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Tighter Monogamy Relations for the Tsallis-q and Rényi-α Entanglement in Multiqubit Systems

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Abstract

Monogamy relations characterize the distributions of quantum entanglement in multipartite systems. In this work, we present some tighter monogamy relations in terms of the power of the Tsallis-q and Rényi-α entanglement in multipartite systems. We show that these new monogamy relations of multipartite entanglement with tighter lower bounds than the existing ones. Furthermore, three examples are given to illustrate the tightness.

Keywords Monogamy relations · The Tsallis-q entanglement · The Rényi-α entanglement

1 Introduction

Quantum entanglement is an essential feature in terms of quantum mechanics, which distinguishes quantum mechanics from the classical world and plays a very important role in communication, cryptography, and computing. A key property of quantum entanglement is the monogamy relations [1, 2], which is a quantum systems entanglement with one of the other subsystems limits its entanglement with the remaining ones, known as the monogamy of entanglement (MoE) [2, 3]. For any tripartite quantum state $\rho_{A|BC}$, MoE can be expressed as the following inequality $\mathcal{E}(\rho_{A|BC}) \geq \mathcal{E}(\rho_{AB}) + \mathcal{E}(\rho_{AC})$, where $\rho_{AB} = \text{tr}_C(\rho_{A|BC})$, $\rho_{AC} = \text{tr}_B(\rho_{A|BC})$, and $\mathcal{E}$ is an quantum entanglement measure. Furthermore, Coffman, Kundu and Wootters expressed that the squared concurrence also satisfies the monogamy relations in multiqubit states [1]. Later the monogamy relations are widely promoted to other entanglement measures such as entanglement of formation [4], entanglement negativity [5], the Tsallis-q and Rényi-α entanglement [6, 7]. These monogamy relations will help us to have a further understanding of the quantum information theory [8], even black-hole physics [9] and condensed-matter physics [10]. In [11, 12], the authors prove that the η th power of Tsallis-q entanglement satisfies

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monogamy relations for $2 \leq q \leq 3$, the power $\eta \geq 1$, the Rényi-α entanglement also satisfies monogamy relations for $\alpha \geq 2$, the power $\eta \geq 1$, and $2 > \alpha \geq \frac{\sqrt{7}-1}{2}$, the power $\eta \geq 2$.

Our paper is organized as follows. In Section 2, we review some basic preliminaries of concurrence, Tsallis-q, and Rényi-α entanglement. In Section 3, we develop a class of monogamy relations in terms of the Tsallis-q entanglement, they are tighter than the results in [11]. In Section 4, we explore a class of monogamy relations based on the Rényi-α entanglement which are tighter than the results in [12]. In Section 5, we summarize our results.

2 Basic Preliminaries

We first recall the definition of concurrence. For a bipartite pure state $|\varphi\rangle_{AB}$, the concurrence can be defined as [13–15]

$$C(|\varphi\rangle_{AB}) = \sqrt{2(1 - tr\rho_A^2)}, \tag{1}$$

where $\rho_A = tr_B(|\varphi\rangle_{AB}\langle\varphi|)$.

For any mixed state ρ_{AB}, its concurrence is defined via the convex-roof extension in [16]

$$C(\rho_{AB}) = min \sum_j p_j C(|\varphi_j\rangle_{AB}), \tag{2}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum\limits_j p_j |\varphi_j\rangle_{AB}\langle\varphi_j|$, and $\sum\limits_j p_j = 1$.

It has been proved that the concurrence $C(\rho_{A|B_1\cdots B_{N-1}})$ of mixed state $\rho_{A|B_1\cdots B_{N-1}}$ has an important property such that [17]

$$C^2(\rho_{A|B_1\cdots B_{N-1}}) \geq C^2(\rho_{A|B_1}) + C^2(\rho_{A|B_2\cdots B_{N-1}}) \geq \cdots \geq \sum_{i=1}^{N-1} C^2(\rho_{A|B_i}), \tag{3}$$

where $\rho_{A|B_i} = \mathrm{tr}_{B_1\cdots B_{i-1}B_{i+1}\cdots B_{N-1}}(\rho_{A|B_1\cdots B_{N-1}})$.

Quantum entanglement plays an important role in quantum information. Another well-known quantum entanglements are Tsallis-q entanglement and Rényi-α entanglement. For any bipartite pure state $|\varphi\rangle_{AB}$, the Tsallis-q entanglement is defined as [18].

$$T_q(|\varphi\rangle_{AB}) = S_q(\rho_A) = \frac{1}{q-1}(1 - tr\rho_A^q), \tag{4}$$

where $q \geq 0$, $q \neq 1$, and $\rho_A = tr_B(|\varphi\rangle_{AB}\langle\varphi|)$. When q tends to 1, the Tsallis-q entropy converges to the von Neumann entropy.

For a mixed state ρ_{AB}, the Tsallis-q entanglement is defined by its convex-roof extension, which can be expressed as

$$T_q(\rho_{AB}) = min \sum_i p_i T_q(|\varphi_i\rangle_{AB}), \tag{5}$$

where the minimum is taken over all possible pure state decomposition of $\rho_{AB} = \sum\limits_i p_i |\varphi_i\rangle_{AB}\langle\varphi_i|$.

When $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$, for any bipartite pure state $|\varphi\rangle_{AB}$, it has been explored that the Tsallis-q entanglement $T_q(|\varphi\rangle_{AB})$ has an analytical formula [19],

$$T_q(|\varphi\rangle_{AB}) = g_q(C^2(|\varphi\rangle_{AB})), \tag{6}$$

where the function $g_q(x)$ is defined as

$$g_q(x) = \frac{1}{q-1}\left[1 - \left(\frac{1+\sqrt{1-x}}{2}\right)^q - \left(\frac{1-\sqrt{1-x}}{2}\right)^q\right], \tag{7}$$

for $0 \leq x \leq 1$, and $g_q(x)$ is an increasing monotonic and convex function in [20]. Specially, for $2 \leq q \leq 3$, the function $g_q(x)$ has an important property [18]

$$g_q(x^2 + y^2) \geq g_q(x^2) + g_q(y^2). \tag{8}$$

When $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$, for any two-qubit mixed state ρ, the Tsallis-q entanglement can be expressed as $T_q(\rho) = g_q(C^2(\rho))$ [20].

Now, we recall some preliminaries of the Rényi-α entanglement. For a bipartite pure state $|\varphi\rangle_{AB}$, the Rényi-α entanglement can be defined as [21]

$$E_\alpha(|\varphi\rangle_{AB}) = \frac{1}{1-\alpha} log_2(tr\rho_A^\alpha), \tag{9}$$

where $\alpha > 0$, and $\alpha \neq 1$, $\rho_A = tr_B(|\varphi\rangle_{AB}\langle\varphi|)$. When α tends to 1, the Rényi-α entropy converges to the von Neumann entropy.

For a bipartite mixed state ρ_{AB}, the Rényi-α entanglement can be defined as

$$E_\alpha(\rho_{AB}) = min \sum_i p_i E_\alpha(|\varphi_i\rangle_{AB}), \tag{10}$$

where the minimum is taken over all possible pure state decompositions $\{p_i, \varphi^i_{AB}\}$ of ρ_{AB}.

When $\alpha \geq \frac{\sqrt{7}-1}{2}$, for any two-qubit state ρ_{AB}, the Rényi-α entanglement has an analytical formula [21, 22]

$$E_\alpha(\rho_{AB}) = f_\alpha(C(\rho_{AB})), \tag{11}$$

where $f_\alpha(x)$ can be expressed as

$$f_\alpha(x) = \frac{1}{1-\alpha} log_2\left[\left(\frac{1-\sqrt{1-x^2}}{2}\right)^\alpha + \left(\frac{1+\sqrt{1-x^2}}{2}\right)^\alpha\right], \tag{12}$$

$0 \leq x \leq 1$, and $f_\alpha(x)$ is a monotonically increasing convexity function.

For $\alpha \geq 2$, the function $f_\alpha(x)$ satisfies the following inequality [22],

$$f_\alpha\left(\sqrt{x^2+y^2}\right) \geq f_\alpha(x) + f_\alpha(y). \tag{13}$$

For $\frac{\sqrt{7}-1}{2} \leq \alpha < 2$, the function $f_\alpha(x)$ has an important property such that [23]

$$f_\alpha^2\left(\sqrt{x^2+y^2}\right) \geq f_\alpha^2(x) + f_\alpha^2(y). \tag{14}$$

3 Tighter Monogamy Relations in Terms of the Tsallis-q Entanglement

To present the tighter monogamy relations of the Tsallis-q entanglement in multipartite systems, we introduce three lemmas as follows.

Lemma 1 *For* $0 \le x \le 1$ *and* $\mu \ge 1$, *we have*

$$\begin{aligned}(1+x)^\mu &\ge 1+\tfrac{\mu^2}{\mu+1}x+\left(2^\mu-\tfrac{\mu^2}{\mu+1}-1\right)x^\mu \\ &\ge 1+\tfrac{\mu}{2}x+\left(2^\mu-\tfrac{\mu}{2}-1\right)x^\mu \ge 1+(2^\mu-1)x^\mu.\end{aligned} \tag{15}$$

Proof If $x=0$, then the inequality is trivial. Otherwise, let $f(\mu,x)=\frac{(1+x)^\mu-\frac{\mu^2}{\mu+1}x-1}{x^\mu}$, then, $\frac{\partial f}{\partial x}=\frac{\mu x^{\mu-1}\left[1+\frac{\mu(\mu-1)}{\mu+1}x-(1+x)^{\mu-1}\right]}{x^{2\mu}}$. When $\mu \ge 1$ and $0 \le x \le 1$, it is obvious that $1+\frac{\mu(\mu-1)}{\mu+1}x \le (1+x)^{\mu-1}$. Thus, $\frac{\partial f}{\partial x} \le 0$, and $f(\mu,x)$ is a decreasing function of x, i.e. $f(\mu,x) \ge f(\mu,1)=2^\mu-\frac{\mu^2}{\mu+1}-1$. Consequently, we have $(1+x)^\mu \ge 1+\frac{\mu^2}{\mu+1}x+\left(2^\mu-\frac{\mu^2}{\mu+1}-1\right)x^\mu$. Since $\frac{\mu^2}{\mu+1} \ge \frac{\mu}{2}$, for $0 \le x \le 1$ and $\mu \ge 1$, one gets

$1+\frac{\mu^2}{\mu+1}x+\left(2^\mu-\frac{\mu^2}{\mu+1}-1\right)x^\mu = 1+\frac{\mu^2}{\mu+1}(x-x^\mu)+(2^\mu-1)x^\mu \ge 1+\frac{\mu}{2}x+\left(2^\mu-\frac{\mu}{2}-1\right)x^\mu = 1$
$+\frac{\mu}{2}(x-x^\mu)+(2^\mu-1)x^\mu \ge 1+(2^\mu-1)x^\mu$

Lemma 2 *For any* $2 \le q \le 3$, $\mu \ge 1$, $g_q(x)$ *defined on the domain* $D=\{(x,y)|0 \le x,y \le 1\}$, *if* $x \ge y$, *then we have*

$$g_q^\mu(x^2+y^2) \ge g_q^\mu(x^2)+\frac{\mu^2}{\mu+1}g_q^{\mu-1}(x^2)g_q(y^2)+\left(2^\mu-\frac{\mu^2}{\mu+1}-1\right)g_q^\mu(y^2). \tag{16}$$

Proof For $2 \le q \le 3$, $\mu \ge 1$, according to inequality (8), we have

$$\begin{aligned}g_q^\mu(x^2+y^2) &\ge (g_q(x^2)+g_q(y^2))^\mu \\ &= g_q^\mu(x^2)(1+\tfrac{g_q(y^2)}{g_q(x^2)})^\mu \\ &\ge g_q^\mu(x^2)+\tfrac{\mu^2}{\mu+1}g_q^{\mu-1}(x^2)g_q(y^2)+(2^\mu-\tfrac{\mu^2}{\mu+1}-1)g_q^\mu(y^2),\end{aligned} \tag{17}$$

where the first inequality is due to inequality (8) and the second inequality is due to Lemma 1.

Lemma 3 *For any N-qubit mixed state* $\rho_{A|B_1\cdots B_{N-1}}$, *we have*

$$T_q(\rho_{A|B_1\cdots B_{N-1}}) \ge g_q(C^2(\rho_{A|B_1\cdots B_{N-1}})). \tag{18}$$

Proof Suppose that $\rho_{A|B_1\cdots B_{N-1}}=\sum_i p_i|\varphi_i\rangle_{A|B_1\cdots B_{N-1}}$ is the optimal decomposition for $T_q(\rho_{A|B_1\cdots B_{N-1}})$, then we have

$$\begin{aligned}T_q(\rho_{A|B_1\cdots B_{N-1}}) &= \sum_i p_i T_q(|\varphi_i\rangle_{A|B_1\cdots B_{N-1}}) \\ &= \sum_i p_i g_q(C^2(|\varphi_i\rangle_{A|B_1\cdots B_{N-1}})) \\ &\ge g_q(\sum_i p_i C^2(|\varphi_i\rangle_{A|B_1\cdots B_{N-1}})) \\ &\ge g_q((\sum_i p_i C(|\varphi_i\rangle_{A|B_1\cdots B_{N-1}}))^2) \\ &\ge g_q(C^2(\rho_{A|B_1\cdots B_{N-1}})),\end{aligned} \tag{19}$$

where the first inequality is due to that $g_q(x)$ is a convex function, the second inequality is due to the Cauchy-Schwarz inequality: $(\sum_i a_i^2)(\sum_i b_i^2) \geq (\sum_i a_i b_i)^2$, with $a_i = \sqrt{p_i}$, and $b_i = \sqrt{p_i} C(|\varphi_i\rangle_{A|B_1 \cdots B_{N-1}})$, and the third inequality is due to the minimum property of $C(\rho_{A|B_1 \cdots B_{N-1}})$.

Now, we give the following theorems of the monogamy inequalities in terms of the Tsallis-q entanglement.

Theorem 1 *For any* $2 \leq q \leq 3$, *the power* $\eta \geq 1$, *and N-qubit mixed state* $\rho_{A|B_1 \cdots B_{N-1}}$, *if* $C(\rho_{A|B_i}) \geq C(\rho_{A|B_{i+1} \cdots B_{N-1}}), i = 1, 2, \cdots, N-2, N > 3$, *we have*

$$T_q^\eta(\rho_{A|B_1 \cdots B_{N-1}}) \geq \sum_{i=1}^{N-3} h^{i-1} T_q^\eta(\rho_{A|B_i}) + h^{N-3} Q_{AB_{N-2}}, \tag{20}$$

where $h = 2^\eta - 1$, $Q_{AB_{N-2}} = T_q^\eta(\rho_{A|B_{N-2}}) + \frac{\eta^2}{\eta+1} T_q^{\eta-1}(\rho_{A|B_{N-2}}) T_q(\rho_{A|B_{N-1}}) + \left(2^\eta - \frac{\eta^2}{\eta+1} - 1\right) T_q^\eta(\rho_{A|B_{N-1}})$.

Proof Let $\rho_{A|B_1 \cdots B_{N-1}}$ be an N-qubit mixed state, from Lemma 3 and inequality (3), we have

$$\begin{aligned} T_q^\eta(\rho_{A|B_1 \cdots B_{N-1}}) &\geq g_q^\eta(C^2(\rho_{A|B_1}) + C^2(\rho_{A|B_2 \cdots B_{N-1}})) \\ &\geq g_q^\eta(C^2(\rho_{A|B_1})) + \frac{\eta^2}{\eta+1} g_q^{\eta-1}(C^2(\rho_{A|B_1})) g_q(C^2(\rho_{A|B_2 \cdots B_{N-1}})) \\ &\quad + (2^\eta - \frac{\eta^2}{\eta+1} - 1) g_q^\eta(C^2(\rho_{A|B_2 \cdots B_{N-1}})) \\ &\geq g_q^\eta(C^2(\rho_{A|B_1})) + h g_q^\eta(C^2(\rho_{A|B_2 \cdots B_{N-1}})), \end{aligned} \tag{21}$$

where the first inequality is due to the monotonically increasing property of the function $g_q(x)$ and inequality (3), the second inequality is due to Lemma 2, and the third inequality is due to the fact that $C^2(\rho_{A|B_1}) \geq C^2(\rho_{A|B_2 \cdots B_{N-1}})$.

Similar calculation procedure can be used to the term $g_q^\eta(C^2(\rho_{A|B_2 \cdots B_{N-1}}))$, by iterative method we can get

$$\begin{aligned} & g_q^\eta(C^2(\rho_{A|B_2 \cdots B_{N-1}})) \\ \geq\ & g_q^\eta(C^2(\rho_{A|B_2})) + h g_q^\eta(C^2(\rho_{A|B_3 \cdots B_{N-1}})) \geq \cdots \\ \geq\ & g_q^\eta(C^2(\rho_{A|B_2})) + h g_q^\eta(C^2(\rho_{A|B_3})) + \cdots + h^{N-5} g_q^\eta(C^2(\rho_{A|B_{N-3}})) \\ & + h^{N-4} \left\{ g_q^\eta(C^2(\rho_{A|B_{N-2}})) + \frac{\eta^2}{\eta+1} g_q^{\eta-1}(C^2(\rho_{A|B_{N-2}})) g_q(C^2(\rho_{A|B_{N-1}})) \right. \\ & \left. + (2^\eta - \frac{\eta^2}{\eta+1} - 1) g_q^\eta(C^2(\rho_{A|B_{N-1}})) \right\}. \end{aligned} \tag{22}$$

According to the fact $T_q(\rho) = g_q(C^2(\rho))$ for any two qubit mixed state ρ, and combining inequality (21) and (22), we complete the proof.

Theorem 2 *For any* $2 \leq q \leq 3$, *the power* $\eta \geq 1$, *and N-qubit mixed state* $\rho_{A|B_1 \cdots B_{N-1}}$, *if* $C(\rho_{A|B_i}) \geq C(\rho_{A|B_{i+1} \cdots B_{N-1}}), i = 1, 2, \cdots, m$, $C(\rho_{A|B_j}) \leq C(\rho_{A|B_{j+1} \cdots B_{N-1}}), j = m+1, m+2, \cdots N-2, N > 3$, *we have*

$$T_q^\eta(\rho_{A|B_1 \cdots B_{N-1}}) \geq \sum_{i=1}^{m} h^{i-1} T_q^\eta(\rho_{AB_i}) + h^{m+1} \sum_{j=m+1}^{N-3} T_q^\eta(\rho_{AB_j}) + h^m Q_{AB_{N-1}}, \tag{23}$$

where $h=2^{\eta}-1$, $Q_{AB_{N-1}}=T_q^{\eta}(\rho_{A|B_{N-1}})+\frac{\eta^2}{\eta+1}T_q^{\eta-1}(\rho_{A|B_{N-1}})T_q(\rho_{A|B_{N-2}})+(2^{\eta}-\frac{\eta^2}{\eta+1}-1)T_q^{\eta}(\rho_{A|B_{N-2}})$.

Proof For $2\le q\le 3$, $\eta\ge 1$, we obtain

$$\begin{aligned}
&T_q^{\eta}(\rho_{A|B_1\cdots B_{N-1}})\\
\ge&\sum_{i=1}^{m}h^{i-1}T_q^{\eta}(\rho_{AB_i})+h^m g_q^{\eta}(C^2(\rho_{A|B_{m+1}\cdots B_{N-1}}))\\
\ge&\sum_{i=1}^{m}h^{i-1}T_q^{\eta}(\rho_{AB_i})+h^m\Big\{g_q^{\eta}(C^2(\rho_{A|B_{m+2}\cdots B_{N-1}}))+(2^{\eta}-\tfrac{\eta^2}{\eta+1}-1)g_q^{\eta}(C^2(\rho_{A|B_{m+1}}))\\
&+\tfrac{\eta^2}{\eta+1}g_q^{\eta-1}(C^2(\rho_{A|B_{m+2}\cdots B_{N-1}}))g_q(C^2(\rho_{A|B_{m+1}}))\Big\}\\
\ge&\sum_{i=1}^{m}h^{i-1}T_q^{\eta}(\rho_{AB_i})+h^{m+1}g_q^{\eta}(C^2(\rho_{A|B_{m+1}}))+h^m g_q^{\eta}(C^2(\rho_{A|B_{m+2}\cdots B_{N-1}}))\ge\cdots\\
\ge&\sum_{i=1}^{m}h^{i-1}T_q^{\eta}(\rho_{AB_i})+h^{m+1}\sum_{j=m+1}^{N-3}g_q^{\eta}(C^2(\rho_{AB_j}))+h^m\Big\{(2^{\eta}-\tfrac{\eta^2}{\eta+1}-1)g_q^{\eta}(C^2(\rho_{A|B_{N-2}}))\\
&+g_q^{\eta}(C^2(\rho_{A|B_{N-1}}))+\tfrac{\eta^2}{\eta+1}g_q^{\eta-1}(C^2(\rho_{A|B_{N-1}}))g_q(C^2(\rho_{A|B_{N-2}}))\Big\},
\end{aligned}\tag{24}$$

where the first inequality is due to Theorem 1, and the second inequality is due to Lemma 2 and the fact that $C(\rho_{A|B_j})\le C(\rho_{A|B_{j+1}\cdots B_{N-1}})$ for $j=m+1,m+2,\cdots N-2$, $N>3$. According to the denotation of $Q_{AB_{N-1}}$ and combining inequality (24), we obtain inequality (23).

Remark 1 We consider a particular case of $N=3$. Note that when $2\le q\le 3$, the power $\eta\ge 1$, if $T_q(\rho_{AB_1})\ge T_q(\rho_{AB_2})$, then we get the following result,

$$\begin{aligned}
T_q^{\eta}(\rho_{A|B_1B_2})\ \ge\ &T_q^{\eta}(\rho_{AB_1})+\tfrac{\eta^2}{\eta+1}T_q^{\eta-1}(\rho_{AB_1})T_q(\rho_{AB_2})\\
&+\Big(2^{\eta}-\tfrac{\eta^2}{\eta+1}-1\Big)T_q^{\eta}(\rho_{AB_2}).
\end{aligned}\tag{25}$$

If $T_q(\rho_{AB_1})\le T_q(\rho_{AB_2})$, then

$$\begin{aligned}
T_q^{\eta}(\rho_{A|B_1B_2})\ \ge\ &T_q^{\eta}(\rho_{AB_2})+\tfrac{\eta^2}{\eta+1}T_q^{\eta-1}(\rho_{AB_2})T_q(\rho_{AB_1})\\
&+\Big(2^{\eta}-\tfrac{\eta^2}{\eta+1}-1\Big)T_q^{\eta}(\rho_{AB_1}).
\end{aligned}\tag{26}$$

To see the tightness of the Tsallis-q entanglement directly, we give the following example.

Example 1 Under local unitary operations, the three-qubit pure state can be written as [24, 25]

$$|\psi\rangle_{A|BC}=\lambda_0|000\rangle+\lambda_1e^{i\varphi}|100\rangle+\lambda_2|101\rangle+\lambda_3|110\rangle+\lambda_4|111\rangle,\tag{27}$$

where $0\le\varphi\le\pi$, $\lambda_i\ge 0, i=0,1,2,3,4$, and $\sum_{i=0}^{4}\lambda_i^2=1$, set $\lambda_0=\frac{\sqrt{5}}{3}$, $\lambda_1=0$, $\lambda_4=0$, $\lambda_2=\frac{\sqrt{3}}{3}$, $\lambda_3=\frac{1}{3}$, $q=2$. From the definition of the Tsallis-q entanglement, after simple computation, we can get $T_q(\rho_{A|BC})=g_q[(2\lambda_0\sqrt{(\lambda_2)^2+(\lambda_3)^2+(\lambda_4)^2})^2]$, $T_q(\rho_{AB})=g_q[(2\lambda_0\lambda_2)^2]$, and $T_q(\rho_{AC})=g_q[(2\lambda_0\lambda_3)^2]$, then we have $T_q(\rho_{A|BC})=0.49383$, $T_q(\rho_{AB})=0.37037$, and $T_q(\rho_{AC})=0.12346$. Consequently,

$T_2^{\eta}(\rho_{A|BC}) = (0.49383)^{\eta} \geq T_2^{\eta}(\rho_{AB}) + (2^{\eta} - \frac{\eta^2}{\eta+1} - 1)T_2^{\eta}(\rho_{AC}) + \frac{\eta^2}{\eta+1}T_2^{\eta-1}(\rho_{AB})T_2(\rho_{AC}) = (0.37037)^{\eta} + \frac{\eta^2}{\eta+1}(37037)^{\eta-1}(0.12346) + (2^{\eta} - \frac{\eta^2}{\eta+1} - 1)(0.12346)^{\eta}$

While the result in [11] is

$T_2^{\eta}(\rho_{AB}) + (2^{\eta} - 1)T_2^{\eta}(\rho_{AC}) + \frac{\eta}{2}T_2(\rho_{AC})((T_2^{\eta-1}(\rho_{AB})) - (T_2^{\eta-1}(\rho_{AC}))) = (0.37037)^{\eta} + (2^{\eta} - 1)(0.12346)^{\eta} + \frac{0.12346\eta}{2}((0.37037)^{\eta-1} - (0.12346)^{\eta-1})$

One can see that our result is tighter than the ones [11] for $\eta \geq 1$. See Fig. 1.

4 Tighter Monogamy Relations in Terms of the Rényi-α Entanglement

In order to present the tighter monogamy relations of the Rényi-α entanglement in multiqubit systems, we introduce three lemmas as follows.

Lemma 4 *For any N-qubit mixed state $\rho_{A|B_1\cdots B_{N-1}}$, we have*

$$E_{\alpha}(\rho_{A|B_1\cdots B_{N-1}}) \geq f_{\alpha}(C(\rho_{A|B_1\cdots B_{N-1}})). \tag{28}$$

Proof Suppose that $\rho_{A|B_1\cdots B_{N-1}} = \sum_i p_i|\varphi_i\rangle_{A|B_1\cdots B_{N-1}}$ is the optimal decomposition for $E_{\alpha}(\rho_{A|B_1\cdots B_{N-1}})$, then we have

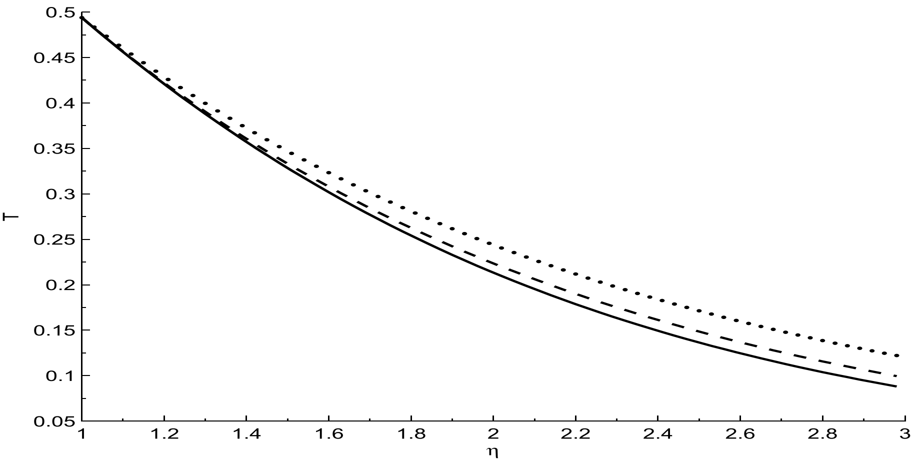

Fig. 1 The axis T stands the Tsallis-q entanglement of $|\psi\rangle_{A|BC}$, which is a function of η $(1 \leq \eta \leq 3)$. The dotted line stands the value of $T_2^{\eta}(\rho_{A|BC})$. The dashed line stands the lower bound given by our improved monogamy relations. The solid black line represents the lower bound given by [11]

(29)

$$\begin{aligned} E_\alpha(\rho_{A|B_1\cdots B_{N-1}}) &= \sum_i p_i E_\alpha(|\varphi_i\rangle_{A|B_1\cdots B_{N-1}}) \\ &= \sum_i p_i f_\alpha(C(|\varphi_i\rangle_{A|B_1\cdots B_{N-1}})) \\ &\geq f_\alpha(\sum_i p_i C(|\varphi_i\rangle_{A|B_1\cdots B_{N-1}})) \\ &\geq f_\alpha(C(\rho_{A|B_1\cdots B_{N-1}})), \end{aligned} \tag{29}$$

where the first inequality is due to the convexity of $f_\alpha(x)$ and the last inequality follows from the definition of concurrence for mixed state.

Lemma 5 *For any $\alpha \geq 2$, $\mu \geq 1$, suppose that the function $f_\alpha(x)$ defined on the domain $D = \{(x,y)|0 \leq x,y \leq 1, 0 \leq x^2 + y^2 \leq 1\}$, if $x \geq y$, then we have*

$$f_\alpha^\mu(\sqrt{x^2+y^2}) \geq f_\alpha^\mu(x) + \frac{\mu^2}{\mu+1} f_\alpha^{\mu-1}(x) f_\alpha(y) + \left(2^\mu - \frac{\mu^2}{\mu+1} - 1\right) f_\alpha^\mu(y). \tag{30}$$

Proof For $\mu \geq 1$, and $\alpha \geq 2$, we have

$$\begin{aligned} f_\alpha^\mu(\sqrt{x^2+y^2}) &\geq (f_\alpha(x) + f_\alpha(y))^\mu \\ &\geq f_\alpha^\mu(x) + \tfrac{\mu^2}{\mu+1} f_\alpha^{\mu-1}(x) f_\alpha(y) + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right) f_\alpha^\mu(y), \end{aligned} \tag{31}$$

where the first inequality is due to inequality (13), and the second inequality is due to Lemma 1.

Lemma 6 *For any $\frac{\sqrt{7}-1}{2} \leq \alpha < 2$, $\mu \geq 1$, $\mu = \frac{\gamma}{2}$, the function $f_\alpha(x)$ defined on the domain $D = \{(x,y)|0 \leq x,y \leq 1, 0 \leq x^2 + y^2 \leq 1\}$, if $x \geq y$, then we have*

$$f_\alpha^\gamma(\sqrt{x^2+y^2}) \geq f_\alpha^\gamma(x) + \frac{\mu^2}{\mu+1} f_\alpha^{\gamma-2}(x) f_\alpha^2(y) + \left(2^\mu - \frac{\mu^2}{\mu+1} - 1\right) f_\alpha^\gamma(y). \tag{32}$$

Proof For $\mu \geq 1$, and $\mu = \frac{\gamma}{2}$, we have

$$\begin{aligned} f_\alpha^\gamma(\sqrt{x^2+y^2}) &\geq (f_\alpha^2(x) + f_\alpha^2(y))^\mu \\ &\geq f_\alpha^\gamma(x) + \tfrac{\mu^2}{\mu+1} f_\alpha^{\gamma-2}(x) f_\alpha^2(y) + (2^\mu - \tfrac{\mu^2}{\mu+1} - 1) f_\alpha^\gamma(y), \end{aligned} \tag{33}$$

where the first inequality can be assured by inequality (14), and the second inequality is due to Lemma 1.

Now, we give the following theorems of the tighter monogamy inequality in terms of the Rényi-α entanglement.

Theorem 3 *For any $\alpha \geq 2$, the power $\mu \geq 1$, and N-qubit mixed state $\rho_{A|B_1\cdots B_{N-1}}$, if $C(\rho_{A|B_i}) \geq C(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $i = 1, 2, \cdots, N-2$, $N > 3$, then we have*

$$E_\alpha^\mu(\rho_{A|B_1\cdots B_{N-1}}) \geq \sum_{i=1}^{N-3} h^{i-1} E_\alpha^\mu(\rho_{A|B_i}) + h^{N-3} Q_{AB_{N-2}}, \tag{34}$$

where $h = 2^\mu - 1$, $Q_{AB_{N-2}} = E_\alpha^\mu(\rho_{A|B_{N-2}}) + \frac{\mu^2}{\mu+1} E_\alpha^{\mu-1}(\rho_{A|B_{N-2}}) E_\alpha(\rho_{A|B_{N-1}}) + (2^\mu - \frac{\mu^2}{\mu+1} - 1) E_\alpha^\mu(\rho_{A|B_{N-1}})$.

Proof We consider an N-qubit mixed state $\rho_{A|B_1\cdots B_{N-1}}$, from Lemma 4, we have

$$
\begin{aligned}
E_\alpha^\mu(\rho_{A|B_1\cdots B_{N-1}}) &\geq f_\alpha^\mu\left(\sqrt{C^2(\rho_{A|B_1}) + C^2(\rho_{A|B_2\cdots B_{N-1}})}\right) \\
&\geq f_\alpha^\mu(C(\rho_{A|B_1})) + \tfrac{\mu^2}{\mu+1} f_\alpha^{\mu-1}(C(\rho_{A|B_1})) f_\alpha(C(\rho_{A|B_2\cdots B_{N-1}})) \\
&\quad + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right) f_\alpha^\mu(C(\rho_{A|B_2\cdots B_{N-1}})) \\
&\geq f_\alpha^\mu(C(\rho_{A|B_1})) + h f_\alpha^\mu(C(\rho_{A|B_2\cdots B_{N-1}})) \\
&\geq \cdots \\
&\geq f_\alpha^\mu(C(\rho_{A|B_1})) + h f_\alpha^\mu(C(\rho_{A|B_2})) + \cdots + h^{N-4} f_\alpha^\mu(C(\rho_{A|B_{N-3}})) \\
&\quad + h^{N-3}\left\{ f_\alpha^\mu(C(\rho_{A|B_{N-2}})) + \tfrac{\mu^2}{\mu+1} f_\alpha^{\mu-1}(C(\rho_{A|B_{N-2}})) f_\alpha(C(\rho_{A|B_{N-1}})) \right. \\
&\quad \left. + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right) f_\alpha^\mu(C(\rho_{A|B_{N-1}})) \right\},
\end{aligned} \tag{35}
$$

where the first inequality is due to the monotonically increasing property of the function $f_\alpha(x)$ and inequality (3), the second inequality is due to Lemma 5, and the third inequality is due to the fact that $C(\rho_{A|B_i}) \geq C(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $i = 1, 2, \cdots, N-2$. Then, according to the denotation of $Q_{AB_{N-2}}$ and the definition of the Rényi-α entanglement, we complete the proof.

Theorem 4 *For any $\alpha \geq 2$, the power $\mu \geq 1$, and N-qubit mixed state $\rho_{A|B_1\cdots B_{N-1}}$, if $C(\rho_{A|B_i}) \geq C(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $i = 1, 2, \cdots, m$, $C(\rho_{A|B_j}) \leq C(\rho_{A|B_{j+1}\cdots B_{N-1}})$, $j = m+1, m+2, \cdots N-2$, $N > 3$, then we have*

$$
E_\alpha^\mu(\rho_{A|B_1\cdots B_{N-1}}) \geq \sum_{i=1}^{m} h^{i-1} E_\alpha^\mu(\rho_{AB_i}) + h^{m+1} \sum_{j=m+1}^{N-3} E_\alpha^\mu(\rho_{AB_j}) + h^m Q_{AB_{N-1}}, \tag{36}
$$

where $h = 2^\mu - 1$, $Q_{AB_{N-1}} = E_\alpha^\mu(\rho_{A|B_{N-1}}) + \frac{\mu^2}{\mu+1} E_\alpha^{\mu-1}(\rho_{A|B_{N-1}}) E_\alpha(\rho_{A|B_{N-2}}) + (2^\mu - \frac{\mu^2}{\mu+1} - 1) E_\alpha^\mu(\rho_{A|B_{N-2}})$.

Proof For any $\alpha \geq 2$, $\mu \geq 1$, $C(\rho_{A|B_i}) \geq C(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $i = 1, 2, \cdots, m$, from Theorem 3, we know that

$$
E_\alpha^\mu(\rho_{A|B_1\cdots B_{N-1}}) \geq \sum_{i=1}^{m} h^{i-1} E_\alpha^\mu(\rho_{AB_i}) + h^m f_\alpha^\mu(C(\rho_{A|B_{m+1}\cdots B_{N-1}})). \tag{37}
$$

When $C(\rho_{A|B_j}) \leq C(\rho_{A|B_{j+1}\cdots B_{N-1}})$, $j = m+1, m+2, \cdots N-2$, $N > 3$, we get that

$$
\begin{aligned}
f_\alpha^\mu(C(\rho_{A|B_{m+1}\cdots B_{N-1}})) &\geq f_\alpha^\mu\left(\sqrt{C^2(\rho_{A|B_{m+1}}) + C^2(\rho_{A|B_{m+2}\cdots B_{N-1}})}\right) \\
&\geq f_\alpha^\mu(C(\rho_{A|B_{m+2}\cdots B_{N-1}})) + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right) f_\alpha^\mu(C(\rho_{A|B_{m+1}})) \\
&\quad + \tfrac{\mu^2}{\mu+1} f_\alpha^{\mu-1}(C(\rho_{A|B_{m+2}\cdots B_{N-1}})) f_\alpha(C(\rho_{A|B_{m+1}})) \\
&\geq f_\alpha^\mu(C(\rho_{A|B_{m+2}\cdots B_{N-1}})) + h f_\alpha^\mu(C(\rho_{A|B_{m+1}})) \\
&\geq \cdots \\
&\geq h\{f_\alpha^\mu(C(\rho_{A|B_{m+1}})) + \cdots + f_\alpha^\mu(C(\rho_{A|B_{N-3}}))\} \\
&\quad + \left\{ f_\alpha^\mu(C(\rho_{A|B_{N-1}})) + \tfrac{\mu^2}{\mu+1} f_\alpha^{\mu-1}(C(\rho_{A|B_{N-1}})) f_\alpha(C(\rho_{A|B_{N-2}})) \right. \\
&\quad \left. + (2^\mu - \tfrac{\mu^2}{\mu+1} - 1) f_\alpha^\mu(C^2(\rho_{A|B_{N-2}})) \right\},
\end{aligned} \tag{38}
$$

where the first inequality is due to the monotonically increasing property of the function $f_\alpha(x)$ and inequality (3), the third inequality is from the fact that $C(\rho_{A|B_j}) \leq C(\rho_{A|B_{j+1}\cdots B_{N-1}})$, $j = m+1, m+2, \cdots N-2$, $N>3$. According to the definition of the Rényi-α entanglement, and combining inequality (37) and (38), we obtain inequality (36).

Remark 2 We consider a particular case of $N=3$. Note that when $\alpha \geq 2$, the power $\mu \geq 1$, if $E_\alpha(\rho_{AB_1}) \geq E_\alpha(\rho_{AB_2})$, then we get the following result,

$$\begin{aligned} E_\alpha^\mu(\rho_{A|B_1B_2}) \geq\ & E_\alpha^\mu(\rho_{AB_1}) + \tfrac{\mu^2}{\mu+1} E_\alpha^{\mu-1}(\rho_{AB_1}) E_\alpha(\rho_{AB_2}) \\ & + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right) E_\alpha^\mu(\rho_{AB_2}), \end{aligned} \tag{39}$$

if $E_\alpha(\rho_{AB_1}) \leq E_\alpha(\rho_{AB_2})$, then

$$\begin{aligned} E_\alpha^\mu(\rho_{A|B_1B_2}) \geq\ & E_\alpha^\mu(\rho_{AB_2}) + \tfrac{\mu^2}{\mu+1} E_\alpha^{\mu-1}(\rho_{AB_2}) E_\alpha(\rho_{AB_1}) \\ & + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right) E_\alpha^\mu(\rho_{AB_1}). \end{aligned} \tag{40}$$

To see the tightness of the Rényi-α entanglement directly, we give the following example.

Example 2 Let us consider the state in (27) given in Example 1. Set $\lambda_0 = \frac{\sqrt{5}}{3}$, $\lambda_1 = \lambda_4 = 0$, $\lambda_2 = \frac{\sqrt{3}}{3}$, $\lambda_3 = \frac{1}{3}$, where $\alpha = 2$. From definition of the Rényi-α entanglement, after simple computation, we get $E_2(\varphi_{A|BC}) = 0.98230$, $E_2(\varphi_{AB}) = 0.66742$, $E_2(\varphi_{AC}) = 0.19010$, and $E_2^\mu(\rho_{A|BC}) = (0.98230)^\mu \geq E_2^\mu(\rho_{AB}) + (2^\mu - \frac{\mu^2}{\mu+1} - 1)E_2^\mu(\rho_{AC}) + \frac{\mu^2}{\mu+1} E_2^{\mu-1}(\rho_{AB}) E_2(\rho_{AC})$ $= (0.66742)^\mu + \frac{\mu^2}{\mu+1}(0.66742)^{\mu-1}(0.19010) + (2^\mu - \frac{\mu^2}{\mu+1} - 1)(0.19010)^\mu$. While the formula in [12] is $E_2^\mu(\rho_{AB}) + \frac{\mu}{2} E_2^{\mu-1}(\rho_{AB}) E_2(\rho_{AC}) + (2^\mu - \frac{\mu}{2} - 1) E_2^\mu(\rho_{AC}) = (0.66742)^\mu + \frac{\mu}{2}(0.66742)^{\mu-1}(0.19010) + (2^\mu - \frac{\mu}{2} - 1)(0.19010)^\mu$. One can see that our result is tighter than the ones in [12] for $\mu \geq 1$. See Fig. 2.

Theorem 5 *For any* $\frac{\sqrt{7}-1}{2} \leq \alpha < 2$, *the power* $\mu \geq 1$, $\mu = \frac{\gamma}{2}$, *and N-qubit mixed state* $\rho_{A|B_1\cdots B_{N-1}}$, *if* $C(\rho_{A|B_i}) \geq C(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $i = 1, 2, \cdots, N-2$, $N>3$, *then we have*

$$E_\alpha^\gamma(\rho_{A|B_1\cdots B_{N-1}}) \geq \sum_{i=1}^{N-3} h^{i-1} E_\alpha^\gamma(\rho_{A|B_i}) + h^{N-3} Q_{AB_{N-2}}, \tag{41}$$

where $h = 2^\mu - 1$, $Q_{AB_{N-2}} = E_\alpha^\gamma(\rho_{A|B_{N-2}}) + \frac{\mu^2}{\mu+1} E_\alpha^{\gamma-2}(\rho_{A|B_{N-2}}) E_\alpha^2(\rho_{A|B_{N-1}}) + (2^\mu - \frac{\mu^2}{\mu+1} - 1) E_\alpha^\gamma(\rho_{A|B_{N-1}})$.

Proof For $\frac{\sqrt{7}-1}{2} \leq \alpha < 2$, $\mu \geq 1$, and $\mu = \frac{\gamma}{2}$, we consider an N-qubit mixed state $\rho_{A|B_1\cdots B_{N-1}}$, from Lemma 4, we have

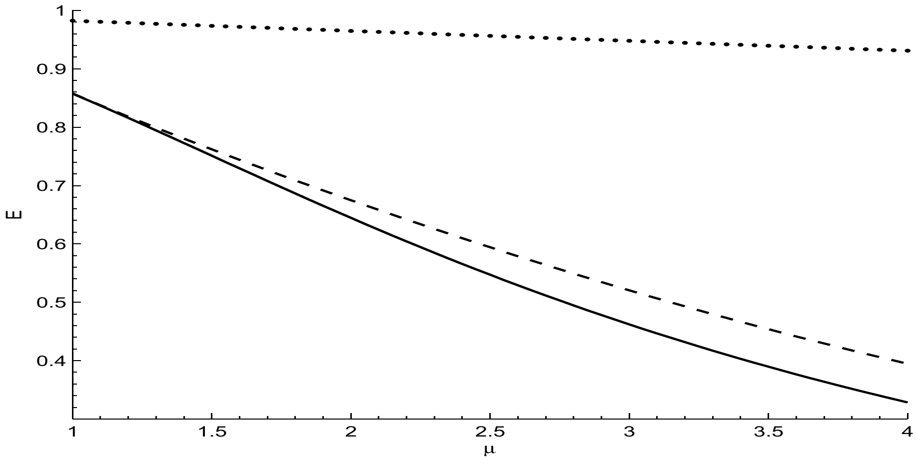

Fig. 2 The axis E stands the Rényi-α entanglement of $|\psi\rangle_{A|BC}$, which is a function of μ ($1 \le \mu \le 4$). The dotted line stands the value of $E_2^{\mu}(\rho_{A|BC})$. The dashed line stands the lower bound given by our improved monogamy relations. The solid black line represents the lower bound given by [12]

$$
\begin{aligned}
E_\alpha^\gamma(\rho_{A|B_1\cdots B_{N-1}}) &\ge f_\alpha^{2\mu}\left(\sqrt{C(\rho_{A|B_1}) + C^2(\rho_{A|B_2\cdots B_{N-1}})}\right)\\
&\ge f_\alpha^\gamma(C(\rho_{A|B_1})) + \tfrac{\mu^2}{\mu+1} f_\alpha^{\gamma-2}(C(\rho_{A|B_1})) f_\alpha^2(C(\rho_{A|B_2\cdots B_{N-1}}))\\
&\quad +(2^\mu - \tfrac{\mu^2}{\mu+1} - 1) f_\alpha^\gamma(C(\rho_{A|B_2\cdots B_{N-1}})\\
&\ge f_\alpha^\gamma(C(\rho_{A|B_1})) + h f_\alpha^\gamma(C(\rho_{A|B_2\cdot B_{N-1}}))\\
&\ge \cdots\\
&\ge f_\alpha^\gamma(C(\rho_{A|B_1})) + h f_\alpha^\gamma(C(\rho_{A|B_2})) + \cdots + h^{N-4} f_\alpha^\gamma(C(\rho_{A|B_{N-3}}))\\
&\quad + h^{N-3}\Big\{ f_\alpha^\gamma(C(\rho_{A|B_{N-2}})) + \tfrac{\mu^2}{\mu+1} f_\alpha^{\gamma-2}(C(\rho_{A|B_{N-2}})) f_\alpha^2(C(\rho_{A|B_{N-1}}))\\
&\quad +(2^\mu - \tfrac{\mu^2}{\mu+1} - 1) f_\alpha^\gamma(C(\rho_{A|B_{N-1}}))\Big\},
\end{aligned}
\tag{42}
$$

where the first inequality comes from the monotonically increasing property of the function $f_\alpha(x)$ and inequality (3), the second inequality is due to Lemma 6, and the third inequality is due to the fact that $C(\rho_{A|B_i}) \ge C(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $i = 1,2,\cdots,N-2$. According to the definition of the Rényi-α entanglement and the denotation of $Q_{AB_{N-2}}$, we obtain inequality (41).

Theorem 6 *For* $\frac{\sqrt{7}-1}{2} \le \alpha < 2$, *the power* $\mu \ge 1$, $\mu = \frac{\gamma}{2}$, *and N-qubit mixed state* $\rho_{A|B_1\cdots B_{N-1}}$, *if* $C(\rho_{A|B_i}) \ge C(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $i = 1, 2, \cdots, m$, $C(\rho_{A|B_j}) \le C(\rho_{A|B_{j+1}\cdots B_{N-1}})$, $j = m+1, m+2, \cdots N-2$, $N > 3$, *then we have*

$$
E_\alpha^\gamma(\rho_{A|B_1\cdots B_{N-1}}) \ge \sum_{i=1}^{m} h^{i-1} E_\alpha^\gamma(\rho_{AB_i}) + h^{m+1} \sum_{j=m+1}^{N-3} E_\alpha^\gamma(\rho_{AB_j}) + h^m Q_{AB_{N-1}},
\tag{43}
$$

where $h = 2^\mu - 1$, $Q_{AB_{N-1}} = E_\alpha^\gamma(\rho_{A|B_{N-1}}) + \frac{\mu^2}{\mu+1} E_\alpha^{\gamma-2}(\rho_{A|B_{N-1}}) E_\alpha^2(\rho_{A|B_{N-2}}) + (2^\mu - \frac{\mu^2}{\mu+1} - 1) E_\alpha^\gamma(\rho_{A|B_{N-2}})$.

Proof When $C(\rho_{A|B_i}) \geq C(\rho_{A|B_{i+1}\cdots B_{N-1}})$, $i=1,2,\cdots,m$, from Theorem 5, we have

$$\begin{aligned}E_\alpha^\gamma(\rho_{A|B_1\cdots B_{N-1}}) &\geq f_\alpha^\gamma(C(\rho_{A|B_1})) + hf_\alpha^\gamma(C(\rho_{A|B_2})) + \cdots + h^{m-1}f_\alpha^\gamma(C(\rho_{A|B_m})) \\ &\quad + h^m f_\alpha^\gamma(C(\rho_{A|B_{m+1}\cdots B_{N-1}})) \\ &= \sum_{i=1}^{m} h^{i-1}E_\alpha^\gamma(\rho_{AB_i}) + h^m f_\alpha^\gamma(C(\rho_{A|B_{m+1}\cdots B_{N-1}})).\end{aligned} \tag{44}$$

When $C(\rho_{A|B_j}) \leq C(\rho_{A|B_{j+1}\cdots B_{N-1}})$, $j = m+1, m+2, \cdots N-2$, $N > 3$, from Lemma 6, we get

$$\begin{aligned}f_\alpha^\gamma(C(\rho_{A|B_{m+1}\cdots B_{N-1}})) &\geq f_\alpha^{2\mu}\left(\sqrt{C^2(\rho_{A|B_{m+1}}) + C^2(\rho_{A|B_{m+2}\cdots B_{N-1}})}\right) \\ &\geq f_\alpha^\gamma(C(\rho_{A|B_{m+2}\cdots B_{N-1}})) + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right)f_\alpha^\gamma(C(\rho_{A|B_{m+1}})) \\ &\quad + \tfrac{\mu^2}{\mu+1}f_\alpha^{\gamma-2}(C(\rho_{A|B_{m+2}\cdots B_{N-1}}))f_\alpha^2(C(\rho_{A|B_{m+1}})) \\ &\geq f_\alpha^\gamma(C(\rho_{A|B_{m+2}\cdots B_{N-1}})) + hf_\alpha^\gamma(C(\rho_{A|B_{m+1}})) \\ &\geq \cdots \\ &\geq hf_\alpha^\gamma(C(\rho_{A|B_{m+1}})) + \cdots + hf_\alpha^\gamma(C(\rho_{A|B_{N-3}})) \\ &\quad + \left\{ f_\alpha^\gamma(C(\rho_{A|B_{N-1}})) + \tfrac{\mu^2}{\mu+1}f_\alpha^{\gamma-2}(C(\rho_{A|B_{N-1}}))f_\alpha^2(C(\rho_{A|B_{N-2}})) \right. \\ &\quad \left. + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right)f_\alpha^\gamma(C(\rho_{A|B_{N-2}})) \right\},\end{aligned} \tag{45}$$

where the first inequality comes from the monotonically increasing property of the function $f_\alpha(x)$ and inequality (3), the second inequality is due to Lemma 6, and the third inequality is due to the fact that $C(\rho_{A|B_j}) \leq C(\rho_{A|B_{j+1}\cdots B_{N-1}})$, $j = m+1, m+2, \cdots N-2$, $N > 3$. According to the denotation of $Q_{AB_{N-1}}$ and combining inequality (44) and (45), we complete the proof.

Remark 3 We consider a particular case of $N=3$. Note that when $\frac{\sqrt{7}-1}{2} \leq \alpha < 2$, the power $\mu \geq 1$ and $\mu = \frac{\gamma}{2}$, if $E_\alpha(\rho_{AB_1}) \geq E_\alpha(\rho_{AB_2})$, then we get the following result,

$$\begin{aligned}E_\alpha^\gamma(\rho_{A|B_1B_2}) &\geq E_\alpha^\gamma(\rho_{AB_1}) + \tfrac{\mu^2}{\mu+1}E_\alpha^{\gamma-2}(\rho_{AB_1})E_\alpha^2(\rho_{AB_2}) \\ &\quad + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right)E_\alpha^\gamma(\rho_{AB_2}),\end{aligned} \tag{46}$$

if $E_\alpha(\rho_{AB_1}) \leq E_\alpha(\rho_{AB_2})$, then

$$\begin{aligned}E_\alpha^\gamma(\rho_{A|B_1B_2}) &\geq E_\alpha^\gamma(\rho_{AB_2}) + \tfrac{\mu^2}{\mu+1}E_\alpha^{\gamma-2}(\rho_{AB_2})E_\alpha^2(\rho_{AB_1}) \\ &\quad + \left(2^\mu - \tfrac{\mu^2}{\mu+1} - 1\right)E_\alpha^\gamma(\rho_{AB_1}).\end{aligned} \tag{47}$$

To see the tightness of the Rényi-α entanglement directly, we give the following example.

Example 3 Let us consider the state in (27) given in Example 1. Suppose that $\lambda_0 = \frac{\sqrt{5}}{3}$, $\lambda_1 = \lambda_4 = 0$, $\lambda_2 = \frac{\sqrt{3}}{3}$, $\lambda_3 = \frac{1}{3}$, and $\alpha = \frac{\sqrt{7}-1}{2}$. From definition of the Rényi-α entanglement, we have $E_\alpha(|\psi\rangle_{A|BC}) = 0.99265$, $E_\alpha(|\psi\rangle_{AB}) = 0.83477$, $E_\alpha(|\psi\rangle_{AC}) = 0.41466$, and $E_\alpha^\gamma(\rho_{A|BC}) = (0.99265)^\gamma \geq E_\alpha^\gamma(\rho_{AB}) + \frac{\gamma^2}{4+2\gamma}E_\alpha^{\gamma-2}(\rho_{AB})E_\alpha^2(\rho_{AC}) + (2^{\frac{\gamma}{2}} - \frac{\gamma^2}{4+2\gamma} - 1)E_\alpha^\gamma(\rho_{AC}) = (0.83477)^\gamma + \frac{\gamma^2}{4+2\gamma}(0.83477)^{\gamma-2}(0.41466)^2$

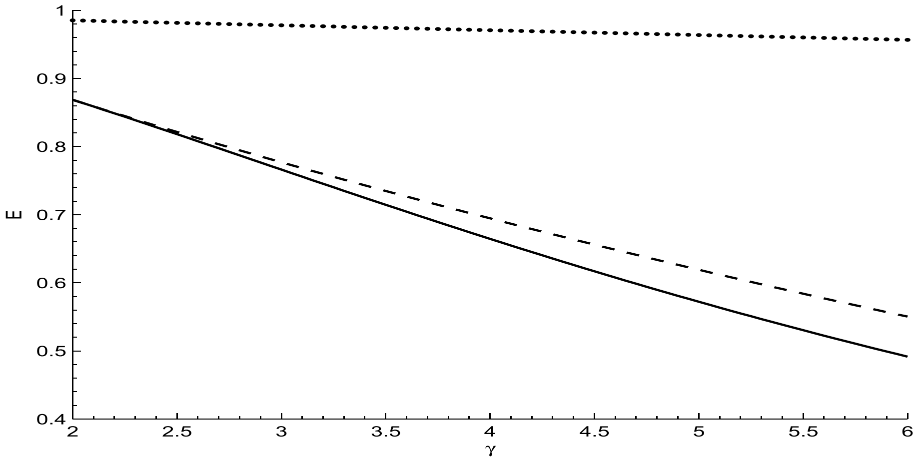

Fig. 3 The axis E stands the Rényi-α entanglement of $|\psi\rangle_{A|BC}$, which is a function of γ ($2 \le \gamma \le 6$). The dotted line stands the value of $E^{\gamma}_{\frac{\sqrt{7}-1}{2}}(\rho_{A|BC})$. The dashed line stands the lower bound given by our improved monogamy relations. The solid black line represents the lower bound given by [12]

$+ (2^{\frac{\gamma}{2}} - \frac{\gamma^2}{4+2\gamma} - 1)(0.41466)^{\gamma}$. While the formula in [12] is $E^{\gamma}_{\alpha}(\rho_{AB}) + \frac{\gamma}{4}E^{\gamma-2}_{\alpha}(\rho_{AB})E^{2}_{\alpha}(\rho_{AC}) + (2^{\frac{\gamma}{2}} - \frac{\gamma}{4} - 1)E^{\gamma}_{\alpha}(\rho_{AC}) = (0.83477)^{\gamma} + \frac{\gamma}{4}(0.83477)^{\gamma-2}(0.41466)^2 + (2^{\frac{\gamma}{2}} - \frac{\gamma}{4} - 1)(0.41466)^{\gamma}$. One can see that our result is tighter than the result in [12] for $\mu = \frac{\gamma}{2}, \gamma \ge 2$. See Fig. 3.

5 Conclusion

Multipartite entanglement can be regarded as a fundamental problem in the theory of quantum entanglement. Our results may contribute to a fuller understanding of the Tsallis-q and Rényi-α entanglement in multipartite systems. In this paper, we have explored some tighter monogamy relations in terms of η th power of the Tsallis-q entanglement $T^{\eta}_{q}(\rho_{A|B_1\cdots B_{N-1}})$ ($\eta \ge 1$, $2 \le q \le 3$) and the Rényi-α entanglement $E^{\mu}_{\alpha}(\rho_{A|B_1\cdots B_{N-1}})$ ($\mu \ge 1$, $\alpha \ge 2$) and $E^{\gamma}_{\alpha}(\rho_{A|B_1\cdots B_{N-1}})$ ($\gamma \ge 2$, $\frac{\sqrt{7}-1}{2} \le \alpha < 2$). We show that these new monogamy relations of multiparty entanglement have larger lower bounds and are tighter than the existing results [11, 12]. Our approach may also be applied to the study of monogamy properties related to other quantum correlations.

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Declarations

Conflict of Interests The authors have no relevant financial or non-financial interests to disclose.

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