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The Transformation From Pure States to X States Under Incoherent Operations

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Abstract
We study the conversion between pure states and X states under incoherent operations. We derive an optimal pure state decomposition of X state such that all pure state decompositions of X state are majorized by it. Then we show a necessary and sufficient condition for pure states which can be converted into X state under incoherent operations. We also obtain an optimal pure state related to X state such that all pure states that can be converted to X state by incoherent operations are majorized by it. The incoherent operations converting the pure state into X state are analyzed. The coherence measure is also calculated for X states.

Keywords X states · Coherence manipulation · Coherence measure

1 Introduction

Quantum coherence is an essential ingredient for a plethora of physical phenomena in quantum optics, quantum information, solid state physics, and nanoscale thermodynamics [1, 2]. As a kind of quantum resource, some basic characterizations of coherence are studied such as quantification and manipulation. The quantification of quantum coherence is initiated in Ref. [3] and some coherence measures such as the l_1 norm of coherence [3], the relative entropy of coherence [3], intrinsic randomness of coherence [4], coherence concurrence [5], distillable coherence [6], robustness of coherence [7], geometric coherence [8], coherence number [9] are proposed from different aspects.

The quantum coherence manipulation is the conversion between quantum states under incoherent operations [10]. For pure states, one can be converted into another by incoherent

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operations if and only if their coherence vectors satisfy a majorization relation [11], which is a counterpart of the celebrated Nielson theorem in entanglement manipulation [12]. Then a necessary and sufficient condition for the transformation between two pure state ensembles is demonstrated [13], but the conversion between ensembles is different from the manipulation of mixed states. The conversion from pure states to mixed states is studied and the necessary and sufficient condition has been provided in terms of a sequence of inequalities about a given coherence measure in Ref. [14].

The quantification and manipulation of coherence are not separate because the coherence is not increased under the incoherent operations. On one hand, some necessary and sufficient conditions of the coherence manipulation is presented in terms of coherence measures [13, 14]. On the other hand, some coherence measures are introduced based on the coherence manipulation [14, 15]. Generally, both the transformation between mixed states under incoherent operations and the quantification of coherence in terms of extremization for mixed states are complex and difficult because of the infinite pure state decompositions of mixed states.

Here we focus on the conversion between pure states and X state and calculate its coherence measure. X state is a class of mixed states with density matrix in X shape. Its quantum correlation such as quantum discord [16–18] and one-way quantum deficit [19, 20], and quantum coherence such as coherence concurrence [21] are investigated broadly. Attributing to the symmetry of X state formally, the study of its quantumness seems to be more feasible than general mixed states.

In the following, we first find an optimal pure state decomposition of X state that can major any pure state decomposition of X state. Then the necessary and sufficient condition for pure states which can be converted to X state is given analytically. We also find the optimal pure state associated with X state such that any pure state which can be converted to X state should be majorized by it. Then we construct the incoherent operations converting the pure state to X state. This process is illustrated by an explicit example in a three dimensional system. At last we calculate the coherence measure for X states.

2 The Conversion from Pure States to *X* States Under Incoherent Operations

2.1 Some Basic Concepts

The resource theory of coherence is composed of two basic elements: free states and free operations. Let H be a finite-dimensional Hilbert space with $dim(H) = d$. Take a set of basis $\{|i\rangle\}_{i=1}^{d}$, we call the diagonal quantum state $\rho = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|$ under this set of basis as the incoherent state. This set of incoherent states is labeled by ∇. Φ is an operation if and only if there exists finite bounded linear operators K_n satisfying $\sum_n K_n^\dagger K_n = I$ and $\Phi(\rho) = \sum_n K_n \rho K_n^\dagger$, where I is the identity operation. Φ is an incoherent operation if it fulfills $K_n \delta K_n^\dagger / Tr(K_n \delta K_n^\dagger) \in \nabla$ for all $\delta \in \nabla$ and for all n [22]. We denote $\rho \xrightarrow{ICO} \rho'$ when the quantum state ρ can be transformed to ρ' by incoherent operations.

The quantum coherence is degreed by a nonnegative function named the coherence measure. A coherence measure should satisfy five conditions as follows [3]: (A1) $C(\rho) = 0$ for all $\rho \in \nabla$; (A2) monotonicity: $C(\varepsilon(\rho)) \leq C(\rho)$ for any incoherent operation ε; (A3) strong monotonicity: $\sum_n p_n C(K_n \rho K_n^\dagger / p_n) \leq C(\rho)$ with $p_n = Tr(K_n \rho K_n^\dagger)$ and

$\rho_n = K_n \rho K_n^\dagger / p_n$; (A4) convexity: $C(\rho) \le \sum_i p_i C(\rho_i)$ for any $\rho = \sum_i p_i \rho_i$; (A5) only maximally coherent states reach the maximum: $C(\rho)$ is maximal only for $\rho = |\Phi_d\rangle\langle\Phi_d|$, where $|\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d e^{\mathrm{i}\theta_i} |i\rangle$ with real θ_i and i the imaginary unit. $C(\cdot)$ will be called as a coherence monotone if it satisfies all above conditions but $(A4)$.

In this paper, we fix the reference basis as $\{|i\rangle\}_{i=1}^d$. For any pure state $|\phi\rangle = \sum_{i=1}^d \phi_i |i\rangle$, its coherence vector is $(|\phi_1|^2, |\phi_2|^2, \cdots, |\phi_d|^2)^T$. Let $\mathcal{R}^\downarrow(|\phi\rangle) = (|\phi_1|^{2\downarrow}, |\phi_2|^{2\downarrow}, \cdots, |\phi_d|^{2\downarrow})^T$ be the vector obtained by rearranging the entries of the coherence vector in descending order $|\phi_1|^{2\downarrow} \ge |\phi_2|^{2\downarrow} \ge \cdots \ge |\phi_d|^{2\downarrow}$. Based on the vector $\mathcal{R}^\downarrow(|\phi\rangle)$, a series of coherence measures are introduced in Ref. [14].

Definition 1 For any pure state $|\phi\rangle = \sum_{i=1}^d \phi_i |i\rangle$, let

$$C_{f_l}(|\phi\rangle\langle\phi|) = \sum_{i=l}^d |\phi_i|^{2\downarrow}, \tag{1}$$

then $C_{f_l}(\rho) = \min \sum_n \mu_n C_{f_l}(|\phi_n\rangle\langle\phi_n|)$ is a coherence measure, where the minimization is taken over all pure state decompositions of $\rho = \sum_n \mu_n |\phi_n\rangle\langle\phi_n|$. We call $C_{f_l}(\rho)$ the l-th order coherence of ρ, $l = 2, \cdots, d$.

Suppose $\mathbf{x} = (x_1,\ x_2, \cdots,\ x_d)^T$ and $\mathbf{y} = (y_1,\ y_2, \cdots,\ y_d)^T$ are two probability distributions in H with coordinates in the decreasing order. $\mathbf{x}$ is majorized by $\mathbf{y}$ denoted as $\mathbf{x} \prec \mathbf{y}$, if $\sum_{i=1}^k x_i \le \sum_{i=1}^k y_i$ for $1 \le k \le d$. By the majorization, the necessary and sufficient condition for the conversion between two pure states is presented [11].

Lemma 1 *For any two pure states* $|\phi\rangle = \sum_{i=1}^d \phi_i |i\rangle$ *and* $|\psi\rangle = \sum_{i=1}^d \psi_i |i\rangle$, $|\psi\rangle \xrightarrow{ICO} |\phi\rangle$ *if and only if* $(|\psi_1|^{2\downarrow}, |\psi_2|^{2\downarrow}, \cdots, |\psi_d|^{2\downarrow})^T \prec (|\phi_1|^{2\downarrow}, |\phi_2|^{2\downarrow}, \cdots, |\phi_d|^{2\downarrow})^T$.

Furthermore, the necessary and sufficient condition for the conversion between pure states and mixed states is shown by a series of inequalities in terms of the l-th order coherence in Ref. [14].

Lemma 2 *For any pure state* $|\phi\rangle$ *and any mixed state* ρ, $|\phi\rangle \xrightarrow{ICO} \rho$ *if and only if there exists a pure state ensemble* $\{p_i, |\psi_i\rangle\}$ *of* ρ *such that*

$$C_{f_l}(|\phi\rangle\langle\phi|) \ge \sum_i p_i C_{f_l}(|\psi_i\rangle\langle\psi_i|), \quad l = 2, \cdots, d. \tag{2}$$

By Lemma 2, if the pure state $|\phi\rangle$ can be converted to the mixed state ρ, then one should find a pure state decomposition of ρ such that the average coherence measured by the l-th order coherence in (1) satisfies the $d-1$ inequalities in (2). So generally it is not easy to determine whether a pure state can be converted to a given mixed state.

2.2 The Transformation from Pure States to *X* States Under Incoherent Operations

Now we study the transformation from pure states to X states by incoherent operations. The d dimensional X states denoted as X_d are quantum states with density matrices in X shape under the reference basis,

$$X_d = \begin{pmatrix} \rho_{11} & 0 & 0 & \cdots & 0 & 0 & \rho_{1,d} \\ 0 & \rho_{22} & 0 & \cdots & 0 & \rho_{2,d-1} & 0 \\ 0 & 0 & \rho_{33} & \cdots & \rho_{3,d-2} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \rho_{d-2,3} & \cdots & \rho_{d-2,d-2} & 0 & 0 \\ 0 & \rho_{d-1,2} & 0 & \cdots & 0 & \rho_{d-1,d-1} & 0 \\ \rho_{d,1} & 0 & 0 & \cdots & 0 & 0 & \rho_{dd} \end{pmatrix}. \quad (3)$$

Now we show an optimal pure state decomposition for X states.

Theorem 1 *For quantum state X_d in (3), let $\mathfrak{D} = \{\mu_{ki}, |\chi_{ki}\rangle\} \cup \left\{\frac{1-(-1)^d}{2}\rho_{[d/2]+1}, [d/2]+1, |[d/2]+1\rangle\right\}$ be the pure state decomposition of X_d, where*

$$\begin{aligned} |\chi_{1i}\rangle\langle\chi_{1i}| &= \begin{pmatrix} \frac{1+z_i}{2} & \frac{\rho_{i,d-i+1}}{\rho_{ii}+\rho_{d-i+1,d-i+1}} \\ \frac{\rho^*_{i,d-i+1}}{\rho_{ii}+\rho_{d-i+1,d-i+1}} & \frac{1-z_i}{2} \end{pmatrix}, \\ |\chi_{2i}\rangle\langle\chi_{2i}| &= \begin{pmatrix} \frac{1-z_i}{2} & \frac{\rho_{i,d-i+1}}{\rho_{ii}+\rho_{d-i+1,d-i+1}} \\ \frac{\rho^*_{i,d-i+1}}{\rho_{ii}+\rho_{d-i+1,d-i+1}} & \frac{1+z_i}{2} \end{pmatrix}, \end{aligned} \quad (4)$$

are pure states on the subspace spanned by $\{|i\rangle\langle i|, |d-i+1\rangle\langle d-i+1|\}$, $z_i = \sqrt{1-4|\frac{\rho_{i,d-i+1}}{\rho_{ii}+\rho_{d-i+1,d-i+1}}|^2}$, $[d/2]$ is the integer part of $d/2$. Let $\mu_{ki} = \tau_i\lambda_{ki}$ with $\tau_i = \rho_{ii} + \rho_{d-i+1,d-i+1}$ for $k = 1, 2$, $\lambda_{1i} = \frac{1}{2} + \frac{\rho_{ii}-\rho_{d-i+1,d-i+1}}{2\sqrt{(\rho_{ii}+\rho_{d-i+1,d-i+1})^2-4|\rho_{i,d-i+1}|^2}}$ and $\lambda_{2i} = 1-\lambda_{1i}$, $i = 1, 2, \cdots, [d/2]$. Then any pure state decomposition $\{p_j, |\psi_j\rangle\}$ of X_d is majorized by the pure state decomposition $\mathfrak{D}$,

$$\sum_j p_j \mathcal{R}^\downarrow(|\psi_j\rangle) \prec \sum_{i=1}^{[d/2]}\sum_{k=1}^{2} \mu_{ki}\mathcal{R}^\downarrow(|\chi_{ki}\rangle) + \frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1}\mathcal{R}^\downarrow(|[d/2]+1\rangle).$$

Proof First we write the state X_d in the form of direct sum,

$$X_d = \tau_1\rho_1 \bigoplus \tau_2\rho_2 \bigoplus \cdots \bigoplus \tau_{[d/2]}\rho_{[d/2]} \bigoplus \frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1}|[d/2]+1\rangle\langle[d/2]+1|,$$

with $\rho_i = \begin{pmatrix} \frac{\rho_{ii}}{\rho_{ii}+\rho_{d-i+1,d-i+1}} & \frac{\rho_{i,d-i+1}}{\rho_{ii}+\rho_{d-i+1,d-i+1}} \\ \frac{\rho^*_{i,d-i+1}}{\rho_{ii}+\rho_{d-i+1,d-i+1}} & \frac{\rho_{d-i+1,d-i+1}}{\rho_{ii}+\rho_{d-i+1,d-i+1}} \end{pmatrix}$ on the subspace spanned by $\{|i\rangle\langle i|, |d-i+1\rangle\langle d-i+1|\}$ and $\tau_i = \rho_{ii} + \rho_{d-i+1,d-i+1}$, $i = 1, 2, \cdots, [d/2]$. Then ρ_i can be decomposed as $\{\lambda_{ki},\ |\chi_{ki}\rangle\}_{k=1}^2$ with $|\chi_{ki}\rangle$ in (4) such that for any pure state decomposition $\{q_{si}, |\psi_{si}\rangle\}$ with $|\psi_{si}\rangle = \sum_j \psi_j^{(si)}|j\rangle$ of ρ_i [15], it has

$$\sum_s q_{si}\mathcal{R}^\downarrow(|\psi_{si}\rangle) \prec \sum_{k=1}^{2} \lambda_{ki}\mathcal{R}^\downarrow(|\chi_{ki}\rangle)$$

for $i = 1, 2, \cdots, [d/2]$. Therefore

$$\textstyle\sum_s q_{si}|\psi_1^{(si)}|^{2\downarrow} \le \lambda_{1i}\frac{1+z_i}{2} + \lambda_{2i}\frac{1+z_i}{2} = \frac{1+z_i}{2},$$

which is equivalent to

$$\textstyle\sum_s q_{si}|\psi_2^{(si)}|^{2\downarrow} \geq \frac{1-z_i}{2}, \tag{5}$$

for $i = 1, 2, \cdots, [d/2]$.

Combining Lemma 2 and Definition 1, for any pure state decomposition $\{p_j, |\psi_j\rangle\}$ of X_d with $|\psi_j\rangle = \sum_i \psi_i^{(j)}|i\rangle$, we have

$$\begin{aligned}\textstyle\sum_j p_j C_{f_2}(|\psi_j\rangle\langle\psi_j|) &\geq C_{f_2}(X_d)\\ &= \textstyle\sum_{i=1}^{[d/2]} \tau_i C_{f_2}(\rho_i) + \frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1} C_{f_2}(|[d/2]+1\rangle)\\ &= \textstyle\sum_{i=1}^{[d/2]} \tau_i \min \sum_s q_{si} C_{f_2}(|\psi_{si}\rangle\langle\psi_{si}|)\\ &\geq \textstyle\sum_{i=1}^{[d/2]} \tau_i \frac{1-z_i}{2}.\end{aligned} \tag{6}$$

The first inequality is the convexity of the coherence measure, the second equality is the additivity of the coherence measure under the direct sum operation [23], the third equality is the definition of the coherence measure C_{f_2}, and the last inequality is due to the (5). Therefore

$$\begin{aligned}&\sum_j p_j |\psi_1^{(j)}|^{2\downarrow} \leq \sum_{i=1}^{[d/2]} \tau_i \frac{1+z_i}{2} + \frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1},\\ &\sum_j p_j |\psi_1^{(j)}|^{2\downarrow} + \sum_j p_j |\psi_2^{(j)}|^{2\downarrow} \leq 1.\end{aligned}$$

Hence $\sum_j p_j \mathcal{R}^{\downarrow}(|\psi_j\rangle) \prec \sum_{i=1}^{[d/2]} \sum_{k=1,2} \mu_{ki} \mathcal{R}^{\downarrow}(|\chi_{ki}\rangle) + \frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1}\mathcal{R}^{\downarrow}(|[d/2]+1\rangle)$. $\square$

We call the pure state decomposition $\mathfrak{D}$ in Theorem 1 the optimal as all other pure state decompositions are majoried by it. Let the vector $\mathbf{v} = \sum_{i=1}^{[d/2]} \sum_{k=1,2} \mu_{ki} \mathcal{R}^{\downarrow}(|\chi_{ki}\rangle) + \frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1}\mathcal{R}^{\downarrow}(|[d/2]+1\rangle)$. In fact, the vector $\mathbf{v}$ has only two nonzero entries

$$\mathbf{v} = (v_1, v_2, 0, \cdots, 0)^T \tag{7}$$

with $v_1 = \frac{1}{2} + \frac{1-(-1)^d}{4}\rho_{[d/2]+1,[d/2]+1} + \frac{1}{2}\sum_{i=1}^{[d/2]}\sqrt{(\rho_{ii}+\rho_{d-i+1,d-i+1})^2 - 4|\rho_{i,d-i+1}|^2}$, $v_2 = \sum_{i=1}^{[d/2]} \frac{\rho_{ii}+\rho_{d-i+1,d-i+1}}{2} - \frac{1}{2}\sum_{i=1}^{[d/2]}\sqrt{(\rho_{ii}+\rho_{d-i+1,d-i+1})^2 - 4|\rho_{i,d-i+1}|^2}$. Then all pure state decompositions of X_d states are majorized by the vector $\mathbf{v}$. By the vector $\mathbf{v}$ and the optimal pure state decomposition $\mathfrak{D}$ of X_d states, we derive the necessary and sufficient condition for the pure state converted to X_d states.

Theorem 2 *A pure state* $|\phi\rangle \xrightarrow{ICO} X_d$ *with* $|\phi\rangle = \sum_{i=1}^{d}\phi_i|i\rangle$ *if and only if*

$$|\phi_1|^{2\downarrow} \leq \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{[d/2]}\sqrt{(\rho_{ii}+\rho_{d-i+1,d-i+1})^2 - 4|\rho_{i,d-i+1}|^2} + \frac{1-(-1)^d}{4}\rho_{[d/2]+1,[d/2]+1}. \tag{8}$$

Proof On one hand, if any pure state $|\phi\rangle$ can be converted to X_d state, according to Lemma 2 and Theorem 1, there exists a pure state decomposition $\{p_j, |\psi_j\rangle\}$ of X_d state such that $C_{f_2}(|\phi\rangle\langle\phi|) \geq \sum_j p_j C_{f_2}(|\psi_j\rangle\langle\psi_j|) \geq \sum_{i=1}^{[d/2]} \tau_i \frac{1-z_i}{2}$. It implies $|\phi_1|^{2\downarrow} \leq \sum_{i=1}^{[d/2]} \tau_i \frac{1+z_i}{2} +$

$\frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1} = \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{[d/2]}\sqrt{(\rho_{ii}+\rho_{d-i+1,d-i+1})^2 - 4|\rho_{i,d-i+1}|^2} + \frac{1-(-1)^d}{4}\rho_{[d/2]+1,[d/2]+1}$.

On the other hand, if the pure state $|\phi\rangle$ satisfies (8), then $C_{f_2}(|\phi\rangle\langle\phi|) = 1 - |\phi_1|^{2\downarrow} \geq 1 - \left(\frac{1}{2} + \frac{1}{2}\sum_{i=1}^{[d/2]}\sqrt{(\rho_{ii}+\rho_{d-i+1,d-i+1})^2 - 4|\rho_{i,d-i+1}|^2} + \frac{1-(-1)^d}{4}\rho_{[d/2]+1,[d/2]+1}\right) = \sum_{i=1}^{[d/2]}\tau_i\frac{1-z_i}{2} = \sum_{ki}\mu_{ki}C_{f_2}(|\chi_{ki}\rangle\langle\chi_{ki}|)$. Furthermore $C_{f_l}(|\phi\rangle\langle\phi|) \geq \sum_{ki}\mu_{ki}C_{f_l}(|\chi_{ki}\rangle\langle\chi_{ki}|) + \frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1}C_{f_l}(|[d/2]+1\rangle\langle[d/2]+1|) = 0$ for $l = 3, 4, \cdots, d$. Therefore the optimal pure state decomposition $\mathfrak{D}$ of X_d satisfies the (2) in Lemma 2 for the pure state $|\phi\rangle$. So (8) is the necessary and sufficient condition for pure states to be converted to X_d states. □

Theorem 2 shows whether a pure state can be converted to X_d state is only decided by its largest magnitude under the reference basis. Let $R(\rho)$ be the set of pure state that can be converted into the given state ρ by incoherent operations. As to X_d state, the pure state $|\phi\rangle = \sum_i \phi_i|i\rangle$ is in the set of $R(X_d)$ if and only if (8) holds true. Denote

$$|\Xi\rangle = (\eta_1, \eta_2, 0, \cdots, 0)^T \tag{9}$$

with $\eta_1 = \sqrt{v_1}$, $\eta_2 = \sqrt{1 - v_1^2}$ with v_1 defined in (7). Then we can express the necessary and sufficient condition of pure states that can be converted to the X_d states by the majorization relation.

Corollary 1 *Any pure state* $|\phi\rangle \xrightarrow{ICO} X_d$ *if and only if* $\mathcal{R}^{\downarrow}(|\phi\rangle) \prec \mathcal{R}^{\downarrow}(|\Xi\rangle)$.

Now we consider the incoherent operations transforming pure states to X states. We accomplish this transformation by two steps. First we convert the pure state $|\phi\rangle$ to the intermediate pure state $|\Xi\rangle$ by an incoherent operation Φ_1. The Kraus operators of the incoherent operation Φ_1 depends on the input state $|\phi\rangle$ and output state $|\Xi\rangle$ which have been constructed in Ref. [11]. Then we convert the intermediate pure state $|\Xi\rangle$ to the given X_d state under an incoherent operation Φ_2 with Kraus operators

$$\begin{aligned}
K_{1i} &= \sqrt{\mu_{1i}}\left(\frac{\sqrt{\frac{1+z_i}{2}}}{\eta_1}|i\rangle\langle 1| + \frac{\sqrt{\frac{1-z_i}{2}}}{\eta_2}|d-i+1\rangle\langle 2|\right),\\
K_{2i} &= \sqrt{\mu_{2i}}\left(\frac{\sqrt{\frac{1-z_i}{2}}}{\eta_2}|i\rangle\langle 2| + \frac{\sqrt{\frac{1+z_i}{2}}}{\eta_1}|d-i+1\rangle\langle 1|\right),\\
K_3 &= \frac{1-(-1)^d}{2}\frac{\sqrt{\rho_{[d/2]+1,[d/2]+1}}}{\eta_1}|[d/2]+1\rangle\langle 1|,\\
K_0 &= \sum_{i=3}^{d}|i\rangle\langle i|
\end{aligned}$$

for $i = 1, 2, \cdots, [d/2]$. In this way any pure state fulfills the (8) can be converted to the X state by incoherent operators Φ_1 and Φ_2.

2.3 An Explicit Example in a Three Dimensional System

Now we illustrate the transformation from pure states to X states by an explicit three dimensional example.

Example 1 Given a three dimensional X state represented by the 3×3 Hermitian matrix $X_3 = \begin{pmatrix} \rho_{11} & 0 & \rho_{13} \\ 0 & \rho_{22} & 0 \\ \rho_{13}^* & 0 & \rho_{33} \end{pmatrix}$, by Theorem 2, the pure state $|\phi\rangle = \phi_1|1\rangle + \phi_2|2\rangle + \phi_3|3\rangle$ can be converted into the X_3 state if and only if $|\phi\rangle$ satisfies that

$$|\phi_1|^{2\downarrow} \leq \frac{1}{2}\left(1 + \rho_{22} + \sqrt{(\rho_{11} + \rho_{33})^2 - 4|\rho_{13}|^2}\right).$$

Now we construct the incoherent operations Φ_1 and Φ_2 such that $|\phi\rangle \xrightarrow{\Phi_1} |\Xi\rangle$ and $|\Xi\rangle \xrightarrow{\Phi_2} X_3$, where $|\Xi\rangle = \eta_1|1\rangle + \eta_2|2\rangle + \eta_3|3\rangle$, $\eta_1 = \sqrt{\frac{1}{2}(1 + \rho_{22} + \sqrt{(\rho_{11} + \rho_{33})^2 - 4|\rho_{13}|^2})}$, $\eta_2 = \sqrt{1 - \eta_1^2}$, $\eta_3 = 0$, respectively. Without loss of generality, we assume $|\phi_1| \geq |\phi_2| \geq |\phi_3|$.

(1) If $\phi_2 \neq 0$, $\phi_3 = 0$, let

$$A = \begin{pmatrix} a & 1-a & 0 \\ 1-a & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$(0 \leq a \leq 1)$ be a doubly stochastic matrix such that $(|\phi_1|^2, |\phi_2|^2, |\phi_3|^2)^T = A(\eta_1^2, \eta_2^2, \eta_3^2)^T$. Define

$$K_1 = \begin{pmatrix} \frac{\sqrt{a}\eta_1}{\phi_1} & 0 & 0 \\ 0 & \frac{\sqrt{a}\eta_2}{\phi_2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, K_2 = \begin{pmatrix} 0 & \frac{\sqrt{1-a}\eta_1}{\phi_2} & 0 \\ \frac{\sqrt{1-a}\eta_2}{\phi_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

It is easy to check that the incoherent operation Φ_1 with Kraus operators K_1 and K_2 converts the pure state $|\phi\rangle$ into the pure state $|\Xi\rangle$.

(2) If $\phi_3 \neq 0$, let A be a doubly stochastic matrix such that $(|\phi_1|^2, |\phi_2|^2, |\phi_3|^2)^T = A\left(\eta_1^2, \eta_2^2, \eta_3^2\right)^T$. Then A can be reduced to a T transform for some indices i, j. Denote

$$\begin{aligned} K_1 &= \sqrt{t}\mathrm{diag}\left(\frac{\eta_1}{\phi_1}, \frac{\eta_2}{\phi_2}, \frac{\eta_3}{\phi_3}\right), \\ K_2 &= \sqrt{1-t}\mathrm{diag}\left(\frac{\eta_1}{\phi_{\pi(1)}}, \frac{\eta_2}{\phi_{\pi(2)}}, \frac{\eta_3}{\phi_{\pi(3)}}\right) P_\pi, \end{aligned}$$

with π a permutation corresponding to T and P_π the matrix corresponding to π, $0 \leq t \leq 1$. According to the assumption $(|\phi_1|^2, |\phi_2|^2, |\phi_3|^2)^T = A\left(\eta_1^2, \eta_2^2, \eta_3^2\right)^T$, it follows $K_1^\dagger K_1 + K_2^\dagger K_2 = I$. Then one can prove the incoherent operation $\Phi_1(\cdot) = \sum_{j=1}^2 K_j(\cdot)K_j^\dagger$ transforms $|\phi\rangle$ to $|\Xi\rangle$.

Now we construct the incoherent operation Φ_2 converting the pure state $|\Xi\rangle$ to X_3 state. For Φ_2, we construct Kraus operators as follows,

$$\begin{aligned} K_1 &= \frac{\sqrt{\mu_{11}}\sqrt{\frac{1+z}{2}}}{\eta_1}|1\rangle\langle 1| + \frac{\sqrt{\mu_{11}}\sqrt{\frac{1-z}{2}}}{\eta_2}|3\rangle\langle 2|, \\ K_2 &= \frac{\sqrt{\mu_{21}}\sqrt{\frac{1-z}{2}}}{\eta_2}|1\rangle\langle 2| + \frac{\sqrt{\mu_{21}}\sqrt{\frac{1+z}{2}}}{\eta_1}|3\rangle\langle 1|, \\ K_3 &= \frac{\sqrt{\rho_{22}}}{\eta_1}|2\rangle\langle 1|, \\ K_4 &= |3\rangle\langle 3|, \end{aligned}$$

where $z = \sqrt{1-4|\frac{\rho_{13}}{\rho_{11}+\rho_{33}}|^2}$, $\mu_{11} = \lambda_{11}(\rho_{11}+\rho_{33})$, $\mu_{21} = \lambda_{21}(\rho_{11}+\rho_{33})$, $\lambda_{11} = \frac{1}{2} + \frac{\rho_{11}-\rho_{33}}{2\sqrt{(\rho_{11}+\rho_{33})^2-4|\rho_{13}|^2}}$. Then we can prove that the incoherent operation Φ_2 with Kraus operators $\{K_j\}_{j=1}^4$ converts the pure state $|\Xi\rangle$ to X_3 state.

3 Coherence Measure of *X* States

In the context of coherence manipulation, some coherence measures or monotone have been introduced [14, 15]. Now we calculate the coherence existed in the X states measured by these quantifiers.

Theorem 3 *The l-th order coherence of X_d state is*

$$C_{f_2}(X_d) = \frac{1}{2}\sum_{i=1}^{[d/2]} \rho_{ii} + \rho_{d-i+1,d-i+1} - \sqrt{(\rho_{ii}+\rho_{d-i+1,d-i+1})^2 - 4|\rho_{i,d-i+1}|^2}$$

and $C_{f_l}(X_d) = 0$ for $l \geq 3$.

Proof For the l-th order coherence, we show the pure state decomposition $\mathfrak{D}$ is the optimal pure state decomposition for X_d state. First, for $l \geq 3$, it has $C_{f_l}(|\chi_{ki}\rangle) = C_{f_l}(|[d/2]+1\rangle) = 0$. Therefore $C_{f_l}(X_d) = 0$ by the definition of l-th order coherence for $l \geq 3$.

Second, for $l = 2$, by (6) in the proof in Theorem 1, for any pure state decomposition $\{p_j, |\psi_j\rangle\}$ of given state X_d, we have

$$\sum_j p_j C_{f_2}(|\psi_j\rangle) \geq \sum_{i=1}^{[d/2]}\sum_{k=1}^{2} \mu_{ki} C_{f_2}(|\chi_{ki}\rangle) + \frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1} C_{f_2}(|[d/2]+1\rangle).$$

Therefore,

$$\begin{aligned} C_{f_2}(X_d) &= \min \textstyle\sum_j p_j C_{f_2}(|\psi_j\rangle) \\ &= \textstyle\sum_{i=1}^{[d/2]}\sum_{k=1}^{2} \mu_{ki} C_{f_2}(|\chi_{ki}\rangle) + \frac{1-(-1)^d}{2}\rho_{[d/2]+1,[d/2]+1} C_{f_2}(|[d/2]+1\rangle) \\ &= \frac{1}{2}\textstyle\sum_{i=1}^{[d/2]} \rho_{ii} + \rho_{d-i+1,d-i+1} - \sqrt{(\rho_{ii}+\rho_{d-i+1,d-i+1})^2 - 4|\rho_{i,d-i+1}|^2}. \end{aligned}$$

□

Another coherence monotone related to the coherence manipulation is introduced as $C_m(\rho) = inf_{|\phi\rangle\in R(\rho)} F(|\phi\rangle)$, where $F(|\phi\rangle)$ is a coherence measure satisfying $(A1)-(A3)$ and $(A5)$, $R(\rho)$ is the set of pure states that can be converted into ρ by incoherent operations [15]. By Corollary 1, one can prove the pure state $|\Xi\rangle$ defined in (9) is the optimal one for C_m of X_d thanks to the equivalence $\mathcal{R}^{\downarrow}(|\phi\rangle) \prec \mathcal{R}^{\downarrow}(|\Xi\rangle)$ if and only if $F(|\phi\rangle) \geq F(|\Xi\rangle)$. So $C_m(X_d) = F(|\Xi\rangle)$.

4 Conclusions

Overall, we have investigated the transformation from pure states to X states under incoherent operations. We derive an optimal pure state decomposition of X state such that all pure state decompositions of X state are majorized by it. Then the necessary and sufficient condition for the pure states to be converted to X state is demonstrated. We find the optimal pure state associated with X state such that any pure state which can be converted to X state should be majorized by it. The incoherent operations transforming pure states to X states are also analyzed. An explicit example is given in a three dimensional system. The coherence measure is also calculated for X states. We hope these results can promote the study of coherence manipulation.

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Declarations

Conflict of Interests The authors declare that they have no conflict of interest.

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