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Demonstrate Absolutely Maximally Entangled of Four- and Eight-qubit States Inexistence via Simple Constraint Condition

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Abstract

A pure multi-qubit state is called absolutely maximally entangled if all reduced states obtained by tracing out at least half of the particles are maximally mixed. Recently, Felix Huber proved that the absolutely maximally seven-qubit entangled state does not exist. In this letter, we investigate the relation of reduced density matrix and the local unitary transformation invariants of four- and eight-qubit entangled states. Using some constraint conditions, for four- and eight-qubit states, we can prove that absolutely maximally entangled states do not exist.

Keywords Absolutely maximally entangled states · Four- and eight-qubit states · Constraint condition

1 Introduction

Quantum entanglement is one of the most fascinating features in quantum physics, with numerous applications in quantum information and computation [1–6]. Maximally entangled states have been shown to be a resource for a variety of quantum information theoretic tasks. Therefore, the research of maximally entangled states has attracted a great deal of attention, especially absolutely maximally entangled (AME) states [7–16]. Then there is a fundamental question: which states are maximally entangled. In the case of 2 qubits, it is known that Bell states are maximally entangled with respect to any measures of entanglement [1]. Note that GHZ-like states are highly entangled, but even more entangled are AME states [10, 17], which are maximally entangled in every bipartition of the system.

The study of AME states has become an intensive area of research along the recent years due to both theoretical foundations and practical applications. Especially, with devel-

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opment and applications of optical quantum computing [18–20], it is possible to build entangle states base on photon. An n-qubit pure state $|\psi\rangle$ is a k-uniform state provided that all of its reductions to k-qubits are maximally mixed [16]. It is known that the integer number k cannot exceed n/2. Particularly interesting are those n-qubit states which are [n/2]-uniform states. Such states are also called absolutely maximally entangled (AME) states. For instance, Bell states and GHZ states are AME states for bipartite and three partite systems respectively.

It is well known that absolutely maximally entangled (AME) states exist only for special values of n (n=2,3,5,6) [9, 21]. Recently, Felix Huber, et al. [21] has proved that there is no AME state for seven qubits. In this note we will give some expressions for four- and eight-qubit states. Furthermore, we prove that AME states for four- and eight-qubit states do not exist via simple constraint condition. We hope this method can be used to demonstrate the more qubits AME states inexistence.

2 The Constraint Condition of Four- and Eight-qubit States

2.1 The Constraint Condition of Four -qubit State

For the wave function of a four-qubit pure state,

$$\begin{aligned}|\psi\rangle_{1234} = \ & a_0|0000\rangle + a_1|0001\rangle + a_2|0010\rangle + a_3|0011\rangle \\ & + a_4|0100\rangle + a_5|0101\rangle + a_6|0110\rangle + a_7|0111\rangle \\ & + a_8|1000\rangle + a_9|1001\rangle + a_{10}|1010\rangle + a_{11}|1011\rangle \\ & + a_{12}|1100\rangle + a_{13}|1101\rangle + a_{14}|1110\rangle + a_{15}|1111\rangle \end{aligned} \tag{1}$$

Then we have density matrix

$$\rho_{1234} = |\psi\rangle_{1234\,1234}\langle\psi| \tag{2}$$

The corresponding reduced density matrix can be shown as [14, 15]

$$Tr_{ijkl}\rho_{ijkl}^2 = \frac{1}{16} + \frac{1}{16}(\sum_u T_u + \sum_{u\neq v} T_{uv} + \sum_{u\neq v\neq w} T_{uvw} + \sum_{i\neq j\neq k\neq l} T_{ijkl}) \tag{3}$$

where

$$T_i = \langle\psi|\hat{\sigma}_{ix}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{iy}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{iz}|\psi\rangle^2 \tag{4}$$

$$\begin{aligned}T_{ij} = \ & \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jy}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jz}|\psi\rangle^2 \\ & + \langle\psi|\hat{\sigma}_{iy}\hat{\sigma}_{jx}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{iy}\hat{\sigma}_{jy}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{iy}\hat{\sigma}_{jz}|\psi\rangle^2 \\ & + \langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jx}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jy}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}|\psi\rangle^2 \end{aligned} \tag{5}$$

$$\begin{aligned}T_{ijk} = \ & \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{kx}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{ky}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{kz}|\psi\rangle^2 \\ & + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jy}\hat{\sigma}_{kx}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jy}\hat{\sigma}_{ky}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jy}\hat{\sigma}_{kz}|\psi\rangle^2 \\ & + \ldots \\ & + \langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}\hat{\sigma}_{kx}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}\hat{\sigma}_{ky}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}\hat{\sigma}_{kz}|\psi\rangle^2 \end{aligned} \tag{6}$$

$$\begin{aligned}T_{ijkl} = \ & \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{kx}\hat{\sigma}_{lx}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{kx}\hat{\sigma}_{ly}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{kx}\hat{\sigma}_{lz}|\psi\rangle^2 \\ & +\langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{ky}\hat{\sigma}_{lx}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{ky}\hat{\sigma}_{ly}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{ix}\hat{\sigma}_{jx}\hat{\sigma}_{ky}\hat{\sigma}_{lz}|\psi\rangle^2 \\ & +\ldots \\ & +\langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}\hat{\sigma}_{kz}\hat{\sigma}_{lx}|\psi\rangle^2 + \langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}\hat{\sigma}_{kz}\hat{\sigma}_{ly}|\psi\rangle^2 \\ & +\langle\psi|\hat{\sigma}_{iz}\hat{\sigma}_{jz}\hat{\sigma}_{kz}\hat{\sigma}_{lz}|\psi\rangle^2 \end{aligned} \tag{7}$$

It is obvious that such invariants satisfy $T_i \geq 0$, $T_{ij} \geq 0$, $T_{ijk} \geq 0$, $T_{ijkl} \geq 0$.

Let

$$\begin{aligned} & C_1 = T_1 + T_2 + T_3 + T_4, C_2 = T_{12} + T_{13} + T_{14} + T_{23} + T_{24} + T_{34}, \\ & C_3 = T_{123} + T_{124} + T_{134} + T_{234}, C_4 = T_{1234}. \end{aligned} \tag{8}$$

Therefore, we have $C_1 \geq 0$, $C_2 \geq 0$, $C_3 \geq 0$, $C_4 \geq 0$.

For four-qubit pure state, it is well know that $Tr\rho_{1234}^2 = 1$. Then, (3) can be written

$$1 = \frac{1}{16} + \frac{1}{16}(C_1 + C_2 + C_3 + C_4) \tag{9}$$

Further, it is known that [22]

$$tr\rho_{1234}\tilde{\rho}_{1234} = \frac{1}{16} + \frac{1}{16}(-C_1 + C_2 - C_3 + C_4) \tag{10}$$

and

$$\tilde{\rho}_{ijkl} = \sigma_2^{\otimes 4}\rho^T\sigma_2^{\otimes 4} \tag{11}$$

For four-qubit pure state, it is well know that

$$tr\rho_{123}^2 = tr\rho_4^2; tr\rho_{124}^2 = tr\rho_3^2; tr\rho_{134}^2 = tr\rho_2^2; tr\rho_{234}^2 = tr\rho_1^2, \tag{12}$$

Using

$$\begin{aligned} & Tr_i\rho_i^2 = \frac{1}{2} + \frac{1}{2}T_i, i = 1, 2, 3, 4. \\ & Tr_{ij}\rho_{ij}^2 = \frac{1}{4} + \frac{1}{4}(T_i + T_j + T_{ij}), ij = 12, 13, 14, 23, 24, 34. \\ & Tr_{ijk}\rho_{ijk}^2 = \frac{1}{8} + \frac{1}{8}(T_i + T_j + T_k + T_{ij} + T_{ik} + T_{jk} + T_{ijk}), \\ & ijk = 123, 124, 134, 234. \end{aligned} \tag{13}$$

Then, we have

$$\begin{aligned} & \frac{1}{8} + \frac{1}{8}(T_1 + T_2 + T_3 + T_{12} + T_{13} + T_{23} + T_{123}) = \frac{1}{2} + \frac{1}{2}T_4, \\ & \frac{1}{8} + \frac{1}{8}(T_1 + T_2 + T_4 + T_{12} + T_{14} + T_{24} + T_{124}) = \frac{1}{2} + \frac{1}{2}T_3, \\ & \frac{1}{8} + \frac{1}{8}(T_1 + T_3 + T_4 + T_{13} + T_{14} + T_{34} + T_{134}) = \frac{1}{2} + \frac{1}{2}T_2, \\ & \frac{1}{8} + \frac{1}{8}(T_2 + T_3 + T_4 + T_{23} + T_{24} + T_{34} + T_{234}) = \frac{1}{2} + \frac{1}{2}T_1 \end{aligned} \tag{14}$$

From (14), we can obtain

$$C_3 = 12 + C_1 - 2C_2 \tag{15}$$

From (10–15), we can also obtain a relation

$$4tr\rho_{1234}\tilde{\rho}_{1234} = -2 - C_1 + C_2 \tag{16}$$

On the other hand, it is well know that $Tr_A\rho_A^2 = 1/2^{n_A}$ for every subsystem A if multi-qubit states is a absolutely maximally entangled (AME) state, where $n_A = [n/2]$. The marginal density matrix $\rho_A = Tr_{\bar{A}}|\psi\rangle\langle\psi|$ is the reduced matrix after partial trace operation over the complementary subsystem $\bar{A}$ is implemented. For four-qubit pure states, $n_A = 2, Tr_A\rho_A^2 = 1/4$.

Therefore, if four-qubit pure state is an absolutely maximally entangled (AME) state, it must have

$$Tr_{ij}\rho_{ij}^2 = \frac{1}{4}, ij = 12, 13, \ldots, 34 \tag{17}$$

Compare with (13), we know it must be $C_1 = 0, C_2 = 0$.

Then from (16) we have $4tr\rho_{1234}\tilde{\rho}_{1234} = -2$. But from Ref [22], we know that $4tr\rho_{1234}\tilde{\rho}_{1234} \geq 0$.

It is contradiction. Therefore, there is no absolutely maximally entangled state of four-qubit state.

2.2 The Constraint Condition of Eight- qubit State

For the wave function of a eight-qubit pure state,

$$\begin{aligned}|\psi\rangle_{12345678} = \; & a_0|00000000\rangle + a_1|00000001\rangle + a_2|00000010\rangle \\ & + \ldots \\ & + a_{253}|11111101\rangle + a_{254}|11111110\rangle + a_{255}|11111111\rangle\end{aligned} \tag{18}$$

Then we have density matrix $\rho_{12345678} = |\psi\rangle_{12345678\,12345678}\langle\psi|$

Similarly, we have

$$1 = \frac{1}{256} + \frac{1}{256}(C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8) \tag{19}$$

$$\begin{aligned}& tr\rho_{12345678}\tilde{\rho}_{12345678} = \\ & \frac{1}{256} + \frac{1}{256}(-C_1 + C_2 - C_3 + C_4 - C_5 + C_6 - C_7 + C_8)\end{aligned} \tag{20}$$

Using $tr\rho_{12345}^2 = tr\rho_{678}^2; tr\rho_{123456}^2 = tr\rho_{78}^2; tr\rho_{1234567}^2 = tr\rho_8^2$, etc, we have

$$16tr\rho_{12345678}\tilde{\rho}_{12345678} = -26 - 9C_1 - C_2 + C_3 + C_4 \tag{21}$$

It is known that absolutely maximally entangled state, it must be $C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 0$.

Thus, from (21), we know that left $4tr\rho_{12345678}\tilde{\rho}_{12345678} \geq 0$, but right $-26 - 9C_1 - C_2 + C_3 + C_4 = -26$.

It is a contradiction. Therefore, there is no absolutely maximally entangled state of eight-qubit state.

3 Conclusions

In summary, we investigate the relation between the reduced density matrix and the local unitary (LU) transformation invariants of four- qubit and eight-qubit states. For four- and eight-qubit states, we obtain some constraint conditions. By using these constraint conditions, we can prove that absolutely maximally entangled four- qubit and eight-qubit states do not exist. In the following, we will try to demonstrate whether more qubits exist when k-body reduced density are maximally mixed for $k < [n/2]$. We believe this constraint condition can play an important role in determining whether absolutely maximally entangled exist.

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Declarations

Conflict of Interests On behalf of all authors, the corresponding author states that there is no conflict of interest.

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