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Stronger Superadditivity Relations for Multiqubit Systems

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Abstract
Superadditivity relations characterize the distributions of coherence in multipartite quantum systems. In this work, we investigate the superadditivity relations related to the l_1-norm of coherence C_{l_1} in multiqubit quantum systems. Tighter superadditivity inequalities based on the α-th ($\alpha \geqslant 1$) power of l_1-norm of coherence are presented for multiqubit states under certain conditions, which include the existing results as special cases. These superadditivity relations give rise to finer characterization of the coherence distributions among the subsystems of a multipartite system. A detailed example is presented.

Keywords Superadditivity relation · l_1-norm of coherence · Multiqubit system

1 Introduction

The quantum nature of superposition, entanglement, and measurement are applicable to the quantum information industry. Entanglement is a very unique feature of the quantum sciences and plays a crucial role in quantum information processing. For example, in quantum computing [1, 2]. Stemming from the principle of quantum superposition, quantum coherence is another essential feature of quantum mechanics. It also plays an important role in quantum information processing [3] such as secret sharing [4], quantum secure direct communication [5], quantum key distribution [6], quantum teleportation [7, 8], quantum steering [9], quantum metrology [10], thermodynamics [11, 12], and quantum biology [13]. As a kind of physical resource, recently Baumgaratz et al. [14] proposed a rigorous framework to quantify the coherence. Two intuitive and easily computable measures of coherence are

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identified, the l_1-norm of coherence and the relative entropy of coherence. Following this seminal work, various operational measures of quantum coherence have been proposed [15–18]. Correspondingly, the dynamics of coherence [26], the distillation of coherence [27, 28] and the relations between quantum coherence and quantum correlations [29–33] have been extensively investigated.

The distribution of coherence in multipartite systems is one of the basic problems in the resource theory of coherence. An interesting subject in the theory of coherence is the superadditivity of coherence measures. A given coherence measure C is said to be superadditive if

$$C(\rho_{AB}) \geqslant C(\rho_A) + C(\rho_B), \tag{1}$$

for all bipartite density matrices ρ_{AB} of a finite-dimensional system with respect to a particular reference basis $\{|i\rangle_A \otimes |j\rangle_B\}$, where $\rho_A = \mathrm{tr}_B(\rho_{AB})$ and $\rho_B = \mathrm{tr}_A(\rho_{AB})$ are the reduced density matrices with respect to the basis $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$, respectively. However, not all coherence measures satisfy such superadditivity relations. The superadditivity for bipartite quantum states based on the relative entropy of coherence has been verified in [24]. Later, the superadditivity was generalized to the case of tripartite pure states [22]. A sufficient condition for the convex roof coherence measures to fulfill the superadditivity relations was provided in [23]. In [19], it has been shown that the l_1-norm of coherence C_{l_1} satisfies the superadditivity relations for all multiqubit states. Then the superadditivity of the l_1-norm of coherence C_{l_1} for multiqubit systems has been deeply studied [20, 21].

In this paper, we show that superadditivity inequalities related to the α-th ($\alpha \geqslant 1$) power of C_{l_1} for multiqubit systems can be further improved. A class of tighter superadditivity inequalities in multiqubit systems based on the α-th($\alpha \geqslant 1$) power of l_1-norm of coherence C_{l_1} are presented with detailed examples.

2 Stronger Superadditivity Relations

We first recall some basic facts related to the l_1-norm of coherence C_{l_1}. Let $\mathcal{H}$ denote a discrete finite-dimensional complex vector space associated with a quantum subsystem. For a quantum state $\rho \in \mathcal{H}$, the l_1-norm of coherence is given by the sum of the absolute values of the off-diagonal entries of the state ρ [14],

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|. \tag{2}$$

The superadditivity relations of the l_1-norm of coherence in multiqubit systems has been proved in [19].

$$C_{l_1}(\rho_{A_1A_2\cdots A_n}) \geqslant C_{l_1}(\rho_{A_1}) + C_{l_1}(\rho_{A_2}) + \cdots + C_{l_1}(\rho_{A_n}). \tag{3}$$

In [20], tighter superadditivity relations in multiqubit systems has been derived,

$$\begin{aligned} C_{l_1}^{\alpha}(\rho_{A_1A_2\cdots A_n}) \geqslant\ & C_{l_1}^{\alpha}(\rho_{A_1}) + (2^{\alpha}-1)C_{l_1}^{\alpha}(\rho_{A_2}) + \cdots + (2^{\alpha}-1)^{m-1}C_{l_1}^{\alpha}(\rho_{A_m}) \\ & +(2^{\alpha}-1)^{m+1}[C_{l_1}^{\alpha}(\rho_{A_{m+1}}) + \cdots + C_{l_1}^{\alpha}(\rho_{A_{n-1}})] \\ & +(2^{\alpha}-1)^{m}C_{l_1}^{\alpha}(\rho_{A_n}) \end{aligned} \tag{4}$$

for all $\alpha \geqslant 1$ and $n \geqslant 3$, if for some m $(1 \leqslant m \leqslant n-2)$ $C_{l_1}(\rho_{A_j}) \leqslant C_{l_1}(\rho_{A_{j+1}\cdots A_n})$ for $j = m+1, \cdots, n-1$. Later, the relation (4) has been improved in [21],

$$\begin{aligned} C_{l_1}^{\alpha}(\rho_{A_1A_2\cdots A_n}) \geqslant\ & C_{l_1}^{\alpha}(\rho_{A_1}) + \left(\tfrac{(1+k)^{\alpha}-1}{k^{\alpha}}\right) C_{l_1}^{\alpha}(\rho_{A_2}) + \cdots + \left(\tfrac{(1+k)^{\alpha}-1}{k^{\alpha}}\right)^{m-1} C_{l_1}^{\alpha}(\rho_{A_m}) \\ & + \left(\tfrac{(1+k)^{\alpha}-1}{k^{\alpha}}\right)^{m+1} [C_{l_1}^{\alpha}(\rho_{A_{m+1}}) + \cdots + C_{l_1}^{\alpha}(\rho_{A_{n-1}})] \\ & + \left(\tfrac{(1+k)^{\alpha}-1}{k^{\alpha}}\right)^{m} C_{l_1}^{\alpha}(\rho_{A_n}), \end{aligned} \tag{5}$$

for all $\alpha \geqslant 1$ and $n \geqslant 3$, conditioned that for a real number k $(0 < k \leqslant 1)$, $C_{l_1}(\rho_{A_i}) \geqslant \frac{1}{k} C_{l_1}(\rho_{A_{i+1}\cdots A_n})$ for $i = 1, 2, \cdots, m$, and $C_{l_1}(\rho_{A_j}) \leqslant \frac{1}{k} C_{l_1}(\rho_{A_{j+1}\cdots A_n})$ for $j = m+1, \cdots, n-1$, $1 \leqslant m \leqslant n-2$.

Improving the above results, we have the following theorems.

Theorem 1 *Let k and δ be real numbers satisfying $0 < k \leqslant 1$ and $\delta \geq 1$. For any n-qubit ($n \geqslant 3$) quantum state $\rho_{A_1A_2\cdots A_n}$ such that, without loss of generality, $C_{l_1}(\rho_{A_i}) \geqslant \frac{1}{k^{\delta}} C_{l_1}(\rho_{A_{i+1}\cdots A_n})$ for $i = 1, 2, \cdots, m$, and $C_{l_1}(\rho_{A_j}) \leqslant \frac{1}{k^{\delta}} C_{l_1}(\rho_{A_{j+1}\cdots A_n})$ for $j = m+1, \cdots, n-1$, $1 \leqslant m \leqslant n-2$, we have*

$$\begin{aligned} C_{l_1}^{\alpha}(\rho_{A_1A_2\cdots A_n}) \geqslant\ & C_{l_1}^{\alpha}(\rho_{A_1}) + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right) C_{l_1}^{\alpha}(\rho_{A_2}) + \cdots + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right)^{m-1} C_{l_1}^{\alpha}(\rho_{A_m}) \\ & + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right)^{m+1} [C_{l_1}^{\alpha}(\rho_{A_{m+1}}) + \cdots + C_{l_1}^{\alpha}(\rho_{A_{n-1}})] \\ & + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right)^{m} C_{l_1}^{\alpha}(\rho_{A_n}), \end{aligned} \tag{6}$$

for all $\alpha \geqslant 1$.

Proof Due to the superadditivity inequality $C_{l_1}(\rho_{AB}) \geqslant C_{l_1}(\rho_A) + C_{l_1}(\rho_B)$ for any $2 \otimes 2^{n-1}$ bipartite states ρ_{AB} [20], and the inequality [25],

$$(1+t)^{\alpha} \geqslant 1 + \frac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}} t^{\alpha},$$

where k and δ are any real numbers satisfying $0 < k \leqslant 1$ and $\delta \geq 1$, $0 \leqslant t \leqslant k^{\delta}$ and $\alpha \geqslant 1$, we have

$$\begin{aligned} C_{l_1}^{\alpha}(\rho_{A_1A_2\cdots A_n}) & \geqslant [C_{l_1}(\rho_{A_1}) + C_{l_1}(\rho_{A_2\cdots A_n})]^{\alpha} \\ & = C_{l_1}^{\alpha}(\rho_{A_1}) \left[1 + \tfrac{C_{l_1}(\rho_{A_2\cdots A_n})}{C_{l_1}(\rho_{A_1})}\right]^{\alpha} \\ & \geqslant C_{l_1}^{\alpha}(\rho_{A_1}) \left\{1 + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right) \left[\tfrac{C_{l_1}(\rho_{A_2\cdots A_n})}{C_{l_1}(\rho_{A_1})}\right]^{\alpha}\right\} \\ & = C_{l_1}^{\alpha}(\rho_{A_1}) + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right) C_{l_1}^{\alpha}(\rho_{A_2\cdots A_n}) \\ & \geqslant C_{l_1}^{\alpha}(\rho_{A_1}) + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right) C_{l_1}^{\alpha}(\rho_{A_2}) + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right)^{2} C_{l_1}^{\alpha}(\rho_{A_3\cdots A_n}) \\ & \geqslant \cdots \\ & \geqslant C_{l_1}^{\alpha}(\rho_{A_1}) + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right) C_{l_1}^{\alpha}(\rho_{A_2}) + \cdots + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right)^{m-1} C_{l_1}^{\alpha}(\rho_{A_m}) \\ & \quad + \left(\tfrac{(1+k^{\delta})^{\alpha}-1}{k^{\delta\alpha}}\right)^{m} C_{l_1}^{\alpha}(\rho_{A_{m+1}\cdots A_n}). \end{aligned} \tag{7}$$

Similarly, as $C_{l_1}(\rho_{A_j}) \leqslant \frac{1}{k^\delta} C_{l_1}(\rho_{A_{j+1}\cdots A_n})$ for $j = m+1, \cdots, n-1$, we get

$$\begin{aligned} C_{l_1}^\alpha(\rho_{A_{m+1}\cdots A_n}) &\geqslant \left(\tfrac{(1+k^\delta)^\alpha-1}{k^{\delta\alpha}}\right) C_{l_1}^\alpha(\rho_{A_{m+1}}) + C_{l_1}^\alpha(\rho_{A_{m+2}\cdots A_n}) \\ &\geqslant \left(\tfrac{(1+k^\delta)^\alpha-1}{k^{\delta\alpha}}\right) [C_{l_1}^\alpha(\rho_{A_{m+1}}) + \cdots + C_{l_1}^\alpha(\rho_{A_{n-1}})] + C_{l_1}^\alpha(\rho_{A_n}). \end{aligned} \tag{8}$$

Combining (7) and (8) we obtain (6). □

Remark 1 The Theorem 4 in [20] is the special case of $k = 1$ and $\delta = 1$ of our Theorem 1. Our Theorem 1 also includes the Theorem 1 given in [21] as a special case of $\delta = 1$.

In Theorem 1 we have generally assumed that $C_{l_1}(\rho_{A_i}) \geqslant \frac{1}{k^\delta} C_{l_1}(\rho_{A_{i+1}\cdots A_n})$ for $i = 1, 2, \cdots, m$, and $C_{l_1}(\rho_{A_j}) \leqslant \frac{1}{k^\delta} C_{l_1}(\rho_{A_{j+1}\cdots A_n})$ for $j = m+1, \cdots, n-1$, for some m satisfying $1 \leqslant m \leqslant n-2$. In particular, if all $C_{l_1}(\rho_{A_i}) \geqslant \frac{1}{k^\delta} C_{l_1}(\rho_{A_{i+1}\cdots A_n})$ for $i = 1, 2, \cdots, n-2$, i.e., $m = n-2$, we have the following conclusion:

Theorem 2 *If $C_{l_1}(\rho_{A_i}) \geqslant \frac{1}{k^\delta} C_{l_1}(\rho_{A_{i+1}\cdots A_n})$ for all $i = 1, 2, \cdots, n-2$, then*

$$\begin{aligned} C_{l_1}^\alpha(\rho_{A_1A_2\cdots A_n}) \geqslant\; & C_{l_1}^\alpha(\rho_{A_1}) + \left(\tfrac{(1+k^\delta)^\alpha-1}{k^{\delta\alpha}}\right) C_{l_1}^\alpha(\rho_{A_2}) + \cdots + \left(\tfrac{(1+k^\delta)^\alpha-1}{k^{\delta\alpha}}\right)^{n-2} C_{l_1}^\alpha(\rho_{A_{n-1}}) \\ & + \left(\tfrac{(1+k^\delta)^\alpha-1}{k^{\delta\alpha}}\right)^{m+1} [C_{l_1}^\alpha(\rho_{A_{m+1}}) + \cdots + C_{l_1}^\alpha(\rho_{A_{n-1}})] \\ & + \left(\tfrac{(1+k^\delta)^\alpha-1}{k^{\delta\alpha}}\right)^{n-1} C_{l_1}^\alpha(\rho_{A_n}). \end{aligned} \tag{9}$$

It is easily verified that our bound (9) is larger than the one from (5),

$$\begin{aligned} C_{l_1}^\alpha(\rho_{A_1}) + \tfrac{(1+k^\delta)^\alpha-1}{k^{\delta\alpha}} C_{l_1}^\alpha(\rho_{A_2}) + \left(\tfrac{(1+k^\delta)^\alpha-1}{k^{\delta\alpha}}\right)^2 C_{l_1}^\alpha(\rho_{A_3}) \geq\; & C_{l_1}^\alpha(\rho_{A_1}) + \tfrac{(1+k)^\alpha-1}{k^\alpha} C_{l_1}^\alpha(\rho_{A_2}) \\ & + \left(\tfrac{(1+k)^\alpha-1}{k^\alpha}\right)^2 C_{l_1}^\alpha(\rho_{A_3}) \end{aligned}$$

for $\alpha \geq 1$. As an example, let us consider the following three-qubit state,

$$|\Psi_{A_1A_2A_3}\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \otimes \frac{|0\rangle + 3|1\rangle}{\sqrt{10}}.$$

For $|\Psi_{A_1A_2A_3}\rangle$ we have $C_{l_1}(\rho_{A_1}) = 1$, $C_{l_1}(\rho_{A_2}) = 0$, $C_{l_1}(\rho_{A_3}) = \frac{3}{5}$, $C_{l_1}(\rho_{A_2A_3}) = \frac{3}{5}$. Hence, we can choose $\delta = 2$ and $k = \frac{4}{5}$. We have the lower bound of (9),

$$y_1 \equiv C_{l_1}^\alpha(\rho_{A_1}) + \frac{(1+k^\delta)^\alpha - 1}{k^{\delta\alpha}} C_{l_1}^\alpha(\rho_{A_2}) + \left(\frac{(1+k^\delta)^\alpha - 1}{k^{\delta\alpha}}\right)^2 C_{l_1}^\alpha(\rho_{A_3}) = 1 + [(\frac{41}{25})^\alpha - 1]^2 (\frac{375}{256})^\alpha.$$

While the lower bound of (5) is given by

$$y_2 \equiv C_{l_1}^\alpha(\rho_{A_1}) + \frac{(1+k)^\alpha - 1}{k^\alpha} C_{l_1}^\alpha(\rho_{A_2}) + \left(\frac{(1+k)^\alpha - 1}{k^\alpha}\right)^2 C_{l_1}^\alpha(\rho_{A_3}) = 1 + [(\frac{9}{5})^\alpha - 1]^2 (\frac{15}{16})^\alpha.$$

Figure 1 *shows that our result is indeed tighter than the one given in* [21].

Inequality (6) can be further generalized to the following theorem, with a similar proof to (6).

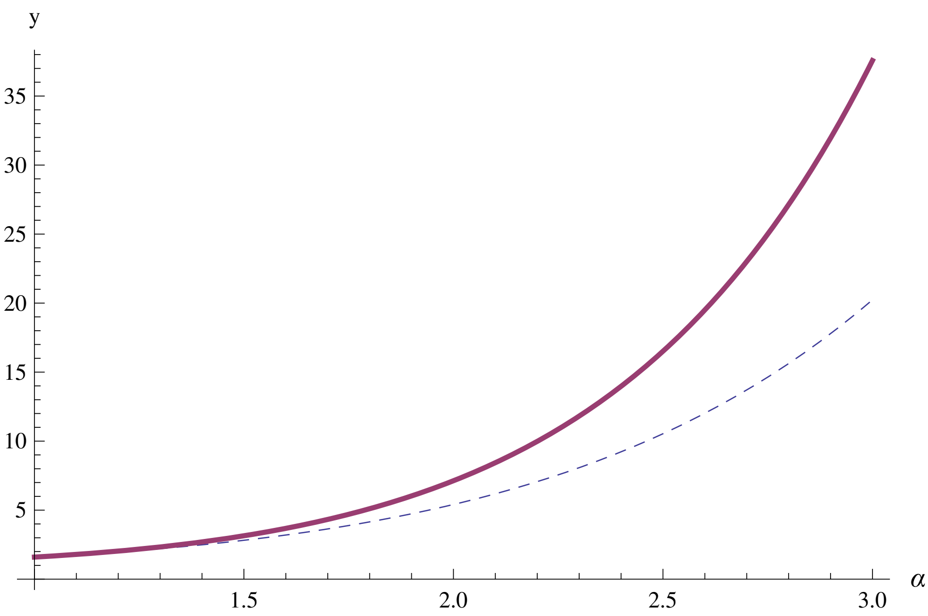

Fig. 1 The l_1-norm of coherence C_{l_1} with respect to α: the solid line is for y_1 and the dashed line for y_2 from the result in [21]

Theorem 3 *Let k, δ and β be real numbers with $0 < k \leqslant 1$ and $\delta, \beta \geqslant 1$. For any n-qubit quantum state such that $C_{l_1}^{\beta}(\rho_{A_i}) \geqslant \frac{1}{k^\delta} C_{l_1}^{\beta}(\rho_{A_{i+1}\cdots A_n})$ for $i = 1, 2, \cdots, m$, and $C_{l_1}^{\beta}(\rho_{A_j}) \leqslant \frac{1}{k^\delta} C_{l_1}^{\beta}(\rho_{A_{j+1}\cdots A_n})$ for $j = m+1, \cdots, n-1$, $1 \leqslant m \leqslant n-2$ and $n \geqslant 3$, we have*

$$\begin{aligned} C_{l_1}^{\alpha\beta}(\rho_{A_1A_2\cdots A_n}) \geqslant & C_{l_1}^{\alpha\beta}(\rho_{A_1}) + \left(\frac{(1+k^\delta)^\alpha - 1}{k^{\delta\alpha}}\right) C_{l_1}^{\alpha\beta}(\rho_{A_2}) + \cdots + \left(\frac{(1+k^\delta)^\alpha - 1}{k^{\delta\alpha}}\right)^{m-1} C_{l_1}^{\alpha\beta}(\rho_{A_m}) \\ & + \left(\frac{(1+k^\delta)^\alpha - 1}{k^{\delta\alpha}}\right)^{m+1} [C_{l_1}^{\alpha\beta}(\rho_{A_{m+1}}) + \cdots + C_{l_1}^{\alpha\beta}(\rho_{A_{n-1}})] \\ & + \left(\frac{(1+k^\delta)^\alpha - 1}{k^{\delta\alpha}}\right)^{m} C_{l_1}^{\alpha\beta}(\rho_{A_n}) \end{aligned} \tag{10}$$

for all $\alpha \geqslant 1$.

Remark 2 Theorem 3 reduces to Theorem 1 when $\beta = 1$. In particular, when $m = n - 2$ Theorem 3 gives rise to a simpler stronger superadditivity relation:

Theorem 4 *If $C_{l_1}^{\beta}(\rho_{A_i}) \geqslant \frac{1}{k^\delta} C_{l_1}^{\beta}(\rho_{A_{i+1}\cdots A_n})$ for all $i = 1, 2, \cdots, n-2$, we have*

$$\begin{aligned} C_{l_1}^{\alpha\beta}(\rho_{A_1A_2\cdots A_n}) \geqslant & C_{l_1}^{\alpha\beta}(\rho_{A_1}) + \left(\frac{(1+k^\delta)^\alpha - 1}{k^{\delta\alpha}}\right) C_{l_1}^{\alpha\beta}(\rho_{A_2}) + \cdots + \left(\frac{(1+k^\delta)^\alpha - 1}{k^{\delta\alpha}}\right)^{n-2} C_{l_1}^{\alpha\beta}(\rho_{A_{n-1}}) \\ & + \left(\frac{(1+k^\delta)^\alpha - 1}{k^{\delta\alpha}}\right)^{n-1} C_{l_1}^{\alpha\beta}(\rho_{A_n}) \end{aligned} \tag{11}$$

for all $\alpha \geqslant 1$.

Note that not all coherence measures satisfy a superadditivity relation like the inequality (1) for all quantum states. The method used in Theorem 4 can be applied to derive tighter superadditivity inequalities for the α-th ($\alpha \geqslant 1$) power of coherence measures satisfying the superadditivity relation.

3 Conclusion

Superadditivity relation is a fundamental property with respect to multipartite quantum systems. In this paper, we have focused on the distributions of quantum coherence characterized by the superadditivity relations. We have proposed a class of tighter superadditivity inequalities related to the α-th ($\alpha \geqslant 1$) power of the l_1-norm coherence C_{l_1} for multiqubit systems. These new inequalities give rise to finer characterizations for the coherence distribution. Our results provide better understanding of multipartite coherence and may highlight related researches on other quantum coherence measures.

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