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Some Measurement-Based Characterizations of Separability of Bipartite States

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Abstract

Importance of quantum entanglement has been demonstrated in various applications. Usually, separability of a bipartite state is defined by its algebraic structure, i.e. a convex combination of product states. But it seems to be hard to check separability (equivalently, entanglement) of a state from its algebraic structure. In this note, we give some characterizations of separability of bipartite states based on POVM measurements. For bipartite pure states, we prove the separability, Bell locality, unsteerability and classical correlation are the same. As a consequence, every entangled pure bipartite state is always Bell nonlocal, steerable and quantum correlated.

Keywords POVM measurement · Separability · Bell locality · Unsteerability · Classical correlation

1 Introduction

Quantum entanglement, as the essence of quantum formalism, was recognized by Einstein, Podolsky, Rosen [1], and Schrödinger [2] in 1935. This holistic property of compound quantum systems involves nonclassical correlations between subsystems and then has potential for many quantum processes, including canonical ones: quantum cryptography, quantum teleportation, and dense coding.

As a special entanglement, Bell nonlocality of a compound quantum system was recognized by Bell [3] in 1964, who accepted the EPR conclusion that the quantum description of physical reality is not complete as a working hypothesis and formalized the EPR's idea of deterministic world in terms of the local hidden variable model [LHVM]. He then showed that the probabilities for the outcomes obtained when some entangled state is suitably measured violate an inequality, which was named the Bell inequality. And this property of quantum state found by Bell is the so-called Bell nonlocality. It is, indeed, demonstrated by some local quantum measurements whose statistics of the measurement outcomes cannot be explained by an LHVM [4]. Bell nonlocality is an important resources in quantum

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information and then has been widely discussed, please refer to Clauser and Shimony [5], Home and Selleri [6], Khalfin and Tsirelson [7], Tsirelson [8], Zeilinger [9], Werner and Wolf [10], Genovese [11], and Buhrman et al. [12], and [13–19].

As an intermediary property between Bell nonlocality and entanglement, EPR steering was first observed by Schrödinger [2] in the context of the well-known EPR paradox [1, 20–22]. It is also is an important resources in quantum information and then has been recently discussed, please refer to [17, 23–31]. Especially, mathematical definitions of Bell nonlocality and EPR steerability of bipartite states were proposed and their characterizations were given in [17] by Cao and Guo.

Although entanglement was first recognized as the characteristic trait of quantum mechanics [1], its role has been debated since it does not capture all the quantum features of a quantum system [32–34]. In the case of bipartite systems, the quantum discord (generally, quantum correlation) has been widely accepted as a fundamental tool due to its relevance in quantum computing tasks not relying on entanglement [32–35]. Luo in [36] established the mathematical definition of classical correlation and quantum correlation of a bipartite state by using measurement-induced disturbance to the state. Guo and Cao [37] established a new characterization of a classical correlated (CC) state and proved that the set of all CC states becomes a perfect, nowhere dense and compact subset of the metric space of all states. Please refer to [38–46] for more researches on quantum correlations.

As usual, separability of a bipartite state is defined by its algebraic structure, i.e. a convex combination of product states. States that are not separable are said to entangled states. But it seems to be hard to check the entanglement (equivalently, the separability) of a state from its algebraic structure. In this note, we give some measurement-based characterizations of separability of bipartite states. In Section 2, measurement-based separability will be discussed; In Sections 3–5, it will be proved that the separability, Bell locality, unsteerability and classical correlation are the same for bipartite pure states.

2 Measurement-Based Separability

According to quantum mechanics, a quantum system S is described by a d_S-dimensional complex Hilbert space $\mathcal{H}_S$ (called the *state space* of S) with a right-linear inner product $\langle\cdot|\cdot\rangle$ and the states of the system are denoted by positive operators of trace 1 on $\mathcal{H}_S$. The set of all states of S is denoted by $D(\mathcal{H}_S)$. Thus,

$$D(\mathcal{H}_S) = \{\rho \in B(\mathcal{H}_S) : \rho \geq 0, \mathrm{tr}\rho = 1\},$$

where $B(\mathcal{H}_S)$ is the C^*-algebra of all bounded linear operators on $\mathcal{H}_S$. The elements of $D(\mathcal{H}_S)$ are called the *mixed states* of S. A unit vector $|\psi\rangle$ in $\mathcal{H}_S$ is said to be a *pure state* of S and the set of all pure states of S is denoted by $PS(\mathcal{H}_S)$. We also use $[d]$ to denote the set $\{1, 2, \ldots, d\}$.

By the postulates of quantum mechanics, the state space of the composite system AB of A and B is given by the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ of the state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of A and B, respectively.

Recall that a state $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be *separable* if it can be written as

$$\rho = \sum_{i=1}^{k} c_i \rho_i^A \otimes \rho_i^B \tag{2.1}$$

for some states $\rho_i^A \in D(\mathcal{H}_A), \rho_i^B \in D(\mathcal{H}_B)$ and $c_i \geq 0 (i \in [k])$ with $\sum_{i=1}^k c_i = 1$. Otherwise, it is said to be *entangled*. Moreover, a pure state $|\psi\rangle \in PS(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be *separable* if it can be written as the form $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ for some $|\psi_A\rangle \in PS(\mathcal{H}_A)$ and $|\psi_B\rangle \in PS(\mathcal{H}_B)$. Otherwise, it is called to be *entangled*.

Next propositions are remarks on the definition of separability and may be well-known.

Proposition 2.1 *For a state $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$, the following statements are equivalent.*

(1) *ρ is separable, i.e. it has the form of (2.1).*

(2) *ρ can be written as*

$$\rho = \sum_{i,j=1}^{k} d_{ij} \rho_i^A \otimes \eta_j^B \tag{2.2}$$

for some states $\rho_i^A \in D(\mathcal{H}_A), \eta_j^B \in D(\mathcal{H}_B)$ and $d_{ij} \geq 0 (i, j \in [k])$ with $\sum_{i,j=1}^k d_{ij} = 1$.

(3) *ρ can be written as*

$$\rho = \sum_{s=1}^{d_1} \sum_{t=1}^{d_2} \gamma_{st} |\psi_s^A\rangle\langle\psi_s^A| \otimes |\psi_t^B\rangle\langle\psi_t^B| \tag{2.3}$$

for some pure states $|\psi_s^A\rangle$ and $|\psi_t^B\rangle$ of the systems A and B, respectively, and $\gamma_{st} \geq 0$ with

$$\sum_{s=1}^{d_1} \sum_{t=1}^{d_2} \gamma_{st} = 1.$$

(4) *ρ can be written as*

$$\rho = \sum_{i=1}^{n} c_i |\psi_i^A\rangle\langle\psi_i^A| \otimes \rho_i^B \tag{2.4}$$

for some pure states $|\psi_i^A\rangle$ and mixed states ρ_i^B of the systems A and B, respectively, and $c_i \geq 0$ with $\sum_{i=1}^n c_i = 1$.

(5) *ρ can be written as*

$$\rho = \sum_{i=1}^{m} q_i \rho_i^A \otimes |\psi_i^B\rangle\langle\psi_i^B| \tag{2.5}$$

for some mixed states ρ_i^A and pure states $|\psi_i^B\rangle$ of the systems A and B, respectively, and $q_i \geq 0$ with $\sum_{i=1}^m q_i = 1$.

Proposition 2.2 *A pure state $|\psi\rangle$ of the system AB is separable if and only if $|\psi\rangle\langle\psi|$ is separable.*

Theorem 2.1 *A state ρ^{AB} is separable if and only if there exists a probability distribution(PD) $\{\pi_k\}_{k=1}^d$, states $\{\rho_k^A\}_{k=1}^d \subset D(\mathcal{H}_A)$ and $\{\rho_k^B\}_{k=1}^d \subset D(\mathcal{H}_B)$ s.t. for every local POVM $\{M_i \otimes N_j\}_{i,j}$, it holds that*

$$tr[(M_i \otimes N_j)\rho^{AB}] = \sum_{k=1}^{d} \pi_k tr(M_i \rho_k^A) tr(N_j \rho_k^B), \quad \forall i, j. \tag{2.6}$$

In that case, $\rho^{AB} = \sum_{\lambda=1}^d \pi_\lambda \rho^{A_k} \otimes \rho^{B_k}$.

Proof *Necessity*. Clearly.

Sufficiency. Suppose that "the sufficient condition" is satisfied, that is, the desired $\{\pi_k\}_{k=1}^d$, $\{\rho_k^A\}_{k=1}^d$ and $\{\rho_k^B\}_{k=1}^d$ exist. Put $\sigma = \rho^{AB} - \sum_{k=1}^d \pi_k \rho_k^A \otimes \rho_k^B$. For all $0 \le X \le I_A$ and $0 \le Y \le I_B$, put $M_1 = X$, $M_2 = I_A - X$ and $N_1 = Y$, $N_2 = I_B - Y$, then we obtain POVMs $\{M_1, M_2\}$ and $\{N_1, N_2\}$ of A and B, respectively. We see from (2.6) that

$$\mathrm{tr}[(M_i \otimes N_j)\rho^{AB}] = \sum_{k=1}^d \pi_k \mathrm{tr}(M_i \rho_k^A)\mathrm{tr}(N_j \rho_k^B) = \mathrm{tr}\left[(M_i \otimes N_j)\sum_{k=1}^d \pi_k \rho_k^A \otimes \rho_k^B\right].$$

Thus, $\mathrm{tr}[(M_i \otimes N_j)\sigma] = 0$ for $i, j = 1, 2$. When $i = j = 1$, we have $\mathrm{tr}[(X \otimes Y)\sigma] = 0$. Hence, $\mathrm{tr}[(X \otimes Y)\sigma] = 0$ for all $0 \le X \le I_A$ and $0 \le Y \le I_B$. Thus, $\mathrm{tr}[(X \otimes Y)\sigma] = 0$ for all hermitian operators X on $\mathcal{H}_A$ and Y on $\mathcal{H}_B$. Hence, $\mathrm{tr}[(X \otimes Y)\sigma] = 0$ for all $X \in B(\mathcal{H}_A)$ and $Y \in B(\mathcal{H}_B)$. Now, every operator $T \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ has a decomposition $T = \sum_n X_n \otimes Y_n$ and so $\mathrm{tr}(T\sigma) = \sum_n \mathrm{tr}[(X_n \otimes Y_n)\sigma] = 0$. By using this fact for $T = \sigma^\dagger$, we conclude that $\sigma = 0$, i.e., $\rho^{AB} = \sum_{k=1}^d \pi_k \rho_k^A \otimes \rho_k^B$. Therefore, ρ^{AB} is separable. □

Corollary 2.1 *A state ρ^{AB} is separable if and only if there exists a PD $\{\pi_k\}_{k=1}^d$, states $\{\rho_k^A\}_{k=1}^d \subset D(\mathcal{H}_A)$ and $\{\rho_k^B\}_{k=1}^d \subset D(\mathcal{H}_B)$ s.t. for all local observables $X \otimes Y$ of AB, it holds that*

$$\langle X \otimes Y\rangle_{\rho^{AB}} = \sum_{k=1}^d \pi_k \langle X\rangle_{\rho^{A_k}} \cdot \langle Y\rangle_{\rho^{B_k}}. \tag{2.7}$$

In that case, $\rho^{AB} = \sum_{\lambda=1}^d \pi_\lambda \rho^{A_k} \otimes \rho^{B_k}$.

Proof The necessity is clear. To prove the sufficiency, we suppose that "the sufficient condition" is satisfied, that is, the desired $\{\pi_k\}_{k=1}^d$, $\{\rho_k^A\}_{k=1}^d$ and $\{\rho_k^B\}_{k=1}^d$ exist. For every POVM $\{M_i \otimes N_j\}_{i,j}$, using (2.7) for $X = M_i$ and $Y = N_j$, we get

$$\mathrm{tr}((M_i \otimes N_j)\rho^{AB}) = \sum_{k=1}^d \pi_k \mathrm{tr}(M_i \rho_k^A)\mathrm{tr}(N_j \rho_k^B),$$

which is just (2.6). So, Theorem 2.1 yields that ρ^{AB} is separable. □

As an application of Corollary 2.1, we have $\rho^{AB} = \rho^A \otimes \rho^B$ if and only if for all local observables $X \otimes Y$ of AB, it holds that $\langle X \otimes Y\rangle_{\rho^{AB}} = \langle X\rangle_{\rho^A} \cdot \langle Y\rangle_{\rho^B}$.

Corollary 2.2 *ρ^{AB} is separable if and only if there exists a PD $\{\pi_k\}_{k=1}^d$, $\{\rho_k^A\}_{k=1}^d \subset D(\mathcal{H}_A)$ and $\{\rho_k^B\}_{k=1}^d \subset D(\mathcal{H}_B)$ s.t. for every pure state $|\psi\rangle \in \mathcal{H}_A$ and $|\varphi\rangle \in \mathcal{H}_B$, it holds that*

$$tr((|\psi\rangle\langle\psi| \otimes |\varphi\rangle\langle\varphi|)\rho^{AB}) = \sum_{k=1}^d \pi_k \langle\psi|\rho_k^A|\psi\rangle \cdot \langle\varphi|\rho_k^B|\varphi\rangle. \tag{2.8}$$

That is,

$$\langle|\Psi\rangle\langle\Psi|\rangle_{\rho^{AB}} = \sum_{k=1}^d \pi_k \langle|\psi\rangle\langle\psi|\rangle_{\rho^{A_k}} \cdot \langle|\varphi\rangle\langle\varphi|\rangle_{\rho^{B_k}},$$

where $|\Psi\rangle = |\psi\rangle|\varphi\rangle$ and $\langle\sigma\rangle_x = \langle x|\sigma|x\rangle$ denotes the expectation of σ at a pure state $|x\rangle$.

Proof The necessity is clear. To prove the sufficiency, we suppose that "the sufficient condition" is satisfied, that is, the desired $\{\pi_k\}_{k=1}^d$, $\{\rho_k^A\}_{k=1}^d$ and $\{\rho_k^B\}_{k=1}^d$ exist. For every $0 \leq X \leq I_A$ and $0 \leq Y \leq I_B$, we consider the spectrum decompositions of X and Y:

$$X = \sum_{i=1}^{d_A} c_i |\psi_i\rangle\langle\psi_i|, \quad Y = \sum_{j=1}^{d_B} d_j |\varphi_j\rangle\langle\varphi_j|,$$

where $\{|\psi_i\rangle\}_{i=1}^{d_A}$ and $\{|\varphi_j\rangle\}_{j=1}^{d_B}$ are orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and $c_i \geq 0, d_j \geq 0$. For each i, j, we see from (2.8) that

$$\begin{aligned}\mathrm{tr}[(|\psi_i\rangle\langle\psi_i| \otimes |\varphi_j\rangle\langle\varphi_j|)\rho^{AB}] &= \sum_{k=1}^{d} \pi_k \langle\psi_i|\rho_k^A|\psi_i\rangle \cdot \langle\varphi_j|\rho^B|\varphi_j\rangle \\ &= \sum_{k=1}^{d} \pi_k \mathrm{tr}(|\psi_i\rangle\langle\psi_i|\rho_k^A)\mathrm{tr}(|\varphi_j\rangle\langle\varphi_j|\rho^B).\end{aligned}$$

Multiplying two sides of above equation by $c_i d_j$ and then finding the sums over i, j, we obtain that

$$\begin{aligned}\mathrm{tr}[(X \otimes Y)\rho^{AB}] &= \sum_{k=1}^{d}\sum_{i=1}^{d_A} \pi_k c_i \mathrm{tr}[(|\psi_i\rangle\langle\psi_i|)\rho_k^A] \sum_{j=1}^{d_B} d_j \mathrm{tr}[(|\varphi_j\rangle\langle\varphi_j|)\rho^B] \\ &= \sum_{k=1}^{d} \pi_k \mathrm{tr}(X\rho_k^A)\mathrm{tr}(Y\rho_k^B).\end{aligned}$$

Thus, (2.7) is valid. It follows from Corollary 2.1 that ρ^{AB} is separable. □

The following conclusion is a direct application of Corollary 2.2, which was pointed out in [15](pp. 140402-2, Eq. (4)) without proof.

Corollary 2.3 *A state ρ^{AB} is separable if and only if there exists a PD $\{\pi_k\}_{k=1}^d$, states $\{\rho_k^A\}_{k=1}^d \subset D(\mathcal{H}_A)$ and $\{\rho_k^B\}_{k=1}^d \subset D(\mathcal{H}_B)$ s.t. for every local von Neumann measurement (projective measurement) $\{P_i \otimes Q_j\}_{i,j}$, it holds that*

$$tr[(P_i \otimes Q_j)\rho^{AB}] = \sum_{k=1}^{d} \pi_k tr(P_i\rho_k^A)tr(Q_j\rho_k^B), \quad \forall i, j. \tag{2.9}$$

Next, we prove a simple characterization of separability of a pure state.

Theorem 2.2 *A pure state $|\psi\rangle$ of AB is separable if and only if for all projective measurements $M = \{M_a\}_{a=1}^m$ of A and $N = \{N_b\}_{b=1}^n$ of B, it holds that*

$$tr[(M_a \otimes N_b)(|\psi\rangle\langle\psi|)] = P_A(a)P_B(b), \quad \forall a \in [m], b \in [n], \tag{2.10}$$

where $\{P_A(a)\}_{a=1}^m$ and $\{P_B(b)\}_{b=1}^n$ are PDs.

Proof The necessity is clear. To prove the sufficiency, we assume that the sufficient condition is satisfied. Let $|\psi\rangle = \sum_{k=1}^r c_k|\psi_k^A\rangle|\psi_k^B\rangle$ be the Schmidt decomposition of $|\psi\rangle$ where

$c_k > 0(k \in [r])$, $\{|\psi_k^A\rangle\}_{k=1}^r$ and $\{|\psi_k^B\rangle\}_{k=1}^r$ are orthonormal sets in H_A and H_B, respectively. Then

$$|\psi\rangle\langle\psi| = \sum_{k,j=1}^{r} c_k c_j |\psi_k^A\rangle|\psi_k^B\rangle\langle\psi_j^A|\langle\psi_j^B| = \sum_{k,j=1}^{r} c_k c_j |\psi_k^A\rangle\langle\psi_j^A| \otimes |\psi_k^B\rangle\langle\psi_j^B|. \quad (2.11)$$

Extending $\{|\psi_k^A\rangle\}_{k=1}^r$ and $\{|\psi_k^B\rangle\}_{k=1}^r$ as orthonormal bases $\{|\psi_k^A\rangle\}_{k=1}^{d_A}$ and $\{|\psi_k^B\rangle\}_{k=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, we obtain projective measurements $M = \{M_a\}_{a=1}^{d_A}$ of A and $N = \{N_b\}_{b=1}^{d_B}$ of B where $M_a = |\psi_a^A\rangle\langle\psi_a^A|$ and $N_b = |\psi_b^B\rangle\langle\psi_b^B|$. By (2.10) and (2.11), we get

$$P_A(a)P_B(b) = \mathrm{tr}[(M_a \otimes N_b)(|\psi\rangle\langle\psi|)] = c_a c_b \delta_{a,b} (a, b \in [r])$$

while $P_A(a)P_B(b) = 0$ when $a > r$ or $b > r$. Thus,

$$\begin{pmatrix} c_1^2 & 0 & \cdots & 0 \\ 0 & c_2^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & c_r^2 \end{pmatrix} = \begin{pmatrix} P_A(1) \\ P_A(2) \\ \vdots \\ P_A(r) \end{pmatrix} (P_B(1), P_B(2), \cdots, P_B(r)).$$

Since the matrix on the right-hand side is of rank 1, we conclude that $r = 1$ and so $|\psi\rangle = |\psi_1^A\rangle|\psi_1^B\rangle$. □

Corollary 2.4 *A pure state $|\psi\rangle$ of AB is entangled if and only if there exist two projective measurements $M = \{M_a\}_{a=1}^m$ of A and $N = \{N_b\}_{b=1}^n$ of B such that the joint PD $\{tr[(M_a \otimes N_b)(|\psi\rangle\langle\psi|)]\}_{a,b}$ of the output results (a, b) can not be factorized a s a product of the PDs of the output results a and b.*

For example, when $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we take

$$M_a = |a\rangle\langle a|(a = 0, 1), N_b = |b\rangle\langle b|(b = 0, 1)$$

and put $P_{ab} = \mathrm{tr}[(M_a \otimes N_b)(|\psi\rangle\langle\psi|)]$. Then

$$[P_{ab}] = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \neq \begin{pmatrix} P_A(0) \\ P_A(1) \end{pmatrix} (P_B(0), P_B(1))$$

for any PDs $\{P_A(0), P_A(1)\}$ and $\{P_B(0), P_B(1)\}$ since the left-hand side has rank 2 while the right-hand side has rank 1. It follows from Corollary 2.4 that $|\psi\rangle$ is entangled.

3 Separability and Bell Locality

Usually, Bell nonlocality was revealed by the violations of various Bell's inequalities and Bell nonlocal states must be entangled. However, entangled states are not necessarily Bell nonlocal [10]. Gisin [48, 49] proved that all entangled pure states are Bell nonlocal, which was referred to as Gisin's theorem and have been generalized to multipartite systems [50–55].

To recall the definition of Bell locality, we use x and y to denote the labels of POVMs of Alice and Bob and use a and b to denote their measurement outcomes, respectively. Thus, their POVM choices are denoted by $M^x = \{M_{a|x}\}_{a=1}^{o_A}$ and $N^y = \{N_{b|y}\}_{b=1}^{o_B}$, respectively, where $x \in [m_A]$, $y \in [m_B]$. These POVMs form *measurement assemblages* of A and B: $\mathcal{M}_A = \{M^x\}_{x=1}^{m_A}$ and $\mathcal{N}_B = \{N^y\}_{y=1}^{m_B}$, respectively.

Definition 3.1 [17, Definition 2.1]

(1) A state ρ^{AB} is said to be *Bell local* for a given measurement scenario $\mathcal{M}_A \otimes \mathcal{N}_B$ if there exists a PD $\{\pi_\lambda\}_{\lambda=1}^d$ such that

$$\mathrm{tr}[(M_{a|x} \otimes N_{b|y})\rho^{AB}] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x, \lambda) P_B(b|y, \lambda), \ \ \forall a, b, x, y, \tag{3.1}$$

where $\{P_A(a|x, \lambda)\}_{a=1}^{o_A}$ and $\{P_B(b|y, \lambda)\}_{b=1}^{o_B}$ are PDs for each (λ, x) and (λ, y), respectively.

Equation (3.1) is said to be a *local hidden variable (LHV) model* of ρ^{AB} w. r. t $\mathcal{M}_A \otimes \mathcal{N}_B$ and λ is said to be an LHV with PD $\{\pi_\lambda\}_{\lambda=1}^d$.

(2) A state ρ^{AB} is said to be *Bell nonlocal* for $\mathcal{M}_A \otimes \mathcal{N}_B$ if it is not Bell local for $\mathcal{M}_A \otimes \mathcal{N}_B$.

(3) A state ρ^{AB} is said to be *Bell local* if it is Bell local for every $\mathcal{M}_A \otimes \mathcal{N}_B$.

(4) A state ρ^{AB} is said to be *Bell nonlocal* if it is not Bell local, i.e., if there exists an $\mathcal{M}_A \otimes \mathcal{N}_B$ such that ρ^{AB} is not Bell local for $\mathcal{M}_A \otimes \mathcal{N}_B$.

Let $\mathcal{BL}(\mathcal{M}_A, \mathcal{N}_B)$ denote the set of all states that are Bell local for $\mathcal{M}_A \otimes \mathcal{N}_B$, $\mathcal{BNL}(\mathcal{M}_A, \mathcal{N}_B)$ denote the set of all states that are Bell nonlocal for $\mathcal{M}_A \otimes \mathcal{N}_B$, $\mathcal{BL}(AB)$ the set of all Bell local states of AB; $\mathcal{BNL}(AB)$ denote the set of all states that are Bell nonlocal. Thus, we see from the definition that

$$\mathcal{BL}(AB) = \bigcap_{\mathcal{M}_A, \mathcal{N}_B} \mathcal{BL}(\mathcal{M}_A, \mathcal{N}_B), \ \mathcal{BNL}(AB) = \bigcup_{\mathcal{M}_A, \mathcal{N}_B} \mathcal{BNL}(\mathcal{M}_A, \mathcal{N}_B).$$

A pure state $|\psi\rangle$ of AB is said to be Bell local (resp. Bell nonlocal) if $|\psi\rangle\langle\psi|$ is Bell local (resp. Bell nonlocal).

A following characterization of Bell locality was proved in [17], in which $\Omega_A = \{J_1, J_2, \ldots, J_{N_A}\}$ with $N_A = o_A^{m_A}$ denotes the set of all possible maps from $[m_A]$ into $[o_A]$, and $\Omega_B = \{K_1, K_2, \ldots, K_{N_B}\}$ with $N_B = o_B^{m_B}$ is the set of all possible maps from $[m_B]$ into $[o_B]$.

Lemma 3.1 [17, Theorem 2.1]. *A state ρ^{AB} is Bell local for $\mathcal{M}_A \otimes \mathcal{N}_B$ if and only if there exists a PD $\{q_{k,j} : 1 \le k \le N_A, 1 \le j \le N_B\}$ satisfying*

$$tr[(M_{a|x} \otimes N_{b|y})\rho^{AB}] = \sum_{k=1}^{N_A} \sum_{j=1}^{N_B} q_{k,j} \delta_{a, J_k(x)} \delta_{b, K_j(y)}, \tag{3.2}$$

where $\{q_{k,j}\}_{(k,j)\in[N_A]\times[N_B]}$ is a PD.

The sufficiency is clearly valid by Definition 3.1 and the necessity was proved in [17] by using the Total Probability Formula. Indeed, when (3.1) holds, the matrix $[P_A(a|x, \lambda)]$ with (x, a)-entry is row-stochastic and it is follows from [47] that $[P_A(a|x, \lambda)]$ can be written as a convex combination of $\{0, 1\}$ row-stochastic matrices $[\delta_{a, J_k(x)}]$:

$$P_A(a|x, \lambda) = \sum_{k=1}^{N_A} p_A(k, \lambda) \delta_{a, J_k(x)}, \ \forall x, a, \tag{3.3}$$

where $\{p_A(k,\lambda)\}_{k=1}^{N_A}$ is a PD. Similarly,

$$P_B(b|y,\lambda) = \sum_{j=1}^{N_B} p_B(j,\lambda)\delta_{b,K_j(y)}, \ \forall y, b, \tag{3.4}$$

where $\{p_B(j,\lambda)\}_{j=1}^{N_B}$ is a PD. Using (3.1)–(3.4) shows that (3.2) holds for

$$q_{k,j} = \sum_{\lambda=1}^{d} \pi_\lambda p_A(k,\lambda) p_B(j,\lambda).$$

With the characterization (3.2), it was proved in [17] that $\mathcal{BL}(AB)$ is a compact convex set and then $\mathcal{BNL}(AB)$ is an open set. The following Bell local inequality was proved essentially in [18] by using (3.1). Indeed, it can be obtained easily in light of (3.2).

Lemma 3.2 [18, Theorem 3.2] *If $\rho \in \mathcal{BL}(AB)$, C, D and P, Q are ± 1-valued observables of A and B, resp., then*

$$|\langle C\otimes P\rangle_\rho + \langle C\otimes Q\rangle_\rho + \langle D\otimes P\rangle_\rho - \langle D\otimes Q\rangle_\rho| \le 2, \tag{3.5}$$

$$\langle C\otimes P\rangle_\rho + \langle C\otimes Q\rangle_\rho + |\langle D\otimes P\rangle_\rho - \langle D\otimes Q\rangle_\rho| \le 2. \tag{3.6}$$

Next theorem shows that the separability and the Bell locality are the same for a pure state. We first discuss the case where $d_A = d_B = N$ (Theorem 3.1) and then deduce the general case (Corollary 3.1). The proof of Theorem 3.1 is motivated by [48–50].

Theorem 3.1 *When $d_A = d_B = N$, a pure state $|\psi\rangle$ of AB is separable if and only if it is Bell local.*

Proof Since every separable state is Bell local [17, Remark 2.2], the necessity holds.

To proof the sufficiency, we let $|\psi\rangle$ be entangled. Let $|\psi\rangle = \sum_{k=1}^N c_k|\psi_k^A\rangle|\psi_k^B\rangle$ be the Schmidt decomposition of $|\psi\rangle$ where $c_1 > c_2 \ge c_k \ge 0 (k = 3, 4 \ldots, N)$, $\{|\psi_k^A\rangle\}_{k=1}^N$ and $\{|\psi_k^B\rangle\}_{k=1}^N$ are orthonormal bases (ONBs) for H_A and H_B.

Case 1. $N = 2n$. Define generalized Pauli X and Z operators by

$$X_k^A = |\psi_{2k-1}^A\rangle\langle\psi_{2k}^A| + |\psi_{2k}^A\rangle\langle\psi_{2k-1}^A|,\ Z_k^A = |\psi_{2k-1}^A\rangle\langle\psi_{2k-1}^A| - |\psi_{2k}^A\rangle\langle\psi_{2k}^A|,$$

$$X_k^B = |\psi_{2k-1}^B\rangle\langle\psi_{2k}^B| + |\psi_{2k}^B\rangle\langle\psi_{2k-1}^B|,\ Z_k^B = |\psi_{2k-1}^B\rangle\langle\psi_{2k-1}^B| - |\psi_{2k}^B\rangle\langle\psi_{2k}^B|,$$

then X_k^A, Z_k^A, X_k^B and Z_k^B are Hermitian unitary operators and satisfy

$$(X_k^L)^2 = (Z_k^L)^2 = |\psi_{2k-1}^L\rangle\langle\psi_{2k-1}^L| + |\psi_{2k}^L\rangle\langle\psi_{2k}^L| (L = A, B),$$

$$X_k^L Z_j^L = Z_k^L X_j^L = X_k^L X_j^L = Z_k^L Z_j^L = 0 (k \ne j)(L = A, B),$$

$$X_k^L Z_k^L = -|\psi_{2k-1}^L\rangle\langle\psi_{2k}^L| + |\psi_{2k}^L\rangle\langle\psi_{2k-1}^L| (L = A, B),$$

$$Z_k^L X_k^L = |\psi_{2k-1}^L\rangle\langle\psi_{2k}^L| - |\psi_{2k}^L\rangle\langle\psi_{2k-1}^L| (L = A, B).$$

Put

$$A(\alpha) = (\sin\alpha)\sum_{k=1}^n X_k^A + (\cos\alpha)\sum_{k=1}^n Z_k^A (-\pi < \alpha < \pi), \tag{3.7}$$

$$B(\beta) = (\sin\beta)\sum_{k=1}^n X_k^B + (\cos\beta)\sum_{k=1}^n Z_k^B (-\pi < \beta < \pi), \tag{3.8}$$

then $A(\alpha)$ and $B(\beta)$ are Hermitian unitary operators on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Therefore, the eigenvalues of $A(\alpha)$ and $B(\beta)$ are ± 1. Clearly, (3.7) and (3.8) imply that

$$\begin{aligned} A(\alpha)\otimes B(\beta) \;=\; &(\sin\alpha\sin\beta)\sum_{k,j=1}^{n} X_k^A\otimes X_j^B + (\sin\alpha\cos\beta)\sum_{k,j=1}^{n} X_k^A\otimes Z_j^B \\ &+(\cos\alpha\sin\beta)\sum_{k,j=1}^{n} Z_k^A\otimes X_j^B + (\cos\alpha\cos\beta)\sum_{k,j=1}^{n} Z_k^A\otimes Z_j^B. \end{aligned}$$

By writing

$$|\xi_i\rangle = c_{2i-1}|\psi_{2i-1}^A\rangle|\psi_{2i-1}^B\rangle + c_{2i}|\psi_{2i}^A\rangle|\psi_{2j}^B\rangle,\ |\psi\rangle = \sum_{i=1}^{n}|\xi_i\rangle,$$

we compute that

$$\begin{aligned} (X_k^A\otimes X_j^B)|\xi_i\rangle \;&=\; c_{2i-1}X_k^A|\psi_{2i-1}^A\rangle\otimes X_j^B|\psi_{2i-1}^B\rangle + c_{2i}X_k^A|\psi_{2i}^A\rangle\otimes X_j^B|\psi_{2j}^B\rangle \\ &=\; c_{2i-1}\delta_{k,i}|\psi_{2k}^A\rangle\otimes\delta_{j,i}|\psi_{2j}^B\rangle + c_{2i}\delta_{k,i}|\psi_{2i-1}^A\rangle\otimes\delta_{j,i}|\psi_{2j-1}^B\rangle. \end{aligned}$$

Hence,

$$\langle\xi_m|(X_k^A\otimes X_j^B)|\xi_i\rangle = 2c_{2i-1}c_{2i},\quad \text{if } m=k=j=i; \tag{3.9}$$

it is 0 otherwise. Similarly,

$$(X_k^A\otimes Z_j^B)|\xi_i\rangle = c_{2i-1}|\psi_{2i}^A\rangle\otimes|\psi_{2i-1}^B\rangle - c_{2i}|\psi_{2i-1}^A\rangle\otimes|\psi_{2i}^B\rangle \quad \text{if } k=j=i;$$

it is 0 otherwise. Therefore,

$$\langle\xi_m|(X_k^A\otimes Z_j^B)|\xi_i\rangle = 0,\ \forall i,j,k,m. \tag{3.10}$$

Likewise,

$$\langle\xi_m|(Z_k^A\otimes X_j^B)|\xi_i\rangle = 0,\ \ \forall i,j,k,m. \tag{3.11}$$

Since

$$\begin{aligned} (Z_k^A\otimes Z_j^B)|\xi_i\rangle \;&=\; c_{2i-1}Z_k^A|\psi_{2i-1}^A\rangle\otimes Z_j^B|\psi_{2i-1}^B\rangle + c_{2i}Z_k^A|\psi_{2i}^A\rangle\otimes Z_j^B|\psi_{2j}^B\rangle \\ &=\; c_{2i-1}\delta_{k,i}|\psi_{2i-1}^A\rangle\otimes\delta_{j,i}|\psi_{2i-1}^B\rangle + c_{2i}\delta_{k,i}|\psi_{2i}^A\rangle\otimes\delta_{j,i}|\psi_{2i}^B\rangle, \end{aligned}$$

we get

$$(Z_k^A\otimes Z_j^B)|\xi_i\rangle = c_{2i-1}|\psi_{2i-1}^A\rangle\otimes|\psi_{2i-1}^B\rangle + c_{2i}|\psi_{2i}^A\rangle\otimes|\psi_{2i}^B\rangle,\ \ \text{if } k=j=i;$$

it is 0 otherwise. Thus,

$$\langle\xi_m|(Z_k^A\otimes Z_j^B)|\xi_i\rangle = c_{2i-1}^2 + c_{2i}^2,\quad \text{if } m=k=j=i; \tag{3.12}$$

it is 0 otherwise. Using (3.9)–(3.12) shows that

$$\langle ab\rangle := \langle A(\alpha)\otimes B(\beta)\rangle_\psi = \cos\alpha\cos\beta + K\sin\alpha\sin\beta, \tag{3.13}$$

where $K = 2(c_1c_2 + c_3c_4 + \ldots + c_{N-1}c_N) > 0$. So,

$$\begin{aligned} \langle ab'\rangle &:= \langle A(\alpha)\otimes B(\beta')\rangle_\psi = \cos\alpha\cos\beta' + K\sin\alpha\sin\beta', \\ \langle a'b\rangle &:= \langle A(\alpha')\otimes B(\beta)\rangle_\psi = \cos\alpha'\cos\beta + K\sin\alpha'\sin\beta, \\ \langle a'b'\rangle &:= \langle A(\alpha')\otimes B(\beta')\rangle_\psi = \cos\alpha'\cos\beta' + K\sin\alpha'\sin\beta'. \end{aligned}$$

Especially, letting $\alpha = 0, \alpha' = \pi/2, \beta = -\beta' = \arctan K$ implies that

$$\langle ab\rangle = \cos\beta,\ \langle ab'\rangle = \cos\beta,\ \langle a'b\rangle = K\sin\beta,\ \langle a'b'\rangle = -K\sin\beta,$$

and so

$$\langle ab\rangle + \langle ab'\rangle + \langle a'b\rangle - \langle a'b'\rangle = 2\cos\beta + 2K\sin\beta = 2(1+K^2)^{1/2} > 2.$$

It follows from Lemma 3.2 that $|\psi\rangle$ is Bell nonlocal.

Case 2. $N = 2n+1$. Let $X_k^L, Z_k^L (L = A, B)$ be as in Case 1 and define

$$A(\alpha) = (\sin\alpha)\sum_{k=1}^{n} X_k^A + (\cos\alpha)\sum_{k=1}^{n} Z_k^A + |\psi_{2n+1}^A\psi_{2n+1}^A\rangle\langle\psi_{2n+1}^A\psi_{2n+1}^A|,$$

$$B(\beta) = (\sin\beta)\sum_{k=1}^{n} X_k^B + (\cos\beta)\sum_{k=1}^{n} Z_k^B + |\psi_{2n+1}^B\psi_{2n+1}^B\rangle\langle\psi_{2n+1}^B\psi_{2n+1}^B|,$$

which are Hermitian unitary operators and then have eigenvalues ± 1. By writing

$$|\xi_i\rangle = c_{2i-1}|\psi_{2i-1}^A\rangle|\psi_{2i-1}^B\rangle + c_{2i}|\psi_{2i}^A\rangle|\psi_{2j}^B\rangle$$

and $|\psi\rangle = \sum_{i=1}^n |\xi_i\rangle + c_{2n+1}|\psi_{2n+1}^A\rangle|\psi_{2n+1}^B\rangle$, we obtain that

$$\langle ab\rangle := \langle A(\alpha)\otimes B(\beta)\rangle_\psi = (1-c_N^2)\cos\alpha\cos\beta + K\sin\alpha\sin\beta + c_N^2.$$

Chose $\alpha = 0, \alpha' = \pi/2, \beta = -\beta' = \arctan(K/(1-c_N^2))$, then

$$\langle ab\rangle = (1-c_N^2)\cos\beta + c_N^2,\ \langle ab'\rangle = (1-c_N^2)\cos\beta + c_N^2,$$

$$\langle a'b\rangle = K\sin\beta + c_N^2,\ \langle a'b'\rangle = -K\sin\beta + c_N^2,$$

$$\begin{aligned}\langle ab\rangle + \langle ab'\rangle + \langle a'b\rangle - \langle a'b'\rangle &= 2(1-c_N^2)\cos\beta + 2c_N^2 + 2K\sin\beta\\ &= 2[(1-c_N^2)^2 + K^2]^{1/2} + 2c_N^2\\ &> 2(1-c_N^2) + 2c_N^2\\ &= 2\end{aligned}$$

since $K > 0$. It follows from Lemma 3.2 that $|\psi\rangle$ is Bell nonlocal. □

Now, we turn to discuss the Bell locality of a pure state $|\psi\rangle$ of $\mathcal{H}_A\otimes\mathcal{H}_B$ with $d_A \neq d_B$. Without loss of generality, we assume that $d_A < d_B$.

Let $|\psi^{AB}\rangle$ be an entangled state of $\mathcal{H}_A\otimes\mathcal{H}_B$. Then it has the Schmidt decomposition $|\psi^{AB}\rangle = \sum_{k=1}^r c_k|\psi_k^A\rangle|\psi_k^B\rangle (c_k > 0, r > 1)$, where $\{|\psi_k^A\rangle\}_{k=1}^r$ and $\{|\psi_k^B\rangle\}_{k=1}^r$ are orthonormal sets in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Extending $\{|\psi_k^A\rangle\}_{k=1}^r$ and $\{|\psi_k^B\rangle\}_{k=1}^r$ as orthonormal bases $\{|\psi_k^A\rangle\}_{k=1}^{d_A}$ and $\{|\psi_k^B\rangle\}_{k=1}^{d_B}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Defining

$$J\left(\sum_{k=1}^{d_A} x_k|\psi_k^A\rangle\right) = \sum_{k=1}^{d_A} x_k|\psi_k^B\rangle, \tag{3.14}$$

we obtain an isometric operator $J : \mathcal{H}_A \to \mathcal{H}_B$. Since

$$|\tilde{\psi}^{BB}\rangle := (J\otimes I_B)|\psi^{AB}\rangle = \sum_{k=1}^{r} c_k|\psi_k^B\rangle|\psi_k^B\rangle$$

is an entangled state of BB, we see from Theorem 3.1 that $|\tilde{\psi}^{BB}\rangle$ is Bell nonlocal. Now, let us prove that $|\psi^{AB}\rangle$ is Bell nonlocal. Otherwise, it is Bell local. For every measurement assemblages $\mathcal{M}_B = \{\{M_{a|x}\}_{a=1}^{o_1}\}_{x=1}^{m_1}$ and $\mathcal{N}_B = \{\{N_{b|y}\}_{b=1}^{o_2}\}_{y=1}^{m_2}$ $\mathcal{N}_B$ of B, put $M'_{a|x} = J^\dagger M_{a|x}J$, then $\mathcal{M}'_A = \{\{M'_{a|x}\}_{a=1}^{o_1}\}_{x=1}^{m_1}$ becomes the a measurement assemblage of A since

$J^\dagger J = I_A$. Thus, we get measurement assemblages $\mathcal{M}'_A$ and $\mathcal{N}_B$ of A and B. Since $|\psi^{AB}\rangle$ was Bell local, there exists a probability distribution $\{\pi_\lambda\}_{\lambda=1}^d$ such that

$$\mathrm{tr}[(M'_{a|x} \otimes N_{b|y})|\psi^{AB}\rangle\langle\psi^{AB}|] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda)P_B(b|y,\lambda), \quad \forall a,b,x,y, \tag{3.15}$$

where $\{P_A(a|x,\lambda)\}_{a=1}^{o_1}$ and $\{P_B(b|y,\lambda)\}_{b=1}^{o_2}$ are probability distributions. Since

$$\begin{aligned}\mathrm{tr}[(M_{a|x} \otimes N_{b|y})|\tilde{\psi}^{BB}\rangle\langle\tilde{\psi}^{BB}|] &= \langle\psi^{AB}|(J^\dagger \otimes I_B)M_{a|x} \otimes N_{b|y})(J \otimes I_B)|\psi^{AB}\rangle\\ &= \langle\psi^{AB}|(J^\dagger M_{a|x} J \otimes N_{b|y})|\psi^{AB}\rangle\\ &= \mathrm{tr}[(M'_{a|x} \otimes N_{b|y})|\psi^{AB}\rangle\langle\psi^{AB}|],\end{aligned}$$

it follows from (3.15) that

$$\mathrm{tr}[(M_{a|x} \otimes N_{b|y})|\tilde{\psi}^{BB}\rangle\langle\tilde{\psi}^{BB}|] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda)P_B(b|y,\lambda), \quad \forall a,b,x,y.$$

This shows that $|\tilde{\psi}^{BB}\rangle$ is Bell local, a contradiction. Consequently, $|\psi^{AB}\rangle$ is Bell nonlocal.

As a conclusion, we have the following corollary.

Corollary 3.1 *A pure state of any bipartite system AB is separable if and only if it is Bell local. Equivalently, a pure state of AB is Bell nonlocal if and only if it is entangled.*

Combining Theorem 2.2 and Corollary 3.1 implies the following conclusion, which gives a brief characterization of Bell locality for a pure state.

Corollary 3.2 *A pure state $|\psi^{AB}\rangle$ of AB is Bell local if and only if for all projective measurements $M = \{M_a\}_{a=1}^m$ of A and $N = \{N_b\}_{b=1}^n$ of B, it holds that*

$$tr[(M_a \otimes N_b)|\psi^{AB}\rangle\langle\psi^{AB}|] = P_A(a)P_B(b), \quad \forall a,b, \tag{3.16}$$

where $\{P_A(a)\}_{a=1}^m$ and $\{P_B(b)\}_{b=1}^n$ are PDs.

As the end of this section, motivated by the deduction of Corollary 3.1, we discuss a relationship between the separability (resp. Bell locality) of a mixed state ρ^{AB} of $\mathcal{H}_A \otimes \mathcal{H}_B (d_A < d_B)$ and that of the corresponding state

$$\tilde{\rho}^{BB} = (J \otimes I_B)\rho^{AB}(J \otimes I_B)^\dagger$$

of $\mathcal{H}_B \otimes \mathcal{H}_B$ where J is given by (3.14).

Let ρ^{AB} be separable. Then it can be written as $\rho^{AB} = \sum_{k=1}^d c_k \rho_k^A \otimes \rho_k^B$ where $c_k \geq 0$, $\sum_k c_k = 1$, ρ_k^A and ρ_k^B are states of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then

$$\tilde{\rho}^{BB} = \sum_{k=1}^{d} c_k (J\rho_k^A J^\dagger) \otimes \rho_k^B.$$

Since $J\rho_k^A J^\dagger (k \in [d])$ are states of $\mathcal{H}_B$, we see that $\tilde{\rho}^{BB}$ is separable and then Bell local.

Let ρ^{AB} be Bell local. Using the identity

$$\mathrm{tr}[(M_{a|x} \otimes N_{b|y})\tilde{\rho}^{BB}] = \mathrm{tr}[((J^\dagger M_{a|x} J) \otimes N_{b|y})\rho^{AB}],$$

we see that $\tilde{\rho}^{BB}$ is Bell local.

This leads to the following conclusion.

Proposition 3.1 *When $d_A < d_B$, it holds that*

(1) *If ρ^{AB} is separable, then $\tilde{\rho}^{BB}$ is separable.*
(2) *If ρ^{AB} is Bell local, then $\tilde{\rho}^{BB}$ is Bell local.*
(3) *A pure state $|\psi^{AB}\rangle$ is separable (resp. entangled, Bell local, Bell nonlocal) if and only if so is $\tilde{\psi}^{BB}$.*

This implies that to detect entanglement or Bell nonlocality of a state ρ^{AB}, it suffices to detect that of $\tilde{\rho}^{BB}$.

An interesting question is whether the separability (Bell locality) of $\tilde{\rho}^{BB}$ implies that of ρ^{AB}.

4 Separability and Unsteerability

According to [17], a state ρ of system AB is said to be *unsteerable* from A to B if for any measurement assemblage $\mathcal{M}_A = \{\{M_{a|x}\}_{a=1}^{o_A} : x \in [m_A]\}$ of A, there exists a PD $\{\pi_\lambda\}_{\lambda=1}^{d}$ and a set of states $\{\sigma_\lambda\}_{\lambda=1}^{d} \subset D(\mathcal{H}_B)$ such that

$$\mathrm{tr}_A[(M_{a|x} \otimes I_B)\rho] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda)\sigma_\lambda, \quad \forall x, a, \tag{4.1}$$

where $\{P_A(a|x,\lambda)\}_{a=1}^{o_A}$ is a PD for each (a, x). A state ρ of system AB is said to be *steerable* from A to B if it is not unsteerable from A to B.

With this definition, the following characterization was established in [17, Theorem 3.1].

Theorem 4.1 *A state ρ^{AB} of the system AB is unsteerable from A to B if and only if for any measurement assemblage $\mathcal{M}_A = \{\{M_{a|x}\}_{a=1}^{o_A} : x \in [m_A]\}$ of A, there exists a PD $\{\pi_\lambda\}_{\lambda=1}^{d}$ and a group of states $\{\sigma_\lambda\}_{\lambda=1}^{d} \subset \mathcal{D}_B$ such that, for every POVM $\{N_b\}_{b=1}^{o_B}$ of B, it holds that*

$$tr[(M_{a|x} \otimes N_b)\rho^{AB}] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a|x,\lambda) tr(N_b\sigma_\lambda), \tag{4.2}$$

where $\{P_A(a|x,\lambda)\}_{a=1}^{o_A} (1 \le x \le m_A, 1 \le \lambda \le d)$ are PDs.

We see from this theorem that every unsteerable state must be Bell local. It was also proved in [17, Theorem 4.3] that every entangled pure state of $\mathbb{C}^n \otimes \mathbb{C}^n$ is steerable from A to B. The theorem below shows that the separability and the unsteerability are the same for a pure state of any bipartite system. Thus, any entangled pure state of any bipartite state is always steerable from A to B and from B to A.

Theorem 4.2 *A pure state $|\psi\rangle$ of a system AB is separable if and only if it is unsteerable from A to B and from B to A.*

We see from Theorem 4.2 that an entangled state must be two-way steerable, i.e. it is steerable from both A to B and B to A. This leads to the following.

Corollary 4.1 *A pure state $|\psi\rangle$ of a system AB is entangled if and only if it is two-way steerable.*

5 Separability and Classical Correlation

Recall that [36, 37] a state $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be *classically correlated* (CC) if there exists a rank-1 projective measurement $\Pi = \{\Pi_s^A \otimes \Pi_t^B : s \in [m], t \in [n]\}$ such that

$$\sum_{s=1}^{m}\sum_{t=1}^{n}(\Pi_s^A \otimes \Pi_t^B)\rho(\Pi_s^A \otimes \Pi_t^B) = \rho,$$

otherwise, ρ is said to be *quantum correlated* (QC). A pure state $|\psi\rangle$ is said to be CC (resp. QC) if $|\psi\rangle\langle\psi|$ is CC (resp. QC).

It was proved [37, Theorem 1.1] that every CC state is separable, but not every separable state is CC. The following theorem shows that separability and classical correlation are the same for pure states.

Theorem 5.1 *A pure state $|\psi\rangle$ of a system AB is separable if and only if it is classically correlated.*

Proof Since every CC state is separable [37, Theorem 1.1], the sufficiency is valid. To prove the necessity, we let $|\psi\rangle$ be a separable state of a system AB. Then it can be written as $|\psi\rangle = |\psi^A\rangle|\psi^B\rangle$ for some pure states $|\psi^A\rangle$ and $|\psi^B\rangle$ of A and B, respectively. Thus, $\rho := |\psi\rangle\langle\psi| = |\psi^A\rangle\langle\psi^A| \otimes |\psi^B\rangle\langle\psi^B|$. Extending $|\psi^A\rangle$ and $|\psi^B\rangle$ as orthonormal bases $\{|e_i\rangle\}_{i=1}^{d_A}$ and $\{|f_j\rangle\}_{j=1}^{d_B}$ for H_A and H_B, respectively, such that $|e_1\rangle = |\psi^A\rangle$ and $|f_1\rangle = |\psi^B\rangle$, we obtain a rank-1 projective measurement $\Pi = \{\Pi_s^A \otimes \Pi_t^B : s \in [m], t \in [n]\}$ with $\Pi_s^A = |e_s\rangle\langle e_s|$, $\Pi_t^B = |f_t\rangle\langle f_t|$. Clearly,

$$\sum_{s=1}^{m}\sum_{t=1}^{n}(\Pi_s^A \otimes \Pi_t^B)\rho(\Pi_s^A \otimes \Pi_t^B) = |e_1\rangle\langle e_1| \otimes |f_1\rangle\langle f_1| = \rho.$$

Thus, ρ is CC. □

Corollary 5.1 *A pure state $|\psi\rangle$ of a system AB is entangled if and only if it is quantum correlated.*

6 Conclusions

We have obtained some measurement-based characterizations of separability of bipartite states. Our results imply that the separability of a bipartite state can be detected by local POVM measurements. Especially, for bipartite pure states, we have proved that the separability, Bell locality, unsteerability and classical correlation are the same. Consequently, every entangled pure bipartite state is always Bell nonlocal, two-way steerable and quantum correlated.

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