.

5.1 Quantum MDS Convolutional Codes of Length $\frac{q^2-1}{\mu}$ with $\mu = \ell^2 - 1$

In this subsection, we will construct some new classes of quantum MDS convolutional codes of length $n = \frac{q^2-1}{\mu}$ from negacyclic codes, where $q = \mu m \pm \ell$, $\mu = \ell^2 - 1$ and $\ell > 1$ is a positive odd integer. We first consider the case $q = \mu m + \ell$ and recall a useful lemma in the following, which will play an important role in our construction.

Table 3 New quantum MDS convolutional codes

n	q	Parameters $[(n, k, \upsilon; \gamma, d_f)]_q$	δ
$\frac{q^2+1}{25}$	43	$[(74, 76 - 4\delta, 1; 2, 2\delta + 2)]_{43}$	$2 \leq \delta \leq 5$
	57	$[(130, 132 - 4\delta, 1; 2, 2\delta + 2)]_{57}$	$2 \leq \delta \leq 7$
	107	$[(458, 460 - 4\delta, 1; 2, 2\delta + 2)]_{107}$	$2 \leq \delta \leq 14$
	157	$[(986, 988 - 4\delta, 1; 2, 2\delta + 2)]_{157}$	$2 \leq \delta \leq 21$
	193	$[(1490, 1492 - 4\delta, 1; 2, 2\delta + 2)]_{193}$	$2 \leq \delta \leq 26$
	257	$[(2642, 2644 - 4\delta, 1; 2, 2\delta + 2)]_{257}$	$2 \leq \delta \leq 35$
	293	$[(3434, 3436 - 4\delta, 1; 2, 2\delta + 2)]_{293}$	$2 \leq \delta \leq 40$
	307	$[(3770, 3772 - 4\delta, 1; 2, 2\delta + 2)]_{307}$	$2 \leq \delta \leq 42$
$\frac{q^2+1}{41}$	73	$[(130, 132 - 4\delta, 1; 2, 2\delta + 2)]_{73}$	$2 \leq \delta \leq 7$
	173	$[(730, 732 - 4\delta, 1; 2, 2\delta + 2)]_{173}$	$2 \leq \delta \leq 18$
	337	$[(2770, 2772 - 4\delta, 1; 2, 2\delta + 2)]_{337}$	$2 \leq \delta \leq 36$
	401	$[(3922, 3924 - 4\delta, 1; 2, 2\delta + 2)]_{401}$	$2 \leq \delta \leq 43$

Lemma 8 [23] *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m + \ell$, $\mu = \ell^2 - 1$, m is a positive integer, and ℓ is a positive odd integer. If $\mathscr{C}$ is a q^2-ary negacyclic code of length n with defining set*

$$\mathscr{Z} = \bigcup_{j=\frac{3\ell+3}{2}m+2}^{s_1} C_{1+2j},$$

where $\frac{3\ell+3}{2}m + 2 \leq s_1 \leq \frac{5\ell+3}{2}m + 1$. Then $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$.

Lemma 9 *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m + \ell$, $\mu = \ell^2 - 1$, m is a positive integer, and ℓ is a positive odd integer. Then there exists an $(n, n-\delta, 1; 1, \delta+2)_{q^2}$ convolutional code which contains its Hermitian dual, where $2 \leq \delta \leq \ell m - 1$.*

Proof As we know, the order of q^2 modulo $2n$ is equal to 1. Hence, the q^2-cyclotomic coset C_i modulo $2n$ contains only one element i. Assume that $b = \frac{3\ell+3}{2}m + 2$. Let $\mathscr{C}$ be a negacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z} = \bigcup_{j=b}^{b+\delta} C_{1+2j}$, where $2 \leq \delta \leq \ell m - 1$. By Lemma 1, the parity-check matrix H of $\mathscr{C}$ can be denoted as

$$H = \begin{pmatrix} 1 & \alpha^{2b+1} & \alpha^{2(2b+1)} & \cdots & \alpha^{(n-1)(2b+1)} \\ 1 & \alpha^{2b+3} & \alpha^{2(2b+3)} & \cdots & \alpha^{(n-1)(2b+3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{2b+2\delta-1} & \alpha^{2(2b+2\delta-1)} & \cdots & \alpha^{(n-1)(2b+2\delta-1)} \\ 1 & \alpha^{2b+2\delta+1} & \alpha^{2(2b+2\delta+1)} & \cdots & \alpha^{(n-1)(2b+2\delta+1)} \end{pmatrix},$$

where α is a primitive $2n$-th root of unity. Hence, H has rank $\delta + 1$, and $\mathscr{C}$ is an MDS code with parameters $[n, n-\delta-1, \delta+2]$. Moreover, $\mathscr{C}^{\perp_H}$ is also an MDS code with parameters $[n, \delta+1, n-\delta]$.

Suppose that $\mathscr{C}_0$ is a negacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z} = \bigcup_{j=b}^{b+\delta-1} C_{1+2j}$, where $2 \leq \delta \leq \ell m - 1$. Then $\mathscr{C}_0$ is an MDS code with parameters $[n, n-\delta, \delta+1]$, and its Hermitian dual code $\mathscr{C}_0^{\perp_H}$ is also an MDS code with parameters $[n, \delta, n-\delta+1]$. Furthermore, the parity-check matrix H_0 of $\mathscr{C}_0$ is the following matrix

$$H_0 = \begin{pmatrix} 1 & \alpha^{2b+1} & \alpha^{2(2b+1)} & \cdots & \alpha^{(n-1)(2b+1)} \\ 1 & \alpha^{2b+3} & \alpha^{2(2b+3)} & \cdots & \alpha^{(n-1)(2b+3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{2b+2\delta-1} & \alpha^{2(2b+2\delta-1)} & \cdots & \alpha^{(n-1)(2b+2\delta-1)} \end{pmatrix}.$$

Particularly, the rank of H_0 is δ.

Suppose that $\mathscr{C}_1$ is a negacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z}_1 = C_{2b+2\delta+1}$, where $2 \leq \delta \leq \ell m - 1$. Then $\mathscr{C}_1$ is an MDS code with parameters $[n, n-1, 2]$. The parity-check matrix H_1 of $\mathscr{C}_1$ is the following matrix

$$H_1 = \left(1,\ \alpha^{2b+2\delta+1},\ \alpha^{2(2b+2\delta+1)},\ \cdots,\ \alpha^{(n-1)(2b+2\delta+1)}\right).$$

According to the above construction, one can see that H has been partitioned into two submatrices H_0 and H_1 such that

$$H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}.$$

Suppose that $G(D) = \widetilde{H}_0 + \widetilde{H}_1 D$, where $\widetilde{H}_0 = H_0$ and $\widetilde{H}_1$ is obtained from H_1 by adding zero-rows at its bottom such that $\widetilde{H}_1$ has the same number of rows as H_0. Due to Theorem 2(i), $G(D)$ is a reduced basic generator matrix, and the convolutional code $\mathscr{V}$ generated by such matrix has dimension δ and degree 1. It follows from Lemma 2 that the Hermitian dual code $\mathscr{V}^{\perp_H}$ has dimension $n-\delta$ and degree 1. According to Theorem 2(iii), the free distance $d_f^{\perp_H}$ of $\mathscr{V}^{\perp_H}$ satisfies that $\min\{d_0+d_1, d\} \leq d_f^{\perp_H} \leq d$, where d_0, d_1 and d are the minimum Hamming distances of the negacyclic codes $\mathscr{C}_0, \mathscr{C}_1$ and $\mathscr{C}$, respectively. Then we can get $d_f^{\perp_H} = \delta+2$. Hence, the convolutional code $\mathscr{V}^{\perp_H}$ has parameters $(n, n-\delta, 1; 1, \delta+2)_{q^2}$. Finally, it follows from Theorem 2(ii) that $\mathscr{V}$ is Hermitian self-orthogonal due to the fact that $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$. □

The construction of quantum MDS convolutional codes can be given by using the above lemma.

Theorem 7 *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m + \ell$, $\mu = \ell^2 - 1$, m is a positive integer, and ℓ is a positive odd integer. Then there exists a quantum MDS convolutional code with parameters $[(n, n-2\delta, 1; 1, \delta+2)]_q$, where $2 \leq \delta \leq \ell m - 1$.*

Proof Due to Lemma 9, a convolutional code $\mathscr{V}$ with parameters $(n, \delta, 1; 1, d_f)_{q^2}$ can be obtained, which also satisfies $\mathscr{V} \subseteq \mathscr{V}^{\perp_H}$ for any $2 \leq \delta \leq \ell m - 1$. According to Theorem 2(iii), $d_f \geq n - \delta$. It is easy to see that $d_f^{\perp_H} = \delta + 2 < n - \delta$. Therefore, a quantum convolutional code with parameters $[(n, n-2\delta, 1; 1, \delta+2)]_q$ can be derived from $\mathscr{V}$ due to Theorem 4. Replacing the parameters of the quantum convolutional code in the quantum generalized Singleton bound (Theorem 3), one has the equality

$$\frac{n-k}{2}\left(\left\lfloor \frac{2\gamma}{n+k} \right\rfloor + 1\right) + \gamma + 1 = \delta(0+1) + 1 + 1 = \delta + 2 = d_f,$$

which implies that such quantum convolutional code is a quantum MDS convolutional code. □

Now we consider the case $q = \mu m - \ell$ and we first recall a useful lemma in the following.

Lemma 10 [23] *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m - \ell$, $\mu = \ell^2 - 1$, m is a positive integer, and ℓ is a positive odd integer. If $\mathscr{C}$ is a q^2-ary negacyclic code of length n with defining set*

$$\mathscr{Z} = \bigcup_{j=\frac{\ell-1}{2}m}^{s_2} C_{1+2j},$$

where $\frac{\ell-1}{2}m \leq s_2 \leq \frac{3\ell-3}{2}m - 3$. Then $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$.

Similar to the discussion of Lemma 9 and Theorem 7, we have the following results.

Lemma 11 *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m - \ell$, $\mu = \ell^2 - 1$, m is a positive integer, and ℓ is a positive odd integer. Then there exists an $(n, n-\delta, 1; 1, \delta+2)_{q^2}$ convolutional code which contains its Hermitian dual, where $2 \leq \delta \leq (\ell-1)m - 3$.*

Theorem 8 *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m - \ell$, $\mu = \ell^2 - 1$, m is a positive integer, and ℓ is a positive odd integer. Then there exists a quantum MDS convolutional code with parameters $[(n, n-2\delta, 1; 1, \delta+2)]_q$, where $2 \leq \delta \leq (\ell-1)m - 3$.*

Table 4 New quantum MDS convolutional codes of length $n = \frac{q^2-1}{\mu}$ with $\mu = \ell^2 - 1$

ℓ	m	q	Parameters $[(n, k, \upsilon; \gamma, d_f)]_q$	δ
3	2	19	$[(45, 45-\delta, 1; 1, \delta+2)]_{19}$	$2 \le \delta \le 5$
	3	27	$[(91, 91-\delta, 1; 1, \delta+2)]_{27}$	$2 \le \delta \le 8$
	4	29	$[(105, 105-\delta, 1; 1, \delta+2)]_{29}$	$2 \le \delta \le 5$
	5	37	$[(171, 171-\delta, 1; 1, \delta+2)]_{37}$	$2 \le \delta \le 7$
	7	53	$[(351, 351-\delta, 1; 1, \delta+2)]_{53}$	$2 \le \delta \le 11$
		59	$[(435, 435-\delta, 1; 1, \delta+2)]_{59}$	$2 \le \delta \le 20$
5	1	29	$[(35, 35-\delta, 1; 1, \delta+2)]_{29}$	$2 \le \delta \le 4$
	2	43	$[(77, 77-\delta, 1; 1, \delta+2)]_{43}$	$2 \le \delta \le 5$
		53	$[(117, 117-\delta, 1; 1, \delta+2)]_{53}$	$2 \le \delta \le 9$
	3	67	$[(187, 187-\delta, 1; 1, \delta+2)]_{67}$	$2 \le \delta \le 9$
	4	101	$[(425, 425-\delta, 1; 1, \delta+2)]_{101}$	$2 \le \delta \le 19$
	5	125	$[(651, 651-\delta, 1; 1, \delta+2)]_{125}$	$2 \le \delta \le 24$
	6	139	$[(805, 805-\delta, 1; 1, \delta+2)]_{139}$	$2 \le \delta \le 21$
		149	$[(925, 925-\delta, 1; 1, \delta+2)]_{149}$	$2 \le \delta \le 29$

Remark 3 Quantum MDS convolutional codes of length n being an odd divisor of $q^2 - 1$ had been constructed by Aly et al. in [2], while quantum MDS convolutional codes of length n being an even divisor of $q^2 - 1$ had been constructed by Zhu et al. in [28]. Both of them use the Piret's construction, which is different from the method here. It is easy to see that our results are not contained in theirs either.

Example 2 Some new quantum MDS convolutional codes obtained from Theorems 7 and 8 are listed in Table 4.

5.2 Quantum MDS Convolutional Codes of Length $\frac{q^2-1}{\mu}$ with $\mu = \frac{\ell^2-1}{2}$

In this subsection, we will construct some new classes of quantum MDS convolutional codes of length $n = \frac{q^2-1}{\mu}$, where $q = \mu m \pm \ell$, $\mu = \frac{\ell^2-1}{2}$, m is a positive integer, and ℓ is an odd positive integer. As $\mu = \frac{\ell^2-1}{2}$ should be an integer, one can easily get $\ell \equiv 1(\mathrm{mod}4)$ or $l \equiv 3(\mathrm{mod}4)$.

In order to proceed further, we first recall some useful results shown in the literature.

Lemma 12 [23] *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m + \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 1(\mathrm{mod}4)$, m is a positive integer, and ℓ is a positive odd integer. If $\mathscr{C}$ is a q^2-ary negacyclic code of length n with defining set*

$$\mathscr{Z} = \bigcup_{j=\frac{\ell+3}{4}m+1}^{s_3} C_{1+2j},$$

where $\frac{\ell+3}{4}m + 1 \le s_3 \le \frac{5\ell+3}{4}m + 1$. Then $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$.

Lemma 13 [23] *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m + \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 3(\mathrm{mod} 4)$, m is a positive integer, and ℓ is a positive odd integer. If $\mathscr{C}$ is a q^2-ary negacyclic code of length n with defining set*

$$\mathscr{Z} = \bigcup_{j=\frac{3\ell+3}{4}m+2}^{s_4} C_{1+2j},$$

where $\frac{3\ell+3}{4}m + 2 \le s_4 \le \frac{5\ell+5}{4}m + 1$. Then $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$.

Lemma 14 [23] *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m - \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 1(\mathrm{mod} 4)$, m is a positive integer, and ℓ is a positive odd integer. If $\mathscr{C}$ is a q^2-ary negacyclic code of length n with defining set*

$$\mathscr{Z} = \bigcup_{j=\frac{\ell-1}{4}m}^{s_5} C_{1+2j},$$

where $\frac{\ell-1}{4}m \le s_5 \le \frac{3\ell-3}{4}m - 3$. Then $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$.

Lemma 15 [23] *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m - \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 3(\mathrm{mod} 4)$, m is a positive integer, and ℓ is a positive odd integer. If $\mathscr{C}$ is a q^2-ary negacyclic code of length n with defining set*

$$\mathscr{Z} = \bigcup_{j=\frac{\ell-3}{4}m}^{s_6} C_{1+2j},$$

where $\frac{\ell-3}{4}m \le s_6 \le \frac{5\ell-3}{4}m - 4$. Then $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$.

Lemma 16 *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m + \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 1(\mathrm{mod} 4)$, m is a positive integer, and ℓ is a positive odd integer. Then there exists an $(n, n-\delta, 1; 1, \delta+2)_{q^2}$ convolutional code which contains its Hermitian dual, where $2 \le \delta \le \ell m$.*

Proof Since the order of q^2 modulo $2n$ is equal to 1. Hence, the q^2-cyclotomic coset C_i modulo $2n$ contains only one element i. Assume that $t = \frac{\ell+3}{4}m + 1$. Let $\mathscr{C}$ be a negacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z} = \bigcup_{j=t}^{t+\delta} C_{1+2j}$, where $2 \le \delta \le \ell m$. By Lemma 1, the parity-check matrix H of $\mathscr{C}$ can be denoted as

$$H = \begin{pmatrix} 1 & \alpha^{2t+1} & \alpha^{2(2t+1)} & \cdots & \alpha^{(n-1)(2t+1)} \\ 1 & \alpha^{2t+3} & \alpha^{2(2t+3)} & \cdots & \alpha^{(n-1)(2t+3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{2t+2\delta-1} & \alpha^{2(2t+2\delta-1)} & \cdots & \alpha^{(n-1)(2t+2\delta-1)} \\ 1 & \alpha^{2t+2\delta+1} & \alpha^{2(2t+2\delta+1)} & \cdots & \alpha^{(n-1)(2t+2\delta+1)} \end{pmatrix},$$

where α is a primitive $2n$-th root of unity. Hence, H has rank $\delta + 1$, and $\mathscr{C}$ is an MDS code with parameters $[n, n - \delta - 1, \delta + 2]$. Moreover, $\mathscr{C}^{\perp_H}$ is also an MDS code with parameters $[n, \delta + 1, n - \delta]$.

Let $\mathscr{C}_0$ be a negacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z} = \bigcup_{j=t}^{t+\delta-1} C_{1+2j}$, where $2 \leq \delta \leq \ell m$. Then $\mathscr{C}_0$ is an MDS code with parameters $[n, n-\delta, \delta+1]$, and its Hermitian dual code $\mathscr{C}_0^{\perp_H}$ is also an MDS code with parameters $[n, \delta, n-\delta+1]$. Furthermore, the parity-check matrix H_0 of $\mathscr{C}_0$ is the following matrix

$$H_0 = \begin{pmatrix} 1 & \alpha^{2t+1} & \alpha^{2(2t+1)} & \cdots & \alpha^{(n-1)(2t+1)} \\ 1 & \alpha^{2t+3} & \alpha^{2(2t+3)} & \cdots & \alpha^{(n-1)(2t+3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{2t+2\delta-1} & \alpha^{2(2t+2\delta-1)} & \cdots & \alpha^{(n-1)(2t+2\delta-1)} \end{pmatrix}.$$

Let $\mathscr{C}_1$ be a negacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $\mathscr{Z}_1 = C_{2t+2\delta+1}$, where $2 \leq \delta \leq \ell m$. Then $\mathscr{C}_1$ is an MDS code with parameters $[n, n-1, 2]$. The parity-check matrix H_1 of $\mathscr{C}_1$ is the following matrix

$$H_1 = \left(1,\ \alpha^{2t+2\delta+1},\ \alpha^{2(2t+2\delta+1)},\ \cdots,\ \alpha^{(n-1)(2t+2\delta+1)}\right).$$

According to the above discussion, one can see that H has been partitioned into two submatrices H_0 and H_1 such that

$$H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}.$$

Let $G(D) = \widetilde{H}_0 + \widetilde{H}_1 D$, where $\widetilde{H}_0 = H_0$ and $\widetilde{H}_1$ is derived from H_1 by adding zero-rows at its bottom such that $\widetilde{H}_1$ has the same number of rows as H_0. Due to Theorem 2(i), $G(D)$ is a reduced basic generator matrix, and the convolutional code $\mathscr{V}$ generated by such matrix has dimension δ and degree 1. It follows from Lemma 2 that the Hermitian dual code $\mathscr{V}^{\perp_H}$ has dimension $n-\delta$ and degree 1. According to Theorem 2(iii), the free distance $d_f^{\perp_H}$ of $\mathscr{V}^{\perp_H}$ satisfies that $\min\{d_0 + d_1, d\} \leq d_f^{\perp_H} \leq d$, where d_0, d_1 and d are the minimum Hamming distances of the negacyclic codes $\mathscr{C}_0, \mathscr{C}_1$ and $\mathscr{C}$, respectively. Then we can get $d_f^{\perp_H} = \delta + 2$. Hence, the convolutional code $\mathscr{V}^{\perp_H}$ has parameters $(n, n-\delta, 1; 1, \delta+2)_{q^2}$. It follows from Theorem 2(ii) that $\mathscr{V}$ is also an Hermitian self-orthogonal code due to the fact that $\mathscr{C}^{\perp_H} \subseteq \mathscr{C}$. □

The construction of quantum MDS convolutional codes can be given by using the above lemma.

Theorem 9 *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m + \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 1(\mathrm{mod}4)$, m is a positive integer, and ℓ is a positive odd integer. Then there exists a quantum MDS convolutional code with parameters $[(n, n-2\delta, 1; 1, \delta+2)]_q$, where $2 \leq \delta \leq \ell m$.*

Proof Due to Lemma 16, a convolutional code $\mathscr{V}$ with parameters $(n, \delta, 1; 1, d_f)_{q^2}$ can be obtained, which also satisfies $\mathscr{V} \subseteq \mathscr{V}^{\perp_H}$ for any $2 \leq \delta \leq \ell m$. According to Theorem 2(iii), $d_f \geq n - \delta$. In addition, $d_f^{\perp_H} = \delta + 2 < n - \delta$. Hence, a quantum convolutional code with parameters $[(n, n-2\delta, 1; 1, \delta+2)]_q$ can be obtained from $\mathscr{V}$ according to Theorem 4. It follows from Theorem 3 that

$$\frac{n-k}{2}\left(\left\lfloor \frac{2\gamma}{n+k} \right\rfloor + 1\right) + \gamma + 1 = \delta(0+1) + 1 + 1 = \delta + 2 = d_f,$$

which means that it is a quantum MDS convolutional code. □

Table 5 New quantum MDS convolutional codes of length $n = \frac{q^2-1}{\mu}$ with $\mu = \frac{\ell^2-1}{2}$

ℓ	m	q	Parameters $[(n, k, \upsilon; \gamma, d_f)]_q$	δ
3	2	11	$[(30, 30-\delta, 1; 1, \delta+2)]_{11}$	$2 \le \delta \le 3$
	3	9	$[(20, 20-\delta, 1; 1, \delta+2)]_{9}$	$2 \le \delta \le 5$
	4	13	$[(42, 42-\delta, 1; 1, \delta+2)]_{13}$	$2 \le \delta \le 8$
		19	$[(90, 90-\delta, 1; 1, \delta+2)]_{19}$	$2 \le \delta \le 7$
	5	17	$[(72, 72-\delta, 1; 1, \delta+2)]_{17}$	$2 \le \delta \le 11$
		23	$[(132, 132-\delta, 1; 1, \delta+2)]_{23}$	$2 \le \delta \le 9$
	7	25	$[(156, 156-\delta, 1; 1, \delta+2)]_{25}$	$2 \le \delta \le 17$
5	1	17	$[(24, 24-\delta, 1; 1, \delta+2)]_{17}$	$2 \le \delta \le 5$
	2	29	$[(70, 70-\delta, 1; 1, \delta+2)]_{29}$	$2 \le \delta \le 10$
	3	31	$[(80, 80-\delta, 1; 1, \delta+2)]_{31}$	$2 \le \delta \le 3$
		41	$[(140, 140-\delta, 1; 1, \delta+2)]_{41}$	$2 \le \delta \le 15$
	4	43	$[(154, 154-\delta, 1; 1, \delta+2)]_{43}$	$2 \le \delta \le 5$
		53	$[(234, 234-\delta, 1; 1, \delta+2)]_{53}$	$2 \le \delta \le 20$
7	1	31	$[(40, 40-\delta, 1; 1, \delta+2)]_{31}$	$2 \le \delta \le 2$
	2	41	$[(70, 70-\delta, 1; 1, \delta+2)]_{41}$	$2 \le \delta \le 10$
	3	79	$[(260, 260-\delta, 1; 1, \delta+2)]_{79}$	$2 \le \delta \le 8$
	4	89	$[(330, 330-\delta, 1; 1, \delta+2)]_{89}$	$2 \le \delta \le 24$
		103	$[(442, 442-\delta, 1; 1, \delta+2)]_{103}$	$2 \le \delta \le 15$

Similar to the discussion of the case $q = \mu m + \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 1(\mathrm{mod}4)$, and ℓ is a positive odd integer. We have the following results for the remaining cases.

Theorem 10 *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m + \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 3(\mathrm{mod}4)$, m is a positive integer, and ℓ is a positive odd integer. Then there exists a quantum MDS convolutional code with parameters $[(n, n-2\delta, 1; 1, \delta+2)]_q$, where $2 \le \delta \le \frac{\ell+1}{2}m - 1$.*

Theorem 11 *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m - \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 1(\mathrm{mod}4)$, m is a positive integer, and ℓ is a positive odd integer. Then there exists a quantum MDS convolutional code with parameters $[(n, n-2\delta, 1; 1, \delta+2)]_q$, where $2 \le \delta \le \frac{\ell-1}{2}m - 3$.*

Theorem 12 *Let $n = \frac{q^2-1}{\mu}$, where q is an odd prime power of the form $q = \mu m - \ell$, $\mu = \frac{\ell^2-1}{2}$, $\ell \equiv 3(\mathrm{mod}4)$, m is a positive integer, and ℓ is a positive odd integer. Then there exists a quantum MDS convolutional code with parameters $[(n, n-2\delta, 1; 1, \delta+2)]_q$, where $2 \le \delta \le \ell m - 4$.*

Example 3 Some new quantum MDS convolutional codes obtained from Theorems 9, 10, 11 and 12 are listed in Table 5.

6 Conclusion

In this paper, we have constructed eight classes of quantum MDS convolutional codes from classical constacyclic codes by using algebraic methods. Compared with the codes available in the literature, most of the obtained quantum MDS convolutional codes are new in the sense that their parameters are not covered by the codes available in the literature. It is interesting to construct more new quantum MDS convolutional codes via constacyclic codes.

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