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Output Tracking and Feedback Controller Design for Nonlinear Stochastic Time-Delay System

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Abstract

This paper considers the design of output tracking and feedback controllers for nonlinear stochastic systems. The system status cannot be measured. The studied system contains nonlinear state functions, noise disturbances and time delays. We use the free weight matrix method to avoid the conservatism caused by the use of model transformation or cross-term bounded techniques. Based on the Lyapunov-Krasovskii functional method, we propose an output tracking and feedback controller design method based on linear matrix inequality (LMI). Numerical examples show the validity of the obtained results.

Keywords Nonlinear stochastic time-delay systems · Lyapunov-Krasovskii functional method · Linear matrix inequality

1 Introduction

Stochastic system theory is a type of theory that combines stochastic process theory with control theory, and now it become an important branch of mathematics and control theory [1–6]. Recently, many scholars have paid attention to the research of robust control for uncertain stochastic time-delay systems, such as the research of robust stabilization and robust H_∞ control for uncertain stochastic time-delay systems, and the research on stochastic time-delay systems with uncertain parameters which satisfy the convex polyhedron structure [7–11]. Most of these studies use Lyapunov function which depends on the parameters and the method combined with the introduction of free matrix variables to obtain sufficient conditions for robust stabilization and robust H_∞ performance of the studied system. On the other hand, the state estimation problem of stochastic systems has always been a relatively important subject in control theory. Since the Luenberger observer [12] was introduced into the research of control systems, there have been many achievements in this area. When the external disturbances do not have accurate statistical characteristics,

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we can consider utilizing H_∞ filtering and $L_2 - L_\infty$ filtering [13–15] to study the system. For the robust control of the system, there are many research methods currently adopted, such as decomposition of singular values, H_∞ control method, Riccati equation method and so on. The above mentioned methods are mainly based on Lyapunov stability theory for robust analysis of uncertain systems. Among them, the advantage of using the Riccati equation processing method is that the structure of the required controller can be obtained, which is very beneficial to the theoretical analysis of the system. Its disadvantage is that it needs to determine some undetermined parameters , which is more conservatism. Then the inner point method for solving convex optimization problems and THE LMI toolbox introduced by Matlab appeared. When these methods were applied to the robust control problem of stochastic systems, they greatly promoted the progress in stability and robust control of stochastic systems. On the basis of these existing results, in view of the universality and practicability of uncertain stochastic time-delay systems, it is of great significance to study the design and control problem of state estimators for stochastic systems, especially uncertain stochastic systems.

The output analog control has a widespread application in industrial production and life. The main purpose of analog control is reappear the system output, and make the output of the reference model and the original system nearly as much as possible. Output simulation is also widely used in robot control and aircraft control, and output control design problems are generally more complicated and more difficult to implement than stability analysis. In this paper, we study the H_∞ output control problem for nonlinear stochastic time-delay systems. Using Lyapunov stability theory and free weight matrix method, sufficient conditions for time-delay correlation are given to ensure that the output of the stochastic nonlinear system approximates the output of the given reference model in the sense of H_∞. Finally, numerical examples are used to verify the feasibility of the conclusions obtained.

2 Model Description

In this paper, we consider nonlinear stochastic time-delay systems

$$\begin{aligned} dx(t) &= [Ax(t) + A_\tau x(t-\tau) + f(x(t), x(t-\tau), t) + Bu(t) \\ &\quad + Ev(t)]dt + Hx(t)d\omega(t) \end{aligned} \tag{1}$$

$$z(t) = C_1 x(t) + C_2 x(t-\tau) \tag{2}$$

In the system, assuming that the nonlinear perturbation satisfies the following boundedness conditions

$$\| f(x(t), x(t-\tau), t)\| \le \alpha_1 \|x(t)\| + \alpha_2 \|x(t-\tau)\| \tag{3}$$

$\omega(t) \in L_2[0, \infty)$ is a square integrable vector function, its norm is defined as

$$\|\omega(t)\|_2 = \sqrt{\int_0^\infty |\omega(t)|^2 dt}$$

$\tau > 0$ is a constant time-delay, the reference model is designed as

$$\dot{x}_r(t) = Dx_r(t) \tag{4}$$

$$z_r(t) = Fx_r(t) \tag{5}$$

Where the dimension of $z_r(t)$ is the same as the dimension of $z(t)$. $x_r(t) \in \mathbb{R}^r$ is the reference state. D, F are constant matrices of appropriate dimensions, and D is a Hurwitz matrix. The state feedback control rate is

$$u(t) = Kx(t) + K_r x_r(t) \tag{6}$$

where K and K_r are the control gain matrix. We define that

$$e(t) = z(t) - z_r(t)$$

The state feedback control rate (6) is substituted into the system (1),we can get

$$\begin{aligned} dx(t) &= [(A+BK)x(t) + BK_r x_r(t) + A_\tau x(t-\tau) + f(x(t), x(t-\tau), t) \\ &\quad +Ev(t)] + Hx(t)d\omega(t) \end{aligned} \tag{7}$$

$$e(t) = C_1 x(t) + C_2 x(t-\tau) - F x_r(t) \tag{8}$$

Then the following generalized system can be obtained

$$\begin{aligned} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} &= \left\{ \begin{bmatrix} A+BK & BK_r \\ 0 & D \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix} + \begin{bmatrix} A_\tau & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ x_r(t-\tau) \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} I \\ 0 \end{bmatrix} f(x(t), x(t-\tau), t) + \begin{bmatrix} E \\ 0 \end{bmatrix} v(t) \right\} dt \\ &\quad + \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} d\omega(t) \end{aligned} \tag{9}$$

3 H_∞ Output Tracking of System

Theorem 3.1 *Consider the system (8), (9) and the state feedback control rate in the form (6), if there is a matrix $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$, suitable dimension matrix M, N_1, N_2, and the constant $\varepsilon > 0$, satisfy the matrix inequality*

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & P_1 & P_1 BK_r & -C_1 F & 0 & 0 & P_1 E & \Pi_{19} \\ * & \Pi_{22} & 0 & 0 & -C_2 F & -M & 0 & 0 & A_\tau^T R_1 \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 & 0 & R_1 \\ * & * & * & \Pi_{44} & -N_1 + N_2 & -N_1 & 0 & 0 & K_r^T B^T R_1 \\ * & * & * & * & \Pi_{55} & 0 & -N_2 & 0 & 0 \\ * & * & * & * & * & -\tau^{-1} R_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\tau^{-1} R_2 & 0 & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & * & -\tau^{-1} R_1 \end{bmatrix} < 0 \tag{10}$$

where

$$\begin{aligned} \Pi_{11} &= (A+BK)^T P_1 + P_1 (A+BK) + H^T P_1 H + Q_1 + \varepsilon \alpha_1^2 I + C_1^T C_1 \\ \Pi_{12} &= P_1 A_\tau + M^T + C_1^T C_2 \\ \Pi_{22} &= \varepsilon \alpha_2^2 I - Q_1 - M - M^T + C_2^T C_2 \\ \Pi_{44} &= D^T P_2 + P_2 D + Q_2 \\ \Pi_{55} &= \tau D^T R_2 D - Q_2 + F^T F \\ \Pi_{19} &= A^T R_1 + K^T B^T R_1 \end{aligned}$$

Then the generalized closed-loop system (8), (9) satisfies the H_∞ output tracking performance indicator γ.

Proof Select Lyapunov-Krasovskii functional as

$$\begin{aligned} V(x(t), x_r(t)) =\ & x^T(t)P_1x(t) + x_r^T(t)P_2x_r(t) \\ & + \int_{t-\tau}^{t} x^T(s)Q_1x(s)ds + \int_{t-\tau}^{t} x_r^T(s)Q_2x_r(s)ds \\ & + \int_{t-\tau}^{t}\int_{s}^{t} \bar{x}^T(\alpha)R_1\bar{x}(\alpha)d\alpha ds \\ & + \int_{t-\tau}^{t}\int_{s}^{t} \bar{x}_r^T(\alpha)R_2\bar{x}_r(\alpha)d\alpha ds \end{aligned} \tag{11}$$

According to the formula $It\hat{o}$, we can get the differential operator as

$$\begin{aligned} \mathcal{L}V(x(t), x_r(t)) =\ & 2x^T(t)P_1\bar{x}(t) + 2x_r^T(t)P_2\bar{x}_r(t) \\ & + x^T(t)H^TPHx(t) + x^T(t)Q_1x(t) - x^T(t-\tau)Q_1x(t-\tau) \\ & + x_r^T(t)Q_2x_r(t) - x_r^T(t-\tau)Q_2x_r(t-\tau) \\ & + \tau\bar{x}^T(t)R_1\bar{x}(t) - \int_{t-\tau}^{t}\bar{x}^T(s)R_1\bar{x}(s)ds \\ & + \tau\bar{x}_r^T(t)R_2\bar{x}_r(t) - \int_{t-\tau}^{t}\bar{x}_r^T(s)R_2\bar{x}_r(s)ds \end{aligned}$$

According to the Leibniz-Newton formula

$$2\mathbb{E}\left[x^T(t-\tau)M\left(x(t) - x(t-\tau) - \int_{t-\tau}^{r}\bar{x}(s)ds\right)\right] = 0$$

$$2\mathbb{E}\left[\left(x_r^T(t)N_1 + x_r^T(t-\tau)N_2\right) - \left(x_r(t) - x_r(t-\tau) - \int_{t-\tau}^{r}\bar{x}_r(s)ds\right)\right] = 0$$

In combination with

$$\int_{t-\tau}^{t}\bar{x}^T(s)R_1\bar{x}(s)ds \le \left(\int_{t-\tau}^{t}\bar{x}^T(s)ds\right)\left(-\frac{R_1}{\tau}\right)\left(\int_{t-\tau}^{t}\bar{x}(s)ds\right)$$

$$\int_{t-\tau}^{t}\bar{x}_r^T(s)R_2\bar{x}_r(s)ds \le \left(\int_{t-\tau}^{t}\bar{x}_r^T(s)ds\right)\left(-\frac{R_2}{\tau}\right)\left(\int_{t-\tau}^{t}\bar{x}_r(s)ds\right)$$

When $v(t) = 0$,

$$\begin{aligned} \mathcal{L}V(x(t), x_r(t)) \le\ & 2x^T(t)P_1\left[(A+BK)x(t) + BK_rx_r(t) + A_\tau x(t-\tau)\right] \\ & + f(x(t), x(t-\tau), t) + 2x_r^T(t)P_2Dx_r(t) \\ & + x^T(t)H^TP_1Hx(t) + x^T(t)Q_1x(t) - x^T(t-\tau)Q_1x(t-\tau) \\ & + x_r^T(t)Q_2x_r(t) - x_r^T(t-\tau)Q_2x_r(t-\tau) \\ & + \tau[(A+BK)x(t) + BK_rx_r(t) + A_\tau x(t-\tau) + f(x(t), x(t-\tau), t)]^T \\ & R_1\left[(A+BK)x(t) + BK_rx_r(t) + A_\tau x(t-\tau) + f(x(t), x(t-\tau), t)\right] \\ & + \left(\int_{t-\tau}^{t}\bar{x}^T(s)ds\right)\left(-\frac{R_1}{\tau}\right)\left(\int_{t-\tau}^{t}\bar{x}(s)ds\right) \\ & + \tau[Dx_r(t)]^TR_2\left[Dx_r(t)\right] \\ & + \left(\int_{t-\tau}^{t}\bar{x}_r^T(s)ds\right)\left(-\frac{R_2}{\tau}\right)\left(\int_{t-\tau}^{t}\bar{x}_r(s)ds\right) \\ & + \varepsilon\alpha_1^2x^T(t)x(t) + \varepsilon\alpha_2^2x^T(t-\tau)x(t-\tau) \\ & - \varepsilon f^T(x(t), x(t-\tau), t)\, f(x(t), x(t-\tau), t) \end{aligned}$$

let

$$\eta(t) = \Big[x^T(t)\, x^T(t-\tau)\, f^T\,(x(t), x(t-\tau), t)\ x_r^T(t)$$
$$x_r^T(t-\tau) \int_{t-\tau}^{t} \bar{x}^T(s)ds \int_{t-\tau}^{t} \bar{x}_r^T(s)ds\Big]^T$$

Then

$$\mathrm{E}\mathcal{L}V\,(x(t), x_r(t)) \le \mathrm{E}\Big[\eta^T(t)\Sigma\eta(t)\Big]$$

Where

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & P_1 & P_1BK_r & 0 & 0 & 0 \\ * & \Sigma_{22} & 0 & 0 & 0 & -M & 0 \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{44} & -N_1+N_2 & -N1 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & -N_2 \\ * & * & * & * & * & -\tau^{-1}R_1 & 0 \\ * & * & * & * & * & * & -\tau^{-1}R_2 \end{bmatrix}$$
$$+ \begin{bmatrix} A^T+K^TB^T \\ A_\tau^T \\ I \\ {K_r}^TB^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot (\tau R_1) \cdot \begin{bmatrix} A^T+K^TB^T \\ A_\tau^T \\ I \\ {K_r}^TB^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

and

$$\Sigma_{11} = (A+BK)^TP_1 + P_1(A+BK) + H^TP_1H + Q_1 + \varepsilon\alpha_1^2 I$$
$$\Sigma_{12} = P_1A_\tau + M^T$$
$$\Sigma_{22} = \varepsilon\alpha_2^2 I - Q_1 - M - M^T$$
$$\Sigma_{44} = D^TP_2 + P_2D + Q_2$$
$$\Sigma_{55} = \tau D^TR_2D - Q_2$$

Using Schur complement lemma, we can find that matrix inequality $\Sigma < 0$ is true only if matrix inequality $\bar{\Sigma} < 0$ is true, where

$$\bar{\Sigma} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & P_1 & P_1BK_r & 0 & 0 & 0 & A^T+K^TB^T \\ * & \Sigma_{22} & 0 & 0 & 0 & -M & 0 & A_\tau^T \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 & I \\ * & * & * & \Sigma_{44} & -N_1+N_2 & -N1 & 0 & {K_r}^TB^T \\ * & * & * & * & \Sigma_{55} & 0 & -N_2 & 0 \\ * & * & * & * & * & -\tau^{-1}R_1 & 0 & 0 \\ * & * & * & * & * & * & -\tau^{-1}R_2 & 0 \\ * & * & * & * & * & * & * & -\tau^{-1}{R_1}^{-1} \end{bmatrix}$$

Then multipling simultaneously the left and right sides of the matrix inequality $\bar{\Sigma} < 0$ by the diagonal matrix diag$\{I, I, I, I, I, I, I, IR_1\}$, we can get

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & P_1 & P_1BK_r & 0 & 0 & 0 & A^TR_1 + K^TB^TR_1 \\ * & \Sigma_{22} & 0 & 0 & 0 & -M & 0 & A_\tau^TR_1 \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 & R_1 \\ * & * & * & \Sigma_{44} & -N_1 + N_2 & -N1 & 0 & {K_r}^TB^TR_1 \\ * & * & * & * & \Sigma_{55} & 0 & -N_2 & 0 \\ * & * & * & * & * & -\tau^{-1}R_1 & 0 & 0 \\ * & * & * & * & * & * & -\tau^{-1}R_2 & 0 \\ * & * & * & * & * & * & * & -\tau^{-1}R_1 \end{bmatrix} < 0 \tag{12}$$

Thus, the system (8), (9) is asymptotically stable.

Next, we continue to prove that the closed-loop system (8), (9) can satisfy the corresponding specified H_∞ output characteristics. Assuming that under zero initial conditions, when the system has a perturbation input $v(t)$, considering the Lyapunov-Krasovskii functional (11), we can get the random differential as

$$\begin{aligned} \mathcal{L}V\,(x(t), x_r(t)) \leq\ & 2x^T(t)P_1[(A + BK)x(t) + BK_rx_r(t) + A_\tau x(t-\tau) \\ & + f\,(x(t), x(t-\tau), t) + Ev(t)] + 2x_r^T(t)P_2Dx_r(t) \\ & + x^T(t)H^TP_1Hx(t) + x^T(t)Q_1x(t) - x^T(t-\tau)Q_1x(t-\tau) \\ & + x_r^T(t)Q_2x_r(t) - x_r^T(t-\tau)Q_2x_r(t-\tau) \\ & + \tau[(A + BK)x(t) + BK_rx_r(t) + A_\tau x(t-\tau) \\ & + f\,(x(t), x(t-\tau), t) + Ev(t)]^TR_1 \\ & [(A + BK)x(t) + BK_rx_r(t) + A_\tau x(t-\tau) \\ & + f\,(x(t), x(t-\tau), t) + Ev(t)] \\ & + \left(\int_{t-\tau}^t \bar{x}^T(s)ds\right)\left(-\frac{R_1}{\tau}\right)\left(\int_{t-\tau}^t \bar{x}(s)ds\right) \\ & + \tau\,[Dx_r(t)]^T\,R_2\,[Dx_r(t)] \\ & + \left(\int_{t-\tau}^t \bar{x}_r^T(s)ds\right)\left(-\frac{R_2}{\tau}\right)\left(\int_{t-\tau}^t \bar{x}_r(s)ds\right) \end{aligned}$$

where

$$\bar{x}_r(s) = Dx_r(s)$$

From the S-process, we can see

$$\mathcal{L}V\,(x(t), x_r(t)) \leq \bar{\eta}^T(t)\Pi\bar{\eta}(t)$$

where

$$\begin{aligned} \bar{\eta}(t) = \Big[& x^T(t)x^T(t-\tau)f^T\,(x(t), x(t-\tau), t)\,x_r^T(t) \\ & x_r^T(t-\tau)\int_{t-\tau}^t \bar{x}^T(s)ds \int_{t-\tau}^t \bar{x}_r^T(s)dsv^T(t)\Big]^T \end{aligned}$$

$$\Pi = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & P_1 & P_1BK_r & 0 & 0 & 0 & P_1E \\ * & \Sigma_{22} & 0 & 0 & 0 & -M & 0 & 0 \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{44} & -N_1+N_2 & -N1 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & -N_2 & 0 \\ * & * & * & * & * & -\tau^{-1}R_1 & 0 & 0 \\ * & * & * & * & * & * & -\tau^{-1}R_2 & 0 \\ * & * & * & * & * & * & * & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} A^T + K^TB^T \\ A_\tau^T \\ I \\ K_r{}^TB^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot (\tau R_1) \cdot \begin{bmatrix} A^T + K^TB^T \\ A_\tau^T \\ I \\ K_r{}^TB^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

Considering the following indicators

$$\rho = \int_0^\infty \Big[e^T(s)e(s) - \gamma^2 v^T(s)v(s)\Big]ds$$

Under zero initial conditions, there are $V(0) = 0$ and $V(\infty) \geq 0$, so

$$\rho = \int_0^\infty \Big[e^T(t)e(t) - \gamma^2 v^T(t)v(t) + \mathcal{L}V(x(t), x_r(t))\Big]dt - V(\infty) \tag{13}$$
$$\leq \int_0^\infty \Big[e^T(t)e(t) - \gamma^2 v^T(t)v(t) + \mathcal{L}V(x(t), x_r(t))\Big]dt \tag{14}$$

However

$$e^T(t)e(t) - \gamma^2 v^T(t)v(t) + \mathcal{L}V(x(t), x_r(t)) \leq \bar{\eta}^T(t)\bar{\Pi}\bar{\eta}(t)$$

where

$$\bar{\Pi} = \begin{bmatrix} \Pi_{11} & \Pi_{12} & P_1 & P_1BK_r & -C_1F & 0 & 0 & P_1E \\ * & \Pi_{22} & 0 & 0 & -C_2F & -M & 0 & 0 \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{44} & -N_1+N_2 & -N_1 & 0 & 0 \\ * & * & * & * & \Pi_{55} & 0 & -N_2 & 0 \\ * & * & * & * & * & -\tau^{-1}R_1 & 0 & 0 \\ * & * & * & * & * & * & -\tau^{-1}R_2 & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I \end{bmatrix}$$
$$+ \begin{bmatrix} A^T + K^TB^T \\ A_\tau^T \\ I \\ K_r{}^TB^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot (\tau R_1) \cdot \begin{bmatrix} A^T + K^TB^T \\ A_\tau^T \\ I \\ K_r{}^TB^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

Here

$$
\begin{aligned}
\Pi_{11} &= (A+BK)^T P_1 + P_1(A+BK) + H^T P_1 H + Q_1 + \varepsilon\alpha_1^2 I + C_1^T C_1 \\
\Pi_{12} &= P_1 A_\tau + M^T + C_1^T C_2 \\
\Pi_{22} &= \varepsilon\alpha_2^2 I - Q_1 - M - M^T + C_2^T C_2 \\
\Pi_{44} &= D^T P_2 + P_2 D + Q_2 \\
\Pi_{55} &= \tau D^T R_2 D - Q_2 + F^T F
\end{aligned}
$$

From Schur lemma, the inequality $\bar{\Pi} < 0$ is equivalent to the matrix inequality

$$
\begin{bmatrix}
\Pi_{11} & \Pi_{12} & P_1 & P_1 B K_r & -C_1 F & 0 & 0 & P_1 E & A^T + K^T B^T \\
* & \Pi_{22} & 0 & 0 & -C_2 F & -M & 0 & 0 & A_\tau^T \\
* & * & -\varepsilon I & 0 & 0 & 0 & 0 & 0 & I \\
* & * & * & \Pi_{44} & -N_1 + N_2 & -N_1 & 0 & 0 & K_r^T B^T \\
* & * & * & * & \Pi_{55} & 0 & -N_2 & 0 & 0 \\
* & * & * & * & * & -\tau^{-1} R_1 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\tau^{-1} R_2 & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma^2 I & 0 \\
* & * & * & * & * & * & * & * & -\tau^{-1} R_1^{-1}
\end{bmatrix}
< 0 \tag{15}
$$

Then multipling simultaneously by the diagonal matrix diag$\{I, I, I, I, I, I, I, R_1\}$ on both sides of the inequality, that is easy to obtain the matrix inequality of Theorem 3.1. □

4 Feedback Controller Design

Next, we consider the design problems of H_∞ output tracking controller, and give some conclusions based on Theorem 3.1.

Theorem 4.1 *Considering system* (8), (9), *if there are matrices* $\bar{P}_1 > 0$, $\bar{P}_2 > 0$, $\bar{Q}_1 > 0$, $\bar{Q}_2 > 0$, $R_1 > 0$, $R_2 > 0$, *the appropriate dimension matrix* M, N_1, N_2, *and constant* $\varepsilon > 0$, *makes the matrix inequality*

$$
\begin{bmatrix}
\Pi_{11} & \Pi_{12} & I & B\tilde{K}_r & -C_1 F & \bar{P}_1 A^T + \tilde{K}^T B^T & \Pi_{17} \\
* & \Pi_{22} & 0 & 0 & -C_2 F & \bar{P}_1 A_\tau^T & \Pi_{27} \\
* & * & -\varepsilon I & 0 & 0 & I & 0 \\
* & * & * & \bar{P}_2 D^T + D\bar{P}_2 + \bar{Q}_2 & -N_1\bar{P}_2 + N_2\bar{P}_2 & \tilde{K}_r^T B^T & \Pi_{47} \\
* & * & * & * & -\bar{Q}_2 + F^T F & 0 & \Pi_{57} \\
* & * & * & * & * & -\bar{\tau} R_1 - 2\bar{\tau} I & 0 \\
* & * & * & * & * & * & \Pi_{77}
\end{bmatrix} < 0 \tag{16}
$$

where

$$
\begin{aligned}
\Pi_{11} &= \bar{P}_1 A^T + \tilde{K}^T B^T + A\bar{P}_1 + B\tilde{K} + \bar{Q}_1 + C_1^T C_1 \\
\Pi_{12} &= A_\tau \bar{P}_1 + \bar{M} + C_1^T C_2 \\
\Pi_{22} &= -\bar{Q}_1 - \bar{M} - \bar{M}^T + C_2^T C_2 \\
\Pi_{17} &= \left[0\, 0\, \bar{P}_1 H^T\ \bar{P}_1\, 0\, 0 \right] \\
\Pi_{27} &= \left[-\bar{P}_1 M\, 0\, 0\, 0\, \bar{P}_1\, 0 \right] \\
\Pi_{47} &= \left[-\bar{P}_1 N_1\, 0\, 0\, 0\, 0\, 0 \right] \\
\Pi_{77} &= diag\left[-\bar{\tau} R_1 - \bar{\tau} R_2 - \bar{P}_1 - \varepsilon^{-1}\alpha_1^{-2} I\ - \varepsilon^{-1}\alpha_2^{-2} I - \bar{\tau} R_2 - 2\bar{\tau} I \right] \\
\bar{\tau} &= \tau^{-1}
\end{aligned}
$$

Then the generalized closed-loop system (8), (9) *satisfies H_∞ output tracking performance γ.*

Proof Let

$$\bar{P}_1 = P_1^{-1}, \quad \bar{P}_2 = P_2^{-1}$$

Then multipling simultaneously both sides of the matrix inequality $\bar{\Sigma} < 0$ by the diagonal matrix diag$\left[\bar{P}_1\ \bar{P}_1\ I\ \bar{P}_2\ \bar{P}_2\ I\ I\ I \right]$, we can get

$$
\begin{bmatrix}
\Pi_{11} & \Pi_{12} & I & BK_r\bar{P}_2 & 0 & 0 & 0 & \bar{P}_1 A^T + K^T B^T \\
* & \Pi_{22} & 0 & 0 & 0 & -\bar{P}_1 M & 0 & \bar{P}_1 A_\tau^T \\
* & * & -\varepsilon I & 0 & 0 & 0 & 0 & I \\
* & * & * & \Pi_{44} & -N_1\bar{P}_2 + N_2\bar{P}_2 & -\bar{P}_1 N_1 & 0 & \bar{P}_2 K_r^T B^T \\
* & * & * & * & \Pi_{55} & 0 & -\bar{P}_2 N_2 & 0 \\
* & * & * & * & * & -\tau^{-1} R_1 & 0 & 0 \\
* & * & * & * & * & * & -\tau^{-1} R_2 & 0 \\
* & * & * & * & * & * & * & -\tau^{-1} R_1^{-1}
\end{bmatrix} < 0
$$

where

$$
\begin{aligned}
\Pi_{11} &= \bar{P}_1 (A + BK)^T + (A + BK)\bar{P}_1 + \bar{P}_1 H^T P_1 H \bar{P}_1 + \bar{P}_1 Q_1 \bar{P}_1 + \varepsilon\alpha_1^2 \bar{P}_1 \bar{P}_1 \\
\Pi_{12} &= A_\tau \bar{P}_1 + \bar{P}_1 M \bar{P}_1 \\
\Pi_{22} &= \varepsilon\alpha_2^2 \bar{P}_1 \bar{P}_1 - \bar{P}_1 Q_1 \bar{P}_1 - \bar{P}_1 M \bar{P}_1 - \bar{P}_1 M^T \bar{P}_1 \\
\Pi_{44} &= \bar{P}_2 D^T + D\bar{P}_2 + \bar{P}_2 Q_2 \bar{P}_2 \\
\Pi_{55} &= \tau \bar{P}_2 D^T R_2 D \bar{P}_2 - \bar{P}_2 Q_2 \bar{P}_2
\end{aligned}
$$

Let

$$
\begin{aligned}
&\tilde{K} = K\bar{P}_1, \tilde{K}_r = K_r \bar{P}_2 \\
&\bar{Q}_1 = \bar{P}_1 Q_1 \bar{P}_1, \bar{Q}_2 = \bar{P}_2 Q_2 \bar{P}_2 \\
&\bar{M} = \bar{P}_1 M \bar{P}_1
\end{aligned}
$$

Then

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & I & B\tilde{K}_r & 0 & 0 & 0 & \bar{P}_1A^T + \tilde{K}^TB^T \\ * & \Pi_{22} & 0 & 0 & 0 & -\bar{P}_1M & 0 & \bar{P}_1A_\tau^T \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 & I \\ * & * & * & \Pi_{44} & -N_1\bar{P}_2 + N_2\bar{P}_2 & -\bar{P}_1N_1 & 0 & \tilde{K}_r^TB^T \\ * & * & * & * & \Pi_{55} & 0 & -\bar{P}_2N_2 & 0 \\ * & * & * & * & * & -\tau^{-1}R_1 & 0 & 0 \\ * & * & * & * & * & * & -\tau^{-1}R_2 & 0 \\ * & * & * & * & * & * & * & -\tau^{-1}R_1^{-1} \end{bmatrix} < 0$$

where

$$\Pi_{11} = \bar{P}_1A^T + \tilde{K}^TB^T + A\bar{P}_1 + B\tilde{K} + \bar{P}_1H^TP_1H\bar{P}_1 + \bar{Q}_1 + \varepsilon\alpha_1^2\bar{P}_1\bar{P}_1$$
$$\Pi_{12} = A_\tau\bar{P}_1$$
$$\Pi_{22} = \varepsilon\alpha_2^2\bar{P}_1\bar{P}_1 - \bar{Q}_1$$
$$\Pi_{44} = \bar{P}_2D^T + D\bar{P}_2 + \bar{Q}_2$$
$$\Pi_{55} = \tau\bar{P}_2D^TR_2D\bar{P}_2 - \bar{Q}_2$$

Then under zero initial conditions, we can consider the H_∞ indicator

$$\rho = \int_0^\infty [e^T(s)e(s) - \gamma^2v^T(s)v(s)]ds$$

so

$$e^T(t)e(t) - \gamma^2v^T(t)v(t) + \mathcal{L}V(x(t), x_r(t)) \le \bar{\eta}^T(t)\bar{\Pi}\bar{\eta}(t)$$

where

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & I & B\tilde{K}_r & -C_1F & \bar{P}_1A^T + \tilde{K}^TB^T & \Pi_{17} \\ * & \Pi_{22} & 0 & 0 & -C_2F & \bar{P}_1A_\tau^T & \Pi_{27} \\ * & * & -\varepsilon I & 0 & 0 & I & 0 \\ * & * & * & \bar{P}_2D^T + D\bar{P}_2 + \bar{Q}_2 & -N_1\bar{P}_2 + N_2\bar{P}_2 & \tilde{K}_r^TB^T & 0 \\ * & * & * & * & -\bar{Q}_2 + F^TF & 0 & \Pi_{57} \\ * & * & * & * & * & -\bar{\tau}R_1{}^{-1} & 0 \\ * & * & * & * & * & * & \Pi_{77} \end{bmatrix} < 0$$

Here

$$\Pi_{11} = \bar{P}_1A^T + \tilde{K}^TB^T + A\bar{P}_1 + B\tilde{K} + \bar{Q}_1 + C_1^TC_1$$
$$\Pi_{12} = A_\tau\bar{P}_1 + C_1^TC_2$$
$$\Pi_{22} = -\bar{Q}_1 + C_2^TC_2$$
$$\Pi_{17} = \left[0\,0\,\bar{P}_1H^T\ \bar{P}_1\,0\,0\,E\right]$$
$$\Pi_{27} = \left[0\,0\,0\,0\,\bar{P}_1\,0\,0\right]$$
$$\Pi_{57} = \left[0\,0\,0\,0\,0 - \bar{P}_2D^T\ 0\right]$$
$$\Pi_{77} = \mathrm{diag}\left\{-\bar{\tau}R_1 - \bar{\tau}R_2 - \bar{P}_1 - \varepsilon^{-1}\alpha_1^{-2}I - \varepsilon^{-1}\alpha_2^{-2}I - \bar{\tau}R_2^{-1} - \gamma^2I\right\}$$
$$\bar{\tau} = \tau^{-1}$$

Then through the inequality $(IR)\left(R^{-1}\right)(IR) \ge 0$ is equivalent to $-R^{-1} \le R - 2I$, we can prove the Theorem 4.1 . □

5 Numerical Example

In this section, numerical examples are given to illustrate that the proposed theoretical method is feasible.

Example 1 The relevant parameters of the random time-delay system (1), (2) are as follows

$$A=\begin{bmatrix}2&1.1&-1&2\\-0.5&-1.8&1.3&0.2\\0.2&-2.1&-0.5&1\\0.1&2&1.8&1.5\end{bmatrix},\quad A_\tau=\begin{bmatrix}-0&1.1&1.1&0.2\\-1.5&-1.2&1.2&1.1\\1.4&-1.2&-2.1&1\\1&1.1&1.8&-1.5\end{bmatrix},\quad H=\begin{bmatrix}-1.2&-0.1&0.1&0.2\\-0.5&0.4&1.6&0.5\\1.3&-1.6&0.5&0.1\\0.9&1&1.8&1.2\end{bmatrix},$$

$$D=\begin{bmatrix}-2.1&1&1.6&0.2\\-1.8&-2.8&0.2&0.5\\1.2&-0.1&-1.6&0.8\\0.2&0.1&0.8&-1.5\end{bmatrix},\quad B=\begin{bmatrix}1&-1&0&0\\2&1&1.1&0.1\\1&0.6&0.6&0.5\\1&0.1&1&0.5\end{bmatrix},\quad E=\begin{bmatrix}0.01\\0.20\\0.2\\0.1\end{bmatrix},\quad C_1=\begin{bmatrix}1&0&1&0\\1&1&1&0.6\end{bmatrix},$$

$$C_2=\begin{bmatrix}1&0&1&1\\1&0&1&1\end{bmatrix},\quad F=\begin{bmatrix}1&0.5&1&1\\1&0&1.1&1\end{bmatrix},\quad \tau=0.5,\quad \alpha_1=\alpha_2=0.2,$$

$$K=\begin{bmatrix}1.7&-3&-70&80\\9&10&-120&100\\-6&-19&250&-240\\-1&18&-70&4\end{bmatrix},\quad K_r=\begin{bmatrix}5.3&3.1&-5.2&0.71\\7.1&5&-6&2\\-15&-10&20&-3\\1&1&-3.2&0.5\end{bmatrix},$$

Using H_∞ output tracking design method for the stochastic time-delay system proposed in Theorem 3.1, using the LMI control toolbox to solve the matrix inequality (10), the solution matrix can be obtained is

$$P_1=\begin{bmatrix}153.4776&-113.0205&118.2967&-17.8939\\-113.0205&206.7780&-131.1372&4.4986\\118.2967&-131.1372&311.3364&-16.6175\\-17.8939&4.4986&-16.6175&31.6135\end{bmatrix}$$

$$P_2=10^3\times\begin{bmatrix}2.7816&0.4694&2.6990&1.3911\\0.4694&1.5261&1.1401&0.8413\\2.6990&1.1401&6.4259&3.2534\\1.3911&0.8413&3.2534&4.1505\end{bmatrix}$$

$$Q_1=10^3\times\begin{bmatrix}0.4833&-0.0251&-0.4119&0.0056\\-0.0251&0.4163&-0.2825&-0.2979\\-0.4119&-0.2825&1.6441&-0.4845\\0.0056&-0.2979&-0.4845&1.5044\end{bmatrix}$$

$$Q_2=10^3\times\begin{bmatrix}4.2361&0.5642&-1.7152&-0.8451\\0.5642&5.1765&0.5647&0.1564\\-1.7152&0.5647&3.2389&-1.3641\\-0.8451&0.1564&-1.3641&3.8856\end{bmatrix}$$

$$R_1=\begin{bmatrix}6.5803&-5.8864&4.8925&-0.8778\\-5.8864&8.1701&-5.9815&0.5194\\4.8925&-5.9815&5.5893&-0.4677\\-0.8778&0.5194&-0.4677&0.2369\end{bmatrix}$$

$$R_2=10^3\times\begin{bmatrix}0.5526&0.0500&0.6127&0.2789\\0.0500&0.2095&0.1651&0.1518\\0.6127&0.1651&1.2327&0.6506\\0.2789&0.1518&0.6506&0.9099\end{bmatrix}$$

$$M = \begin{bmatrix} 0.7146 & 0.1395 & 0.1066 & -0.1570 \\ -2.3178 & 5.2264 & -3.1670 & 0.0312 \\ 2.4612 & -2.6956 & 2.3350 & -0.2840 \\ -1.5296 & -1.2299 & -0.0110 & 0.4434 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} -0.5263 & 2.2511 & -1.1853 & -0.0775 \\ -0.2998 & 1.6682 & -0.8913 & -0.0781 \\ -1.7187 & 6.3412 & -3.3680 & -0.1676 \\ -0.2062 & 1.2850 & -0.6820 & -0.0627 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} -0.0158 & 0.0696 & -0.0409 & -0.0030 \\ -0.0385 & 0.0683 & -0.0492 & 0.0009 \\ 0.0019 & 0.0502 & -0.0226 & -0.0041 \\ -0.0336 & 0.0975 & -0.0591 & -0.0018 \end{bmatrix}$$

$$\varepsilon = 2235.5$$

Thus we know that the feasible solution exists, that is, when the time-delay $\tau \leq 0.5$, the closed-loop system satisfies H_∞ output tracking performance γ in the mean square sense, which shows that our proposed method is very effective.

References

1. Bucy R.S., Joseph P.D.: Filtering for Stochastic Processes with Applications to Guidance[M]. Interscience, New York (1968)
2. Gelb A.: Applied Optimal Estimation[M]. Cambridge University Press, Cambridge (1974)
3. Hsiao, F., Pan, S.: Robust kalman filter synthesis for uncertain multiple time-delay stochastic systems [J]. J. Dynam. Syst. Meas. Control **118**, 803–808 (1996)
4. Hsieh, C., Skelton, R.E.: All covariance vontrollers for linear discrete time systems [J]. IEEE Trans. Autom. Control **35**, 908–915 (1990)
5. Ito, K., Rozovskii, B.: Approximation of the kushner equation for nonlinear filtering [J]. Soc. Indust. Appl. Math. J. Control. Opt. **38**, 893–915 (2000)
6. Jazwinski A.H.: Stochastic Processes and Filtering Theory[M]. Academic Press, New York (1970)
7. Wang, Z., Burnham, K.J.: Robust filtering for a class of stochastic uncertain nonlinear time-delay systems via exponential state estimation [J]. IEEE Trans. Signal Process. **49**, 794–804 (2001)
8. Shaikhet, L.: Stability of stochastic hereditary systems with markov switching [J]. Theory Stochastic Process. **2**, 180–184 (1996)
9. Willems, J.L., Willems, J.C.: Feedback stabilizability for stochastic systems with state and control dependent noise [J]. Automatica **12**, 343–357 (1976)
10. Basin, M.V., Rodkina, A.E.: On delay-dependent stability for a class of nonlinear stochastic systems with multiple state delays [J]. Nonlinear Anal. Theory Meth. Appl. **68**, 2147–2157 (2008)
11. Huang, L., Deng, F.: Robust exponential stabilization of stochastic largescale delay systems [J]. In: Proceedings of 2007 IEEE international conference on control and automation, pp. 107–112 (2007)
12. Luenberger, D.G.: Observers for multi-variable systems [J]. IEEE Trans. Auto. Con. **11**, 190–197 (1996)
13. He, Y., Wang, Q.G., Lin, C., et al.: Delay-range-dependent stability for systems with timevarying delay [J]. Automatica **43**, 371–376 (2007)
14. Li, H., Chen, B., Zhou, Q., et al.: Delay dependent robust stability for stochastic time-delay systems with polytopic uncertainties [J]. Int. J. Robust Nonlinear Control **18**, 1482–1492 (2008)
15. Li, H., Chen, B., Zhou, Q., et al.: A delay dependent approach to robusth_∞ control for uncertain stochastic systems with state and input delays [J]. Circ. Syst. Signal Process. **28**, 169–183 (2009)

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