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Partial Steerability and Nonlocality of Multipartite Quantum States

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Abstract

In this paper, we discuss partial steerability and nonlocality of multipartite quantum states. For a state ρ of an n-partite system $A_1A_2\cdots A_n$, we introduce the concepts of the steerability of ρ from i to j and the (i, j)-Bell nonlocality of ρ. By establishing necessary conditions for a state ρ to be unsteerable from i to j (resp. (i, j)-Bell local), we derive sufficient conditions for a state ρ to be steerable from i to j (resp. (i, j)-Bell nonlocal). We prove that if there are some $1 \le i < j \le n$ such that ρ is steerable from i to j (resp. (i, j)-Bell nonlocal), then it is steerable from A to B (resp. (A, B)-Bell nonlocal) provided that $A = A_1A_2\cdots A_k$ and $B = A_{k+1}A_{k+2}\cdots A_n$ with $1 \le i \le k$ and $k < j \le n$, leading to new methods for detecting steerability and (A, B)-Bell nonlocal of multipartite states.

Keywords Quantum steering · Bell locality · Multipartite quantum state

1 Introduction

In 1935 the famous EPR paradox was introduced by Einstein, Podolsky and Rosen [1] and developed to quantum steering by Schrödinger [2]. Quantum steering as a special quantum entanglement is another type of quantum correlations. An experimental about quantum steering was first performed by Ou et al. [3] and then by [4–6]. Various steering criteria give yes/no answers to the question of steerability. To study steering, we must understand the standard provided by Reid [7], which developed by Cavalcanti [8], Foster, Reid and Drummond [9], and Walborn et al [10]. Cao and Guo [11] discussed EPR steering of bipartite states, including mathematical definition and characterizations, the convexity as well as the closedness of the set of all EPR unsteerable states. Li et al. in [12] obtained some characterizations of EPR steerability of bipartite states by proving some necessary and sufficient conditions for a state to be unsteerable with a measurement assemblage of Alice. Based on one of the obtained characterizations, they derived an EPR steering inequality, which

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serves to check EPR steerability of the maximally entangled states. See [13] for more steering inequalities, [14, 15] for the steering of tripartite systems and some applications of the steerable states [16–18].

Bell nonlocality [19–23] is usually detected by a violation of some Bell inequalities, such as the CHSH inequality [24]. Dong and Cao obtained a Hardy Paradox-based method for detecting Bell nonlocality [25]. Chen et al [26] showed that Bell nonlocality can be detected through a violation of EPR steering inequality. Cao and Guo [11] discussed mathematical definition and characterizations of Bell locality and proved the convexity as well as the closedness of the sets of all Bell local states.

In this work, we discuss partial steerability and nonlocality of multipartite quantum states. For a state ρ of an n-partite system $A_1A_2\cdots A_n$, we introduce the concepts of the steerability of ρ from i to j and the (i, j)-Bell nonlocality of ρ in Sections 2 and 3, respectively. By establishing necessary conditions for a state ρ to be unsteerable from i to j and (i, j)-Bell local, respectively, we derive sufficient conditions for a state ρ to be steerable from i to j and (i, j)-Bell nonlocal, respectively. We also prove that if there are some $1 \le i < j \le n$ such that ρ is steerable from i to j (resp. (i, j)-Bell nonlocal), then it is steerable from A to B (resp. (A, B)-Bell nonlocal) provided that $A = A_1A_2\cdots A_k$ and $B = A_{k+1}A_{k+2}\cdots A_n$ with $1 \le i \le k$ and $k < j \le n$. This suggests new methods for detecting steerability and (A, B)-Bell nonlocal of multipartite states.

2 Steering from *i* to *j*

Consider an n-partite system $A_1A_2\cdots A_n$ described by Hilbert space $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n$. We use I_k to denote the identity operator on H_k and tr_j to denote the partial-trace operation tr_{A_j}, and use $\|T\|$ and $\|T\|_1$ to denote the operator-norm and the trace-norm of an operator T. For a nonempty proper subset E of $[n] = \{1, 2, \dots, n\}$, we use tr_E to denote the partial-trace operation on subsystem $\Pi_{i\in E}A_i$. Thus, $\hat{E} := [n]\setminus E$, $\mathrm{tr}_{\hat{E}}$ denotes the partial-trace operation on subsystem $\Pi_{i\in\hat{E}}A_i = \Pi_{i\in[n]\setminus E}A_i$. Specially, when $E = \{i\}$, we write tr_E and $\mathrm{tr}_{\hat{E}}$ as tr_i and $\mathrm{tr}_{\hat{i}}$, respectively.

Let

$$\mathcal{M}_k = \left\{M^{x_k} = \left\{M^{(k)}_{a_k|x_k}\right\}_{a_k=1}^{o_k} : x_k = 1, 2, \dots, m_k\right\} \tag{2.1}$$

be a POVM measurement assemblages (a set of POVMs) of system A_k described by a Hilbert space $\mathcal{H}_k$ of dimension d_k for all $k = 1, 2, \dots, n$. For a state $\rho \equiv \rho^{A_1A_2\cdots A_n}$ of an n-partite system $A_1A_2\cdots A_n$ and two subsystems A_i and $A_j (i < j)$, we denote $N_{a_i|x_i} = \otimes_{k=1}^n T_{ik}$ where $T_{ii} = M^{(i)}_{a_i|x_i}$ and $T_{ik} = I_k (k \neq i)$.

Definition 2.1 For a state $\rho \equiv \rho^{A_1A_2\cdots A_n}$ of an n-partite system $A_1A_2\cdots A_n$ and two subsystems A_i and $A_j (i < j)$, we say that ρ is *unsteerable* from i to j (or i can not steer j) with $\mathcal{M}_i$ if there exists a PD $\{\pi_\lambda\}_{\lambda=1}^d$ and a set of states $\{\sigma_\lambda^{(j)}\}_{\lambda=1}^d$ of A_j such that

$$\mathrm{tr}_{\hat{j}}[N_{a_i|x_i}\rho] = \sum_{\lambda=1}^{d} \pi_\lambda P_i(a_i|x_i, \lambda)\sigma_\lambda^{(j)}, \quad \forall x_i \in [m_i], a_i \in [o_i], \tag{2.2}$$

where $\{P_i(a_i|x_i, \lambda)\}_{a_i=1}^{o_i}$ is a PD for each (λ, x_i). Equation (2.2) is said to be an LHS model of ρ with respect to $\mathcal{M}_i$.

A state ρ is said to *unsteerable* from i to j if it is unsteerable from i to i with *any* $\mathcal{M}_i$; A state ρ is said to be *steerable* from i to j with $\mathcal{M}_i$ if it is not unsteerable from i to j with $\mathcal{M}_i$. It is said to be steerable from i to j if it is steerable from i to j with *some* $\mathcal{M}_i$. Moreover, a pure state $|\psi\rangle$ of $A_1A_2\cdots A_n$ is said to be unsteerable (resp. steerable) from i to j with $\mathcal{M}_i$ if $|\psi\rangle\langle\psi|$ is. Furthermore, we also call the unsteerability and steerability defined here the partial unsteerability and steerability.

Clearly,

$$\mathrm{tr}_{\hat{j}}[N_{a_i|x_i}\rho] = \mathrm{tr}_i\left[\left(M^{(i)}_{a_i|x_i}\otimes I_j\right)\mathrm{tr}_{\widehat{i,j}}\rho\right] = \mathrm{tr}_i\left[\left(M^{(i)}_{a_i|x_i}\otimes I_j\right)\rho_{ij}\right], \tag{2.3}$$

where $\rho_{ij} = \mathrm{tr}_{\widehat{i,j}}\rho$, the reduced state of ρ on the subsystem A_iA_j. Thus, ρ is unsteerable (resp. steerable) from i to j with $\mathcal{M}_i$ if and only if ρ_{ij} is unsteerable (resp. steerable) from A_i to A_j with $\mathcal{M}_i$ in the sense of [11, Definition 3.1].

We use $\mathcal{US}(i \to j, \mathcal{M}_i)$ to denote the set of all states $\rho \equiv \rho^{A_1A_2\cdots A_n}$ of an n-partite system $A_1A_2\cdots A_n$ that are unsteerable from i to j with $\mathcal{M}_i$. Then we see from [11, Corollary 3.1] that $\mathcal{US}(i \to j, \mathcal{M}_i)$ is a compact convex subset of the set $\mathcal{D}(A_1A_2\cdots A_n)$ of all states of $A_1A_2\cdots A_n$. Therefore, the set $\mathcal{S}(i \to j, \mathcal{M}_i)$ of all states of $A_1A_2\cdots A_n$ that are steerable from i to j with $\mathcal{M}_i$ becomes an open set. Also, we use $\mathcal{US}(i \to j)$ and $\mathcal{S}(i \to j)$ to denote the set of all unsteerable and steerable states from i to j of $A_1A_2\cdots A_n$. Thus, we see from Definition 2.1 that

$$\mathcal{US}(i \to j) = \bigcap_{\mathcal{M}_i}\mathcal{US}(i \to j, \mathcal{M}_i),\ \mathcal{S}(i \to j) = \bigcup_{\mathcal{M}_i}\mathcal{S}(i \to j, \mathcal{M}_i). \tag{2.4}$$

This implies that $\mathcal{US}(i \to j)$ is a compact subset of the set $\mathcal{D}(A_1A_2\cdots A_n)$ and that $\mathcal{S}(i \to j)$ is an open subset of $\mathcal{D}(A_1A_2\cdots A_n)$.

Let us discuss a relationship between the unsteerability (steerability) defined by Definition 2.1 and the unsteerability (steerability) introduced in [11] of an n-partite quantum system $A_1A_2\cdots A_n$ as a bipartite system AB where $A = A_1A_2\cdots A_k$, $B = A_{k+1}A_{k+2}\cdots A_n$ and $1 \le i \le k$ and $k < j \le n$. Suppose that a state ρ of $A_1A_2\cdots A_n$ is unsteerable from A to B in the sense of [11, Definition 3.1]. Then for any indices $1 \le i \le k$ and $k < j \le n$, and any POVM measurement assemblage

$$\mathcal{N}_i = \left\{M^{(i)} = \left\{M^{(i)}_{a_i|x_i}\right\}_{a_i=1}^{o_i} : x_i \in [m_i]\right\}$$

of A_i, we denote $M_{a|x} = \otimes_{n=1}^{k} T_{in}$ with $T_{ii} = M^{(i)}_{a|x}$ and $T_{in} = I_n (n \ne i)$ for each $a \in [o_i]$ and $x \in [m_i]$. Then we get a measurement assemblage $\mathcal{M}_A = \{\{M_{a|x}\}_{a=1}^{o_i} : x \in [m_i]\}$ of system A. From [11, Definition 3.1], there exists a PD $\{\pi_\lambda\}_{\lambda=1}^{d}$ and a set $\{\sigma_\lambda\}_{\lambda=1}^{d}$ of states of B such that for all $x \in [m_i]$, $a \in [o_i]$, it holds that

$$\mathrm{tr}_A[(M_{a|x}\otimes I_B)\rho] = \sum_{\lambda=1}^{d}\pi_\lambda P_A(a|x,\lambda)\sigma_\lambda,$$

where $\{P_A(a|x,\lambda)\}_{a=1}^{o_i}$is a PD for each (λ, x). Hence, for all $x_i \in [m_i]$, $a_i \in [o_i]$, it holds that

$$\mathrm{tr}_i\left[\left(M^{(i)}_{a_i|x_i}\otimes I_j\right)\rho_{ij}\right] = \mathrm{tr}_{\hat{j}}\left(\mathrm{tr}_A[(M_{a|x}\otimes I_B)\rho]\right) = \sum_{\lambda=1}^{d}\pi_\lambda P_A(a_i|x_i,\lambda)\sigma_\lambda^{(j)},$$

where $\hat{j} = \{k+1, k+2, \ldots, n\}\backslash\{j\}$ and $\sigma_\lambda^{(j)} = \mathrm{tr}_{\hat{j}}(\sigma_\lambda)$. It follows from (2.3) and Definition 2.1 that ρ is unsteerable from i to j.

Consequently, if there are some $1 \le i < j \le n$ such that ρ is steerable from i to j, then it is steerable from from A to B provided that $A = A_1 A_2 \cdots A_k$ and $B = A_{k+1} A_{k+2} \cdots A_n$ with $1 \le i \le k$ and $k < j \le n$. This leads a method for detecting steerability of multipartite states.

Next, we derive a necessary condition for a state $\rho \equiv \rho^{A_1 A_2 \cdots A_n}$ ro be unsteerable from i to j. To this, we let $\rho \in \mathcal{US}(i \to j)$. Then the reduced state $\rho_{ij} = \mathrm{tr}_{\widehat{i,j}}\rho$ is unsteerable from i to j in the sense of [11, Definition 3.1].

Next, we aim to deduce necessary conditions for unsteerability from i to j. To do so, we let X_t, Y_t be observables of $A_t (t = i, j)$ and $F_t^{\pm} = X_t \pm \imath Y_t (t = i, j)$. Since X_t and Y_t have the spectral decompositions:

$$X_t = \sum_{k=1}^{d_t} x_k^{(t)} P_k^{(t)}, Y_t = \sum_{k=1}^{d_t} y_k^{(t)} Q_k^{(t)}, t = i, j, \tag{2.5}$$

we get a decomposition of $F_t^{s_t}$ where $s_t = \pm \equiv \pm 1$:

$$F_t^{s_t} = \sum_{k=1}^{d_t} \left(x_k^{(t)} P_k^{(t)} + \imath s_t y_k^{(t)} Q_k^{(t)} \right) (t = i, j). \tag{2.6}$$

Consider the projective POVM measurement assemblages induced by (2.5): $\mathcal{M}_i = \{P_i, Q_i\}$ where $P_i = \left\{P_k^{(i)}\right\}_{k=1}^{d_i}$ and $Q_i = \left\{Q_k^{(i)}\right\}_{k=1}^{d_i}$. Since ρ is unsteerable from i to j, we see by Definition 2.1 that there exists a PD $\{\pi_\lambda\}_{\lambda=1}^{d}$ and a set of states $\left\{\sigma_\lambda^{(j)}\right\}_{\lambda=1}^{d} \subset \mathcal{D}_{A_j}$ such that

$$\mathrm{tr}_i \left[\left(P_k^{(i)} \otimes I_j \right) \rho_{ij} \right] = \sum_{\lambda=1}^{d} \pi_\lambda P_i(k|P_i, \lambda) \sigma_\lambda^{(j)}, \quad \forall k = 1, 2, \ldots, d_i, \tag{2.7}$$

$$\mathrm{tr}_i \left[\left(Q_k^{(i)} \otimes I_j \right) \rho_{ij} \right] = \sum_{\lambda=1}^{d} \pi_\lambda P_i(k|Q_i, \lambda) \sigma_\lambda^{(j)}, \quad \forall k = 1, 2, \ldots, d_i, \tag{2.8}$$

where $\{P_i(k|P_i, \lambda)\}_{k=1}^{d_i}$ and $\{P_i(k|Q_i, \lambda)\}_{k=1}^{d_i}$ are PDs. Hence,

$$\mathrm{tr} \left[\left(P_k^{(i)} \otimes P_\ell^{(j)} \right) \rho_{ij} \right] = \sum_{\lambda=1}^{d} \pi_\lambda P_i(k|P_i, \lambda) \mathrm{tr} \left(P_\ell^{(j)} \sigma_\lambda^{(j)} \right), \quad \forall k \in [d_i], \ell \in [d_j], \tag{2.9}$$

$$\mathrm{tr} \left[\left(Q_k^{(i)} \otimes P_\ell^{(j)} \right) \right) \rho_{ij} \right] = \sum_{\lambda=1}^{d} \pi_\lambda P_i(k|Q_i, \lambda) \mathrm{tr} \left(P_\ell^{(j)} \sigma_\lambda^{(j)} \right), \quad \forall k \in [d_i], \ell \in [d_j], \tag{2.10}$$

and so on. With these identities, we compute that

$$\begin{aligned}\langle X_i \otimes X_j\rangle_{\rho_{ij}} &= \sum_{k,\ell} x_k^{(i)} x_\ell^{(j)} \langle P_k^{(i)} \otimes P_\ell^{(j)}\rangle_{\rho_{ij}} \\ &= \sum_{k,\ell} x_k^{(i)} x_\ell^{(j)} \sum_{\lambda=1}^{d} \pi_\lambda P_i(k|P_i,\lambda) \mathrm{tr}\left(P_\ell^{(j)}\sigma_\lambda^{(j)}\right) \\ &= \sum_{\lambda=1}^{d} \pi_\lambda \sum_k x_k^{(i)} P_i(k|P_i,\lambda) \sum_\ell x_\ell^{(j)} \mathrm{tr}\left(P_\ell^{(j)}\sigma_\lambda^{(j)}\right) \\ &= \sum_{\lambda=1}^{d} \pi_\lambda \langle X_i\rangle_\lambda \cdot \langle X_j\rangle_\lambda,\end{aligned}$$

where

$$\langle X_i\rangle_\lambda = \sum_{k=1}^{d_i} x_k^{(i)} P_i(k|P_i,\lambda), \langle X_j\rangle_\lambda = \sum_{\ell=1}^{d_j} x_\ell^{(j)} \mathrm{tr}\left(P_\ell^{(j)}\sigma_\lambda^{(j)}\right).$$

Similarly,

$$\langle X_i \otimes Y_j\rangle_{\rho_{ij}} = \sum_{\lambda=1}^{d} \pi_\lambda \langle X_i\rangle_\lambda \cdot \langle Y_j\rangle_\lambda,\ \langle Y_i \otimes X_j\rangle_{\rho_{ij}} = \sum_{\lambda=1}^{d} \pi_\lambda \langle Y_i\rangle_\lambda \cdot \langle X_j\rangle_\lambda,\ \langle Y_i \otimes Y_j\rangle_{\rho_{ij}} = \sum_{\lambda=1}^{d} \pi_\lambda \langle Y_i\rangle_\lambda \cdot \langle Y_j\rangle_\lambda.$$

Consequently,

$$\begin{aligned}\langle F_i^{s_i} \otimes F_j^{s_j}\rangle_{\rho_{ij}} &= \langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}} \\ &= \sum_{\lambda=1}^{d} \pi_\lambda \left[\langle X_i\rangle_\lambda \cdot \langle X_j\rangle_\lambda + \imath s_j \langle X_i\rangle_\lambda \cdot \langle Y_j\rangle_\lambda + \imath s_i \langle Y_i\rangle_\lambda \cdot \langle X_j\rangle_\lambda - s_i s_j \langle Y_i\rangle_\lambda \cdot \langle Y_j\rangle_\lambda\right] \\ &= \sum_{\lambda=1}^{d} \pi_\lambda \langle F_i^{s_i}\rangle_\lambda \cdot \langle F_j^{s_j}\rangle_\lambda,\end{aligned}$$

where

$$\langle F_k^{s_k}\rangle_\lambda = \langle X_k\rangle_\lambda + \imath s_k \langle Y_k\rangle_\lambda (k = i, j).$$

Convexity of $f(t) = t^2$ implies that

$$|\langle F_i^{s_i} \otimes F_j^{s_j}\rangle_{\rho_{ij}}|^2 \le \sum_{\lambda=1}^{d} \pi_\lambda |\langle F_i^{s_i}\rangle_\lambda|^2 |\langle F_j^{s_j}\rangle_\lambda|^2. \tag{2.11}$$

By using convexity of $f(t) = t^2$ again, we have

$$|\langle F_i^{s_i}\rangle_\lambda|^2 = |\langle X_i\rangle_\lambda + \imath s_i \langle Y_i\rangle_\lambda|^2 = |\langle X_i\rangle_\lambda|^2 + |\langle Y_i\rangle_\lambda|^2 \le \langle X_i^2\rangle_\lambda + \langle Y_i^2\rangle_\lambda = \langle X_i^2 + Y_i^2\rangle_\lambda, \tag{2.12}$$

where

$$\langle X_i^2\rangle_\lambda = \sum_{k=1}^{d_i} |x_k^{(i)}|^2 P_i(k|P_i,\lambda), \langle Y_i^2\rangle_\lambda = \sum_{k=1}^{d_i} |y_k^{(i)}|^2 P_i(k|P_i,\lambda).$$

By introducing

$$\Delta_\lambda^2 T = \langle T^2\rangle_\lambda - (\langle T\rangle_\lambda)^2 = \mathrm{tr}(T^2\sigma_\lambda) - (\mathrm{tr}(T\sigma_\lambda))^2 (T = X_j, Y_j) \tag{2.13}$$

and letting that

$$C_j = \min_{\lambda\in[d]} \left(\Delta_\lambda^2 X_j + \Delta_\lambda^2 Y_j\right), \tag{2.14}$$

we have

$$|\langle F_j^{s_j}\rangle_\lambda|^2 = |\langle X_j\rangle_\lambda|^2+|\langle Y_j\rangle_\lambda|^2 = \langle X_j^2\rangle_\lambda+\langle Y_j^2\rangle_\lambda-\left(\Delta_\lambda^2 X_j + \Delta_\lambda^2 Y_j\right) \le \langle X_j^2\rangle_\lambda+\langle Y_j^2\rangle_\lambda-C_j. \tag{2.15}$$

Combining (2.11), (2.12) and (2.15), we obtain that

$$|\langle F_i^{s_i} \otimes F_j^{s_j}\rangle_{\rho_{ij}}|^2 \le \sum_{\lambda=1}^{d} \pi_\lambda \langle X_i^2 + Y_i^2\rangle_\lambda \langle X_j^2 + Y_j^2 - C_j\rangle_\lambda. \tag{2.16}$$

By (2.9) and (2.10), we get

$$\langle (X_i^2 + Y_i^2) \otimes (X_j^2 + Y_j^2 - C_j)\rangle_{\rho_{ij}} = \sum_{\lambda=1}^{d} \pi_\lambda \langle X_i^2 + Y_i^2\rangle_\lambda \langle X_j^2 + Y_j^2 - C_j\rangle_\lambda,$$

and so

$$|\langle F_i^{s_i} \otimes F_j^{s_j}\rangle_{\rho_{ij}}|^2 \le \langle (X_i^2 + Y_i^2) \otimes (X_j^2 + Y_j^2 - C_j)\rangle_{\rho_{ij}}. \tag{2.17}$$

With the discussion above, we arrive at the following.

Theorem 2.1 *Let $\rho \in \mathcal{US}(i \to j)$ and let X_t, Y_t be observables of $A_t (t = i, j)$ and $F_t^{\pm} = X_t \pm \imath Y_t (t = i, j)$. Then the inequality (2.17) holds for all $s_i, s_j = \pm$. Equivalently, for all $s_i, s_j = \pm 1$, it holds that*

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| \le \sqrt{\langle (X_i^2 + Y_i^2) \otimes (X_j^2 + Y_j^2 - C_j)\rangle_{\rho_{ij}}}, \tag{2.18}$$

where C_j is defined by (2.14). In addition, if $X_t^2 = Y_t^2 = I_t (t = i, j)$, then for all $s_i, s_j = \pm 1$, it holds that

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| \le \sqrt{2(2 - C_j)}. \tag{2.19}$$

Recall [27, (5)] that

$$\Delta_\lambda^2 \sigma^x + \Delta_\lambda^2 \sigma^y + \Delta_\lambda^2 \sigma^z \ge 2, 0 \le \Delta_\lambda^2 \sigma^t \le 1 (t = x, y, x).$$

Thus,

$$\Delta_\lambda^2 \sigma^x + \Delta_\lambda^2 \sigma^y \ge 1, \Delta_\lambda^2 \sigma^x + \Delta_\lambda^2 \sigma^z \ge 1, \Delta_\lambda^2 \sigma^y + \Delta_\lambda^2 \sigma^z \ge 1.$$

With these inequalities, we have the following.

Corollary 2.1 *Let $H_i = H_j = \mathbb{C}^2$, $\rho \in \mathcal{US}(i \to j)$ and let X_i, Y_i be hermitian unitary operators on H_i, $X_j + \imath Y_j \in \{\sigma^x \pm \imath\sigma^y, \sigma^x \pm \imath\sigma^z, \sigma^y \pm \imath\sigma^z\}$. Then*

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| \le \sqrt{2}, \quad \forall s_i, s_j = \pm 1. \tag{2.20}$$

Especially,

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| + |s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} + s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}}| \le 2, \tag{2.21}$$

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| \le 2. \tag{2.22}$$

Clearly, the last inequality is just the famous Bell inequality.

Corollary 2.2 *Let $H_i = H_j = \mathbb{C}^2$. If there exist hermitian unitary operators X_i, Y_i on H_i and $X_j + \imath Y_j \in \{\sigma^x \pm \imath\sigma^y, \sigma^x \pm \imath\sigma^z, \sigma^y \pm \imath\sigma^z\}$, and $s_i, s_j = \pm$ such that*

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j\langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i\langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j\langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| > \sqrt{2}, \quad (2.23)$$

then $\rho \in \mathcal{S}(i \to j)$.

Corollary 2.3 *For a state ρ of an n-qubit system, let the reduced state $\rho_{ij}(i < j)$ of ρ be*

$$\rho_{ij} = tr_{\widehat{i,j}}(\rho) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & xx^* & xy^* & 0 \\ 0 & x^*y & yy^* & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = a|00\rangle\langle 00| + (1-a)|\varphi\rangle\langle\varphi|, \quad (2.24)$$

where $a = 1 - |x|^2 - |y|^2 \geq 0$ and

$$|\varphi\rangle = \frac{1}{\sqrt{x^*x + y^*y}}(0, y, x, 0)^T = \frac{1}{\sqrt{x^*x + y^*y}}(x|01\rangle + y|10\rangle).$$

If $|xy^*| > \frac{\sqrt{2}}{4}$, then ρ is steerable from i to j.

Proof Let $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ and let $\boldsymbol{r}_k = (a_k, b_k, c_k)$ be unit vectors in $\mathbb{R}^3$. Then

$$X_1 := \boldsymbol{r}_1 \cdot \boldsymbol{\sigma} = a_1\sigma^x + b_1\sigma^y + c_1\sigma^z, Y_1 := \boldsymbol{r}_2 \cdot \boldsymbol{\sigma} = a_2\sigma^x + b_2\sigma^y + c_2\sigma^z$$

are Hermitian unitary operators on $\mathbb{C}^2$. Put $F_1^{s_1} = X_1 + \imath s_1 Y_1$ and $F_2^{s_2} = \sigma^x + \imath s_2\sigma^y$ where $s_1, s_2 = \pm 1 \equiv \pm$, and define

$$\delta(\rho_{ij}) = \max\left\{|\langle F_1^{s_1} \otimes F_2^{s_2}\rangle_{\rho_{ij}}| : \|\boldsymbol{r}_1\|_2 = \|\boldsymbol{r}_2\|_2 = 1, s_1, s_2 = \pm 1 \equiv \pm\right\}.$$

We see from Corollary 2.2 that when $\delta(\rho_{ij}) > \sqrt{2}$, ρ is steerable from i to j. When $s_1 = 1$ and $s_2 = -1$, we have

$$F_1^+ = X_1 + \imath Y_1 = \begin{bmatrix} c_1 + \imath c_2 & a_1 + \imath a_2 - \imath b_1 + b_2 \\ a_1 + \imath a_2 + \imath b_1 - b_2 & -c_1 - \imath c_2 \end{bmatrix}, F_2^- = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix},$$

and so

$$F_1^+ \otimes F_2^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2c_1 + 2\imath c_2 & 0 & 2(a_1 + \imath a_2 - \imath b_1 + b_2) & 0 \\ 0 & 0 & 0 & 0 \\ 2(a_1 + \imath a_2 + \imath b_1 - b_2) & 0 & -2c_1 - 2\imath c_2 & 0 \end{bmatrix}.$$

Thus,

$$\langle F_1^+ \otimes F_2^-\rangle_{\rho_{ij}} = \mathrm{tr}\left[\left(F_1^+ \otimes F_2^-\right)\rho_{12}\right] = 2xy^*(a_1 + \imath a_2 - \imath b_1 + b_2).$$

Since $a_k^2 + b_k^2 \leq 1 (k = 1, 2)$, we have

$$2|xy^*(a_1 + \imath a_2 - \imath b_1 + b_2)| \leq 2|xy^*|(|a_1 - \imath b_1| + |\imath a_2 + b_2|)| \leq 4|xy^*| = 2|xy^*|f(1, 0, 0, 1),$$

where $f(a_1, b_1, a_2, b_2) = |a_1 + \imath a_2 - \imath b_1 + b_2| \leq 2$. Hence, $\delta(\rho_{ij}) = 2|xy^*|f(1, 0, 0, 1) = 4|xy^*|$. We conclude from Corollary 2.2 that ρ is steerable from i to j if $|xy^*| > \sqrt{2}/4$. The proof is completed. □

Example 2.1 Consider the four-qubit state $\rho(I) = |\psi^{(I)}\rangle\langle\psi^{(I)}|$ where

$$|\psi^{(I)}\rangle = C_{0001}|0001\rangle + C_{0010}|0010\rangle + C_{0100}|0100\rangle + C_{1000}|1000\rangle, \quad (2.25)$$

with the condition that

$$|C_{0001}|^2 + |C_{0010}|^2 + |C_{0100}|^2 + |C_{1000}|^2 = 1.$$

Case 1 Qubit 1 steers qubit 2, that is, the steerability of $\rho^{(I)}$ from 1 to 2.

First, we have

$$\rho_{12} = \mathrm{tr}_{34}(\rho^{(I)}) = \begin{bmatrix} Q_{0001} + Q_{0010} & 0 & 0 & 0 \\ 0 & Q_{0100} & C_{0100}C_{1000}^* & 0 \\ 0 & C_{0100}^*C_{1000} & Q_{1000} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.26}$$

where $Q_{ijkl} = C_{ijkl}^*C_{ijkl}$. Since ρ_{12} has the form (2.24), Corollary 2.3 implies that $|\psi^{(I)}\rangle$ is steerable from 1 to 2 provided that $|C_{0100}C_{1000}^*| > \sqrt{2}/4$. For example, when $C_{0001} = C_{0010} = 0$ and $|C_{0100}| = |C_{1000}| = \sqrt{2}/2$, we have $\rho_{12} = |\beta_{01}\rangle\langle\beta_{01}|$ where $|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(e^{\imath\theta_1}|01\rangle + e^{\imath\theta_2}|10\rangle)$ with $\theta_1, \theta_2 \in \mathbb{R}$. Since $|C_{0100}C_{1000}^*| = 1/2 > \sqrt{2}/4$, $\rho^{(I)}$ from 1 to 2.

Case 2 Qubit 2 steers qubit 3, that is, the steerability of $\rho^{(I)}$ from 2 to 3.

First, we have

$$\rho_{23} = \mathrm{tr}_{14}(\rho^{(I)}) = \begin{bmatrix} Q_{0001} + Q_{1000} & 0 & 0 & 0 \\ 0 & Q_{0010} & C_{0010}C_{0100}^* & 0 \\ 0 & C_{0010}^*C_{0100} & Q_{0100} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.27}$$

where $Q_{ijkl} = C_{ijkl}^*C_{ijkl}$. Since ρ_{23} has the form (2.24), Corollary 2.3 implies that $|\psi^{(I)}\rangle$ is steerable from 2 to 3 provided that $|C_{0100}C_{0010}^*| > \sqrt{2}/4$.

Case 3 Qubit 3 steers qubit 4, that is, the steerability of $\rho^{(I)}$ from 3 to 4.

First, we have

$$\rho_{34} = \mathrm{tr}_{12}(\rho^{(I)}) = \begin{bmatrix} Q_{0100} + Q_{1000} & 0 & 0 & 0 \\ 0 & Q_{0001} & C_{0001}C_{0010}^* & 0 \\ 0 & C_{0001}^*C_{0010} & Q_{0010} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.28}$$

where $Q_{ijkl} = C_{ijkl}^*C_{ijkl}$. Since ρ_{34} has the form (2.24), Corollary 2.3 implies that $|\psi^{(I)}\rangle$ is steerable from 3 to 4 provided that $|C_{0001}C_{0010}^*| > \sqrt{2}/4$.

Case 4 Qubit 1 steers qubit 4, that is, the steerability of $\rho^{(I)}$ from 1 to 4.

First, we have

$$\rho_{14} = \mathrm{tr}_{23}(\rho^{(I)}) = \begin{bmatrix} Q_{0010} + Q_{0100} & 0 & 0 & 0 \\ 0 & Q_{0001} & C_{0001}C_{1000}^* & 0 \\ 0 & C_{0001}^*C_{1000} & Q_{1000} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.29}$$

where $Q_{ijkl} = C^*_{ijkl}C_{ijkl}$. Since ρ_{14} has the form (2.24), Corollary 2.3 implies that $|\psi^{(I)}\rangle$ is steerable from 1 to 4 provided that $|C^*_{1000}C_{0001}| > \sqrt{2}/4$.

Next, let us discuss a relationship between the steerability defined by Definition 2.1 and the steerability of a bipartite system. To this, let us divide an n quantum system $A_1A_2\cdots A_n$ as a bipartite system AB where $A = A_1A_2\cdots A_k$ and $B = A_{k+1}A_{k+2}\cdots A_n$. Suppose that a state ρ of $A_1A_2\cdots A_n$ is unsteerable from A to B as a state of AB in the sense of [11, Definition 3.1]. Then for any indices $1 \le i \le k$ and $k < j \le n$, ρ is unsteerable from i to j. Indeed, for any POVM measurement assemblage

$$\mathcal{M}_i = \left\{ M^{x_i} = \left\{ M^{(i)}_{a_i|x_i} \right\}_{a_i=1}^{o_i} : x_i = 1, 2, \ldots, m_i \right\}$$

of A_i, we denote $M_{a_i|x_i} = \otimes_{t=1}^{k} T_{it}$ where $T_{ii} = M^{(i)}_{a_i|x_i}$ and $T_{it} = I_k (t \ne i)$. Then we get a POVM measurement assemblage

$$\mathcal{M}_A = \left\{ \{M_{a_i|x_i}\}_{a_i=1}^{o_i} : x_i = 1, 2, \ldots, m_i \right\}$$

of system A with $M_{a_i|x_i} \otimes I_B = N_{a_i|x_i}$. Thus, there exists a PD $\{\pi_\lambda\}_{\lambda=1}^{d}$ and s set $\{\sigma_\lambda\}_{\lambda}^{d}$ of system B such that

$$\mathrm{tr}_A[(M_{a_i|x_i} \otimes I_B)\rho] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a_i|x_i, \lambda)\sigma_\lambda, \quad \forall x_i = 1, 2, \ldots, m_i, a_i = 1, 2, \ldots, o_i,$$

where $\{P_i(a_i|x_i, \lambda)\}_{a_i=1}^{o_i}$ is a PD for each (λ, x_i). Hence, for all $x_i = 1, 2, \ldots, m_i, a_i = 1, 2, \ldots, o_i$, it holds that

$$\mathrm{tr}_{\hat{j}}[N_{a_i|x_i}\rho] = \mathrm{tr}_{\hat{j}}\mathrm{tr}_A[(M_{a_i|x_i} \otimes I_B)\rho] = \sum_{\lambda=1}^{d} \pi_\lambda P_A(a_i|x_i, \lambda)\sigma_\lambda^{(j)},$$

where $\sigma_\lambda^{(j)} = \mathrm{tr}_{\hat{j}}\sigma_\lambda$. It follows from Definition 2.1 that ρ is unsteerable from i to j.

Consequently, if there are some $1 \le i < j \le n$ such that ρ is steerable from i to j, then it is steerable from A to B provided that $1 \le i \le k$ and $k < j \le n$, where $A = A_1A_2\cdots A_k$ and $B = A_{k+1}A_{k+2}\cdots A_n$. This leads a method for detecting steerability a bipartite system AB.

3 (i, j)-Nonlocality

Let $1 \le i < j \le n$. For measurement assemblages $\mathcal{M}_i$ and $\mathcal{M}_j$ given by (1), we denote

$$N_{a_i|x_i} = \otimes_{k=1}^{n} T_{ik}, N_{a_j|x_j} = \otimes_{k=1}^{n} S_{jk},$$

where

$$T_{ii} = M^{(i)}_{a_i|x_i}, T_{jj} = M^{(j)}_{a_j|x_j}, S_{ik} = I_k (k \ne i), S_{jk} = I_k (k \ne j).$$

Then

$$M^{(x_i,x_j)} = \{N_{a_i|x_i}N_{a_j|x_j} : (a_i, a_j) \in [o_i] \times [o_j]\}$$

forms a POVM of $A_1A_2\cdots A_n$ for each label (x_i, x_j) in $[m_i] \times [m_j]$.

Definition 3.1 Let ρ be a state of an n-partite system $A_1A_2\cdots A_n$ and let A_i and $A_j (i < j)$ be given two subsystems.

(1) ρ is said to be (i, j)-*Bell local with respect to* $(\mathcal{M}_i, \mathcal{M}_j)$ if there exists a PD $\{\pi_\lambda\}_{\lambda=1}^d$ such that

$$\mathrm{tr}[N_{a_i|x_i} N_{a_j|x_j} \rho] = \sum_{\lambda=1}^{d} \pi_\lambda P_i(a_i|x_i, \lambda) P_j(a_j|x_j, \lambda) \tag{3.30}$$

for all $x_i \in [m_i], a_i \in [o_i], x_i \in [m_j], a_j \in [o_j]$, where $\{P_i(a_i|x_i, \lambda)\}_{a_i=1}^{o_i}$ and $\{P_j(a_j|x_j, \lambda)\}_{a_j=1}^{o_j}$ are probability distributions (PDs). Equation (3.30) is said to be a LHV model of ρ with respect to $(\mathcal{M}_i, \mathcal{M}_j)$.

(2) ρ is said to be (i, j)-*Bell local* if it is (i, j)- Bell local w.r.t *any* $(\mathcal{M}_i, \mathcal{M}_j)$.

(3) ρ is said to be (i, j)-*Bell nonlocal w.r.t.* $(\mathcal{M}_i, \mathcal{M}_j)$ if it is not (i, j)-Bell local w.r.t. $(\mathcal{M}_i, \mathcal{M}_j)$.

(4) ρ is said to be (i, j)-*Bell nonlocal* if it is not (i, j)-*Bell local* w.r.t. some $(\mathcal{M}_i, \mathcal{M}_j)$.

(5) A pure state $|\psi\rangle$ of $A_1 A_2 \cdots A_n$ is said to be (i, j)-*Bell local* (resp. (i, j)-*Bell local*) if $|\psi\rangle\langle\psi|$ is (i, j)-*Bell local* (resp. (i, j)-*Bell nonlocal*).

Furthermore, we also call the Bell locality and Bell nonlocality defined here the partial Bell locality and Bell nonlocality.

Clearly,

$$\mathrm{tr}[N_{a_i|x_i} N_{a_j|x_j} \rho] = \mathrm{tr}\left[\mathrm{tr}_{\widehat{ij}}\left(N_{a_i|x_i} N_{a_j|x_j} \rho\right)\right] = \mathrm{tr}\left[\left(M^i_{a_i|x_i} \otimes M^j_{a_j|x_j}\right) \rho_{ij}\right], \tag{3.31}$$

where $\rho_{ij} = \mathrm{tr}_{\widehat{i,j}}\rho$, the reduced state of ρ on the subsystem $A_i A_j$. Thus, ρ is (i, j)-*Bell local* if and only if ρ_{ij} is Bell local in the sense of [11, Definition 2.1].

We use $\mathcal{BL}(i, j, \mathcal{M}_i)$ (resp. $\mathcal{BNL}(i, j, \mathcal{M}_i)$) to denote the set of all states $\rho \equiv \rho^{A_1 A_2 \cdots A_n}$ of an n-partite system $A_1 A_2 \cdots A_n$ that are (i, j)-Bell local (resp. (i, j)-Bell nonlocal) w.r.t $\mathcal{M}_i$. Then we see from [11, Corollary 3.1] that $\mathcal{BL}(i, j, \mathcal{M}_i)$ is a compact convex subset of the set $\mathcal{D}(A_1 A_2 \cdots A_n)$ of all states of $A_1 A_2 \cdots A_n$. Therefore, the set $\mathcal{BNL}(i, j, \mathcal{M}_i)$ becomes an open subset of $\mathcal{D}(A_1 A_2 \cdots A_n)$. Also, we use $\mathcal{BL}(i, j)$ and $\mathcal{BNL}(i, j)$ to denote the set of all (i, j)-Bell local and (i, j)-Bell nonlocal states of $A_1 A_2 \cdots A_n$, respectively. Thus, we see from Definition 3.1 that

$$\mathcal{BL}(i, j) = \bigcap_{\mathcal{M}_i} \mathcal{BL}(i, j, \mathcal{M}_i), \ \mathcal{BNL}(i, j) = \bigcup_{\mathcal{M}_i} \mathcal{BNL}(i, j, \mathcal{M}_i). \tag{3.32}$$

This implies that $\mathcal{BL}(i, j)$ is a compact subset of the set $\mathcal{D}(A_1 A_2 \cdots A_n)$ and that $\mathcal{BNL}(i, j)$ is an open subset of $\mathcal{D}(A_1 A_2 \cdots A_n)$.

When $\rho \in \mathcal{BL}(i, j)$, the reduced state $\rho_{ij} = \mathrm{tr}_{\widehat{i,j}}\rho$ is Bell local in the sense of [11, Definition 2.1]. It follows from [11] that ρ_{ij} is unsteerable from i to j and so ρ is unsteerable from i to j. Thus,

$$\mathcal{US}(i \to j) \subset \mathcal{BL}(i, j), \mathcal{BNL}(i, j) \subset \mathcal{S}(i \to j). \tag{3.33}$$

Example 3.1 Consider the tripartite pure state

$$|\psi\rangle = \sum_{i,j=0}^{1} c_{ij} |ij\rangle |ij\rangle |0\rangle$$

of $\in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^2$ with $c_{ij} \geq 0 (i, j = 0, 1)$ and $K = 2(c_{00}c_{01} + c_{10}c_{11}) > 0$ and obtain $\rho_{12} := \mathrm{tr}_3(|\psi\rangle\langle\psi|) = |\varphi\rangle\langle\varphi|$ where $|\varphi\rangle = \sum_{i,j=0}^{1} c_{ij} |ij\rangle |ij\rangle$.

Put

$$A(\alpha) = (\sin\alpha)\begin{pmatrix} \sigma^x & 0 \\ 0 & \sigma^x \end{pmatrix} + (\cos\alpha)\begin{pmatrix} \sigma^y & 0 \\ 0 & \sigma^y \end{pmatrix},$$
$$B(\beta) = (\sin\beta)\begin{pmatrix} \sigma^x & 0 \\ 0 & \sigma^x \end{pmatrix} + (\cos\beta)\begin{pmatrix} \sigma^y & 0 \\ 0 & \sigma^y \end{pmatrix},$$

where $\alpha, \beta \in [-\pi, \pi]$. Then $A(\alpha)$ and $B(\beta)$ are ± 1-valued observables of $\mathbb{C}^4$. we take $s_1 = s_2 = 1$ and

$$X_1 = A(\alpha), Y_1 = A(\alpha'), X_2 = B(\beta), Y_2 = B(\beta'),$$

then

$$\langle X_1 \otimes X_2\rangle_\varphi = \langle A(\alpha) \otimes B(\beta)\rangle_\varphi = \cos\alpha\cos\beta + K\sin\alpha\sin\beta,$$
$$\langle X_1 \otimes Y_2\rangle_\varphi = \langle A(\alpha) \otimes B(\beta')\rangle_\varphi = \cos\alpha\cos\beta' + K\sin\alpha\sin\beta',$$
$$\langle Y_1 \otimes X_2\rangle_\varphi = \langle A(\alpha') \otimes B(\beta)\rangle_\varphi = \cos\alpha'\cos\beta + K\sin\alpha'\sin\beta,$$
$$\langle Y_1 \otimes Y_2\rangle_\varphi = \langle A(\alpha') \otimes B(\beta')\rangle_\varphi = \cos\alpha'\cos\beta' + K\sin\alpha'\sin\beta'.$$

Especially, letting $\alpha = 0, \alpha' = \pi/2, \beta = -\beta' = \arctan(K)$ yields that

$$\langle X_1 \otimes X_2\rangle_\varphi + \langle X_1 \otimes Y_2\rangle_\varphi + \langle Y_1 \otimes X_2\rangle_\varphi - \langle Y_1 \otimes Y_2\rangle_\varphi = 2(\cos\beta + K\sin\beta) = 2(1+K^2)^{1/2} > 2.$$

We conclude from [24, Theorem 3.2] that $|\psi\rangle$ is Bell nonlocal and then it is $(1, 2)$-Bell nonlocal.

Let us discuss a relationship between the (i, j)-Bell locality defined by Definition 3.1 and the (A, B)-Bell locality [24] of an n-partite quantum system $A_1A_2\cdots A_n$ as a bipartite system AB where $A = A_1A_2\cdots A_k$, $B = A_{k+1}A_{k+2}\cdots A_n$ and $1 \le i \le k$ and $k < j \le n$. Suppose that a state ρ of $A_1A_2\cdots A_n$ is (A, B)-Bell local in the sense of [24], i.e., it is Bell local as a bipartite state of AB in the sense of [11, Definition 2.1]. Then for any indices $1 \le i \le k$ and $k < j \le n$, and any POVM measurement assemblages

$$\mathcal{N}_t = \left\{ M^{(t)} = \left\{ M^{(t)}_{a_t|x_t} \right\}_{a_t=1}^{o_t} : x_t \in [m_t] \right\} \ (t = i, j)$$

of $A_t (t = i, j)$, we denote $M_{a|x} = \otimes_{n=1}^k T_{in}$ with $T_{ii} = M^{(i)}_{a|x}$ and $T_{in} = I_n (n \ne i)$ for each $a \in [o_i]$ and $x \in [m_i]$; $N_{b|y} = \otimes_{m=k+1}^n S_{jm}$ with $S_{jj} = M^{(j)}_{b|y}$ and $S_{jm} = I_m (m \ne j)$ for each $b \in [o_j]$ and $y \in [m_j]$. Then we get measurement assemblages

$$\mathcal{M}_A = \{\{M_{a|x}\}_{a=1}^{o_i} : x \in [m_i]\}, \mathcal{N}_B = \{\{N_{b|y}\}_{b=1}^{o_j} : y \in [m_j]\}$$

of systems A and B, respectively. From [11, Definition 2.1], there exists a PD $\{\pi_\lambda\}_{\lambda=1}^d$ such that for all $x \in [m_i]$, $y \in [m_j]$, $a \in [o_i]$, $b \in [o_j]$, it holds that

$$\mathrm{tr}[(M_{a|x} \otimes N_{b|y})\rho] = \sum_{\lambda=1}^d \pi_\lambda P_A(a|x, \lambda) P_B(b|y, \lambda)$$

where $\{P_A(a|x, \lambda)\}_{a=1}^{o_i}$ and $\{P_B(b|y, \lambda)\}_{b=1}^{o_j}$ are PDs for each (λ, x) and each (λ, y), respectively. Hence, for all $x_i \in [m_i]$, $a_i \in [o_i]$, $x_j \in [m_j]$ and $a_j \in [o_j]$, it holds that

$$\mathrm{tr}\left[M^{(i)}_{a_i|x_i} \otimes M^{(j)}_{a_j|x_j} \rho_{ij} \right] = \mathrm{tr}[(M_{a_i|x_i} \otimes N_{a_j|x_j})\rho] = \sum_{\lambda=1}^d \pi_\lambda P_A(a_i|x_i, \lambda) P_B(a_j|x_j, \lambda).$$

It follows from (3.31) and Definition 3.1 that ρ is (i, j)-Bell local.

Consequently, if there are some $1 \le i < j \le n$ such that ρ is (i, j)-Bell nonlocal, then it is Bell nonlocal as a state of AB provided that $A = A_1A_2 \cdots A_k$ and $B = A_{k+1}A_{k+2} \cdots A_n$ with $1 \le i \le k$ and $k < j \le n$. This leads a method for detecting Bell nonlocality of multipartite states.

It is well-known that Bell inequality is a very useful tool for detecting Bell nonlocality. Next, we deduce a complex Bell inequality for detecting (i, j)-Bell nonlocality. To do this, we let ρ be any state of $A_1A_2 \cdots A_n$ and X_t, Y_t be hermitian operators on $\mathcal{H}_t(t = i, j)$. Since $|\langle T\rangle_{\rho_{ij}}|^2 \le \langle |T|^2\rangle_{\rho_{ij}} \le \||T|^2\|_1$, we have for all $s_i, s_j = \pm 1$, it holds that

$$\begin{aligned}
|\langle F_1^{s_1} \otimes F_2^{s_2}\rangle_{\rho_{ij}}| &= |\langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j\langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i\langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j\langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| \\
&= |\langle (X_i + \imath s_i Y_i) \otimes (X_j + \imath s_j Y_j)\rangle_{\rho_{ij}}| \\
&\le \langle |(X_i + \imath s_i Y_i) \otimes (X_j + \imath s_j Y_j)|^2\rangle_{\rho_{ij}}^{\frac{1}{2}} \\
&= \langle |(X_i + \imath s_i Y_i)|^2 \otimes |(X_j + \imath s_j Y_j)|^2\rangle_{\rho_{ij}}^{\frac{1}{2}}.
\end{aligned}$$

This shows that

$$|\langle F_1^{s_1} \otimes F_2^{s_2}\rangle_{\rho_{ij}}| \le \langle |(X_i + \imath s_i Y_i)|^2 \otimes |(X_j + \imath s_j Y_j)|^2\rangle_{\rho_{ij}}^{\frac{1}{2}}. \tag{3.34}$$

Similarly,

$$\sqrt{\left\langle \left(X_i^2 + Y_i^2\right) \otimes \left(X_j^2 + Y_j^2\right)\right\rangle_{\rho_{ij}}} \le \langle |(X_i + \imath s_i Y_i)|^2 \otimes |(X_j + \imath s_j Y_j)|^2\rangle_{\rho_{ij}}^{\frac{1}{2}}.$$

If in addition, $X_t^2 = Y_t^2 = I_t(t = i, j)$, then

$$\begin{aligned}
&\langle |(X_i + \imath s_i Y_i)|^2 \otimes |(X_j + \imath s_j Y_j)|^2\rangle_{\rho_{ij}}^{\frac{1}{2}} \\
=\ &\langle (2I_i - \imath s_i[X_i, Y_i]) \otimes (2I_j - \imath s_j[X_j, Y_j])\rangle_{\rho_{ij}}^{\frac{1}{2}} \\
=\ &\left\langle 4I_i \otimes I_j - 2\imath s_i[X_i, Y_i] \otimes I_j - 2\imath s_j I_i \otimes [X_j, Y_j] + s_i s_j[X_i, Y_i] \otimes [X_j, Y_j]\right\rangle_{\rho_{ij}}^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\|s_i[X_i, Y_i]) \otimes I_j\| \le 2, \|s_j I_i \otimes [X_j, Y_j]\| \le 2, \|[X_i, Y_i] \otimes [X_j, Y_j]\| \le 4,$$

we have

$$\|4 - 2\imath s_i[X_i, Y_i] \otimes I_j - 2\imath s_j I_i \otimes [X_j, Y_j] + s_i s_j[X_i, Y_i] \otimes [X_j, Y_j]\| \le 16$$

and therefore,

$$|\langle F_1^{s_1} \otimes F_2^{s_2}\rangle_{\rho_{ij}}| \le 4. \tag{3.35}$$

Indeed, the last inequality can be obtained from the fact that

$$\|F_1^{s_1} \otimes F_2^{s_2}\| = \|F_1^{s_1}\| \cdot \|F_2^{s_2}\| \le 4$$

when $X_t^2 = Y_t^2 = I_t(t = i, j)$. This shows that a quantum upper bound for $|\langle F_1^{s_1} \otimes F_2^{s_2}\rangle_{\rho_{ij}}|$ is 4.

Similar to the derivation of Theorem 2.1, we can obtain the following conclusion, which is a necessary condition for a state ρ to be (i, j)-Bell local.

Theorem 3.1 *Let $\rho \in \mathcal{BL}(i,j)$ and let X_t, Y_t be hermitian operators on $\mathcal{H}_t (t=i,j)$. Then for all $s_i, s_j = \pm 1$, it holds that*

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| \le \sqrt{\left\langle \left(X_i^2 + Y_i^2\right) \otimes \left(X_j^2 + Y_j^2\right)\right\rangle_{\rho_{ij}}}. \tag{3.36}$$

If in addition, $X_t^2 = Y_t^2 = I_t (t=i,j)$, then for all $s_i, s_j = \pm 1$, it holds that

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| \le 2, \tag{3.37}$$

and

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| \le 2. \tag{3.38}$$

Proof The proof of (3.36) is similar to the derivation of (2.18), and (3.37) is the special case of (3.36). Inequality (3.38) is essentially given in ref. [24, Theorem 3.1]. The proof is completed.

It is remarkable to note that the famous Tsirelson's inequality [28, Problem 2.3, pp.118] shows that in the case that $\mathcal{H}_i = \mathcal{H}_j = \mathbb{C}^2$, the inequality

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| \le 2\sqrt{2} \tag{3.39}$$

holds for all two-qubit states ρ_{ij}. Moreover, the validity of (3.37) is just a necessary condition for a state ρ to be (i,j)-Bell local, but not a sufficient one. For example, when $\rho_{ij} = |\psi\rangle\langle\psi|$ where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we take $s_i = s_j = 1$ and

$$X_i = \sigma^x, Y_i = -\sigma^z, X_j = \frac{1}{\sqrt{2}}(\sigma^x - \sigma^z), Y_j = \frac{1}{\sqrt{2}}(\sigma^x + \sigma^z)$$

and compute that

$$\langle X_i \otimes X_j\rangle = \langle X_i \otimes Y_j\rangle_{\rho_{ij}} = \langle Y_i \otimes X_j\rangle_{\rho_{ij}} = -\langle Y_i \otimes Y_j\rangle_{\rho_{ij}} = \frac{\sqrt{2}}{2}.$$

Thus,

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| = 2,$$

and

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| = 2\sqrt{2}.$$

Thus, (3.37) holds while ρ_{ij} is well-known to be Bell nonlocal.

As consequences of Theorem 3.1, we have the following two corollaries, which are sufficient conditions for a state to be (i,j)-Bell nonlocal. □

Corollary 3.1 *If there exist hermitian operators on $\mathcal{H}_t (t=i,j)$ and $s_i, s_j = \pm 1$ such that*

$$|\langle X_i \otimes X_j\rangle_{\rho_{ij}} + \imath s_j \langle X_i \otimes Y_j\rangle_{\rho_{ij}} + \imath s_i \langle Y_i \otimes X_j\rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j\rangle_{\rho_{ij}}| > \sqrt{\langle (X_i^2 + Y_i^2) \otimes (X_j^2 + Y_j^2)\rangle_{\rho_{ij}}},$$

then $\rho \in \mathcal{BNL}(i,j)$.

Corollary 3.2 *If there exist hermitian unitary operators* X_k, Y_k *on* $H_k (k = i, j)$ *and* $s_i, s_j = \pm 1$ *such that*

$$|\langle X_i \otimes X_j \rangle_{\rho_{ij}} + \imath s_j \langle X_i \otimes Y_j \rangle_{\rho_{ij}} + \imath s_i \langle Y_i \otimes X_j \rangle_{\rho_{ij}} - s_i s_j \langle Y_i \otimes Y_j \rangle_{\rho_{ij}}| > 2,$$

then $\rho \in \mathcal{BNL}(i, j)$.

4 Conclusions

In this paper, we have discussed partial steerability and nonlocality of multipartite quantum states, named steerability from i to j and (i, j)-Bell nonlocality, n-partite states. By establishing necessary conditions for a state ρ to be unsteerable from i to j (resp. (i, j)-Bell local), we derive sufficient conditions for a state ρ to be steerable from i to j (resp. (i, j)-Bell nonlocal). We have proved that if there are some $1 \leq i < j \leq n$ such that ρ is steerable from i to j, then it is steerable from from A to B provided that $A = A_1 A_2 \cdots A_k$ and $B = A_{k+1} A_{k+2} \cdots A_n$ with $1 \leq i \leq k$ and $k < j \leq n$. This leads a method for detecting steerability of multipartite states. Moreover, we have checked that if there are some $1 \leq i < j \leq n$ such that ρ is (i, j)-Bell nonlocal, then it is Bell nonlocal as a state of AB provided that $A = A_1 A_2 \cdots A_k$ and $B = A_{k+1} A_{k+2} \cdots A_n$ with $1 \leq i \leq k$ and $k < j \leq n$. This leads a method for detecting Bell nonlocality of multipartite states.

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