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Enhanced Monogamy Relations in Multiqubit Systems

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Abstract
We investigate the monogamy relations of multipartite entanglement in terms of the αth power of concurrence, entanglement of formation, negativity and Tsallis-q entanglement. Enhanced new monogamy relations of multipartite entanglement with tighter lower bounds than the existing monogamy relations are presented, together with detailed examples showing the tightness. These monogamy relations give rise to finer characterization of the entanglement distributions among the subsystems of a multipartite system.

Keywords Monogamy relations · Multipartite entanglement · Bipartite entanglement measure

1 Introduction

Quantum entanglement is an essential feature of quantum mechanics which can enhance quantum technologies such as communication, cryptography and computing beyond classical limitations. A key property of multipartite entanglement is the monogamous relations [1, 2], which are important correlations with fundamental differences from the classical

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ones. They restrict the sharability of quantum correlations in multipartite quantum states. For example, for three qubit quantum systems, denoted by A, B and C, if A and B are in a maximally entangled state, then A cannot be entangled with C at all. This indicates that it should obey some trade-off relation on the amount of entanglement between the pairs AB and AC.

The monogamy relations give rise to the quantification and characterization of entanglement distribution among the multipartite systems. The first mathematical characterization of the monogamy of entanglement (MOE) was expressed as a form of inequality for three-qubit state [3]: the entanglement $E_{A|BC}$ between A and BC, the entanglement E_{AB} (E_{AC}) between A and B (C) satisfy $E_{A|BC} \geq E_{AB} + E_{AC}$. Further, Coffman, Kundu and Wootters (CKW) proposed that the squared concurrence also satisfies the monogamy relations for multiqubit states [2]. Osborne and Verstraete [4] proved the CKW monogamy inequality, which quantifies the frustration of entanglement between different parties. Later, the monogamy inequalities are generalized to other entanglement measures [5–10]. The monogamy property is of importance in many quantum information tasks, particularly, in quantum cryptography [11]. In the context of quantum cryptography, such monogamy property quantifies how much information an eavesdropper could potentially obtain about the secret key to be extracted. In the context of condensed-matter physics [12], the monogamy property gives rise to the frustration effects observed in, e.g., Heisenberg antiferromagnets. In addition to the monogamy of entanglement, the concept of monogamy has also appeared when discussing the violation of Bell's inequalities [13]. They also play an important role in the security analysis of quantum key distribution [14], even in black-hole physics [15].

In Ref. [4, 6] the authors showed that the αth concurrence and the convex-roof extended negativity (CREN) satisfy the monogamy inequalities in multiqubit systems for $\alpha \geq 2$. It has also been shown that the αth entanglement of formation (EoF), the Tsallis-q entanglement and the Rényi-α entanglement satisfies the monogamy relations when $\alpha \geq \sqrt{2}$, $\alpha \geq 1$, respectively [16–20].

In this paper, we establish some new monogamy relations of multipartite entanglement for arbitrary quantum states, based on the α-th power of the bipartite entanglement. We show that these new monogamy relations are tighter than the existing ones given in [16, 21–28].

2 Enhanced Monogamy Relations for Concurrence

We first consider the monogamy inequalities for concurrence. For a bipartite pure state $|\psi\rangle_{AB}$ in Hilbert space $\mathbb{H}_A \otimes \mathbb{H}_B$, the concurrence is defined as $C(|\psi\rangle_{AB}) = \sqrt{2(1 - \mathrm{tr}(\rho_A^2))}$ with $\rho_A = \mathrm{tr}_B(|\psi\rangle_{AB}\langle\psi|)$ [29, 30]. The concurrence for a bipartite mixed state ρ_{AB} is defined by the convex roof extension, $C(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$, with the minimum taking over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\sum p_i = 1$ and $p_i \geq 0$. For an N-qubit state $\rho_{A|B_1\cdots B_{N-1}} \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, the concurrence $C(\rho_{A|B_1\cdots B_{N-1}})$ of the state $\rho_{A|B_1\cdots B_{N-1}}$ under bipartite partition A and $B_1 \cdots B_{N-1}$ satisfies [17]

$$C^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \geq C^\alpha(\rho_{AB_1}) + C^\alpha(\rho_{AB_2}) + \cdots + C^\alpha(\rho_{AB_{N-1}}), \tag{1}$$

for $\alpha \geq 2$, where ρ_{AB_j} denote the two-qubit reduced density matrices of subsystems AB_j, $j = 1, 2, \ldots, N-1$. The relation (1) is further improved, with the conditions of Theorem 1 in [16], as follows,

$$\begin{aligned} &C^\alpha(\rho_{A|B_1\cdots B_{N-1}}) \\ \geq\ & C^\alpha(\rho_{AB_1}) + \left(2^{\frac{\alpha}{2}}-1\right) C^\alpha(\rho_{AB_2}) + \cdots + \left(2^{\frac{\alpha}{2}}-1\right)^{m-1} C^\alpha(\rho_{AB_m}) \\ &+ \left(2^{\frac{\alpha}{2}}-1\right)^{m+1} [C^\alpha(\rho_{AB_{m+1}}) + \cdots + C^\alpha(\rho_{AB_{N-2}})] + \left(2^{\frac{\alpha}{2}}-1\right)^{m} C^\alpha(\rho_{AB_{N-1}}), \end{aligned} \quad (2)$$

where $\alpha \geq 2$.

Generally, a bipartite entanglement measure E is said to be monogamous if

$$E^{\alpha_c}(\rho_{A|B_1\cdots B_{N-1}}) \geq \sum_{i=1}^{N-1} E^{\alpha_c}(\rho_{AB_i}), \quad (3)$$

where $\rho_{A|B_i} = \mathrm{tr}_{B_1\cdots B_{i-1}B_{i+1}\cdots B_{N-1}}(\rho_{A|B_1\cdots B_{N-1}})$, α_c is the minimum exponent for E^{α_c} to be monogamous [31]. It has been shown in [31] that for $0 \leq x \leq 1$ and $t \geq 1$,

$$(1+x)^t \geq 1 + \frac{t}{2}(x - x^t) + (2^t - 1)x^t \geq 1 + (2^t-1)x^t. \quad (4)$$

Lemma 1 *For any $2 \otimes 2 \otimes 2^{N-2}$ mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B \otimes \mathbb{H}_C$, assuming that $C_{AB} \geq C_{AC}$, we have*

$$C^\alpha_{A|BC} \geq C^\alpha_{AB} + \frac{\alpha}{4} C^2_{AC}(C^{\alpha-2}_{AB} - C^{\alpha-2}_{AC}) + (2^{\frac{\alpha}{2}} - 1) C^\alpha_{AC}, \quad (5)$$

for all $\alpha \geq 2$, where N stands for the number of qubit systems, A and B are qubit systems, C is a 2^{N-2}-dimensional qudit system, consisting of $N-2$ qubit systems.

Proof It has been shown that $C^2_{A|BC} \geq C^2_{AB} + C^2_{AC}$ for arbitrary $2 \otimes 2 \otimes 2^{N-2}$ tripartite state $\rho_{A|BC}$ [4, 32]. In terms of $C_{AB} \geq C_{AC}$, we have

$$\begin{aligned} C^\alpha_{A|BC} &\geq (C^2_{AB} + C^2_{AC})^{\frac{\alpha}{2}} \\ &= C^\alpha_{AB}\left(1 + \frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}} \\ &\geq C^\alpha_{AB}\left[1 + \frac{\alpha}{4}\frac{C^2_{AC}}{C^2_{AB}} + [2^{\frac{\alpha}{2}} - (1 + \frac{\frac{\alpha}{2}}{2})]\left(\frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}}\right] \\ &= C^\alpha_{AB} + \frac{\alpha}{4}C^2_{AC}(C^{\alpha-2}_{AB} - C^{\alpha-2}_{AC}) + (2^{\frac{\alpha}{2}} - 1)C^\alpha_{AC}, \end{aligned}$$

where the second inequality is due to (4). Moreover, if $C_{AB} = 0$, then $C_{AC} = 0$. That is to say the lower bound becomes trivially zero. □

From Lemma 1 we have the following proposition.

Proposition 1 *For an N-qubit mixed state* $\rho \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, *if* $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ *for* $i = 1, 2, \cdots, m$, *and* $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ *for* $j = m+1, \cdots, N-2$ $(1 \leq m \leq N-3)$, *we have*

$$C^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{m} h^{i-1}(C^{\alpha}_{AB_i} + J_{AB_i}) + h^m \sum_{j=m+1}^{N-2} (hC^{\alpha}_{AB_j} + \bar{J}_{AB_j}) + h^m C^{\alpha}_{AB_{N-1}}, \quad (6)$$

for $N \geq 4$ *and* $\alpha \geq 2$, *where* $h = 2^{\frac{\alpha}{2}} - 1$, $J_{AB_i} = \frac{\alpha}{4} C^2_{A|B_{i+1}\cdots B_{N-1}}(C^{\alpha-2}_{AB_i} - C^{\alpha-2}_{A|B_{i+1}\cdots B_{N-1}})$, $\bar{J}_{AB_j} = \frac{\alpha}{4} C^2_{AB_j}(C^{\alpha-2}_{A|B_{j+1}\cdots B_{N-1}} - C^{\alpha-2}_{AB_j})$.

Proof From the inequality (5), we have

$$\begin{aligned}
&C^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \\
&\geq C^{\alpha}_{AB_1} + hC^{\alpha}_{A|B_2\cdots B_{N-1}} + \frac{\alpha}{4} C^2_{A|B_2\cdots B_{N-1}}(C^{\alpha-2}_{AB_1} - C^{\alpha-2}_{A|B_2\cdots B_{N-1}}) \\
&\geq C^{\alpha}_{AB_1} + h[C^{\alpha}_{AB_2} + hC^{\alpha}_{A|B_3\cdots B_{N-1}} + \frac{\alpha}{4} C^2_{A|B_3\cdots B_{N-1}}(C^{\alpha-2}_{AB_2} - C^{\alpha-2}_{A|B_3\cdots B_{N-1}})] \\
&\quad + \frac{\alpha}{4} C^2_{A|B_2\cdots B_{N-1}}(C^{\alpha-2}_{AB_1} - C^{\alpha-2}_{A|B_2\cdots B_{N-1}}) \\
&\geq \cdots \\
&\geq \sum_{i=1}^{m} h^{i-1}(C^{\alpha}_{AB_i} + J_{AB_i}) + h^m C^{\alpha}_{A|B_{m+1}\cdots B_{N-1}}. \qquad (7)
\end{aligned}$$

Similarly, as $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, we get

$$\begin{aligned}
&C^{\alpha}_{A|B_{m+1}\cdots B_{N-1}} \\
&\geq C^{\alpha}_{A|B_{m+2}\cdots B_{N-1}} + hC^{\alpha}_{AB_{m+1}} + \frac{\alpha}{4} C^2_{AB_{m+1}}(C^{\alpha-2}_{A|B_{m+2}\cdots B_{N-1}} - C^{\alpha-2}_{AB_{m+1}}) \\
&\geq \sum_{j=m+1}^{N-2} (hC^{\alpha}_{AB_j} + \bar{J}_{AB_j}) + C^{\alpha}_{AB_{N-1}}. \qquad (8)
\end{aligned}$$

By combining (7) and (8), we come to the conclusion. □

Remark We have assumed $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ and $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ in Proposition 1. These constraints are most generally given by relabeling the subsystems. Due to the conditions of inequality (4), the second inequality of (7) and (8) hold, respectively. As J_{AB_i}s and $\bar{J}_{AB_j}$s are great than 0, we obtain the tighter lower bound than corresponding monogamy inequalities in [16]. Particularly, we have the following proposition.

Proposition 2 *For any N-qubit mixed state, if* $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$, *for* $i = 1, 2, \cdots, N-2$, *we have*

$$C^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-2} h^{i-1}(C^{\alpha}_{AB_i} + J_{AB_i}) + h^{N-2} C^{\alpha}_{AB_{N-1}}, \quad (9)$$

for $\alpha \geq 2$ *and* $N \geq 3$, *where* $h = 2^{\frac{\alpha}{2}} - 1$, $J_{AB_i} = \frac{\alpha}{4} C^2_{A|B_{i+1}\cdots B_{N-1}}(C^{\alpha-2}_{AB_i} - C^{\alpha-2}_{A|B_{i+1}\cdots B_{N-1}})$.

Proof From the inequality (5), we have

$$\begin{aligned}
&C^{\alpha}_{A|B_1B_2\cdots B_{N-1}}\\
\geq\ &C^{\alpha}_{AB_1}+hC^{\alpha}_{A|B_2\cdots B_{N-1}}+\frac{\alpha}{4}C^2_{A|B_2\cdots B_{N-1}}(C^{\alpha-2}_{AB_1}-C^{\alpha-2}_{A|B_2\cdots B_{N-1}})\\
\geq\ &C^{\alpha}_{AB_1}+h[C^{\alpha}_{AB_2}+hC^{\alpha}_{A|B_3\cdots B_{N-1}}+\frac{\alpha}{4}C^2_{A|B_3\cdots B_{N-1}}(C^{\alpha-2}_{AB_2}-C^{\alpha-2}_{A|B_3\cdots B_{N-1}})]\\
&+\frac{\alpha}{4}C^2_{A|B_2\cdots B_{N-1}}(C^{\alpha-2}_{AB_1}-C^{\alpha-2}_{A|B_2\cdots B_{N-1}})\\
\geq\ &\cdots\\
\geq\ &C^{\alpha}_{AB_1}+hC^{\alpha}_{AB_2}+\cdots+h^{N-2}C^{\alpha}_{AB_{N-1}}+h^{N-3}\cdot\frac{\alpha}{4}C^2_{AB_{N-1}}(C^{\alpha-2}_{AB_{N-2}}-C^{\alpha-2}_{AB_{N-1}})\\
&+\cdots+h\cdot\frac{\alpha}{4}C^2_{A|B_3\cdots B_{N-1}}(C^{\alpha-2}_{AB_2}-C^{\alpha-2}_{A|B_3\cdots B_{N-1}})+\frac{\alpha}{4}C^2_{A|B_2\cdots B_{N-1}}(C^{\alpha-2}_{AB_1}-C^{\alpha-2}_{A|B_2\cdots B_{N-1}}).
\end{aligned}\tag{10}$$

According to the denotation of J_{AB_i}, we obtain the result. □

Example 1 Let us consider the three-qubit state $|\psi\rangle$ in the generalized Schmidt decomposition form [33, 34],

$$|\psi\rangle=\lambda_0|000\rangle+\lambda_1e^{i\varphi}|100\rangle+\lambda_2|101\rangle+\lambda_3|110\rangle+\lambda_4|111\rangle,\tag{11}$$

where $\lambda_i\geq 0$, $i=0,1,2,3,4$, $\sum_{i=0}^{4}\lambda_i^2=1$. From the definition of concurrence, we have $C_{A|BC}=2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $C_{AB}=2\lambda_0\lambda_2$ and $C_{AC}=2\lambda_0\lambda_3$. Set $\lambda_0=\lambda_2=\frac{1}{2}$, $\lambda_1=\lambda_3=\lambda_4=\frac{\sqrt{6}}{6}$, one has $C_{A|BC}=\sqrt{\frac{7}{12}}$, $C_{AB}=\frac{1}{2}$, $C_{AC}=\frac{\sqrt{6}}{6}$. Then $C^{\alpha}_{A|BC}=(\frac{7}{12})^{\frac{\alpha}{2}}\geq C^{\alpha}_{AB}+hC^{\alpha}_{AC}+\frac{\alpha}{4}C^2_{AC}(C^{\alpha-2}_{AB}-C^{\alpha-2}_{AC})=(\frac{1}{2})^{\alpha}+h\cdot(\frac{\sqrt{6}}{6})^{\alpha}+\frac{\alpha}{4}\cdot(\frac{\sqrt{6}}{6})^2[(\frac{1}{2})^{\alpha-2}-(\frac{\sqrt{6}}{6})^{\alpha-2}]$. While the result in [16] is $C^{\alpha}_{AB}+hC^{\alpha}_{AC}=(\frac{1}{2})^{\alpha}+h\cdot(\frac{\sqrt{6}}{6})^{\alpha}$. One can see that our lower bound is tighter than theirs in [16], see Fig. 1.

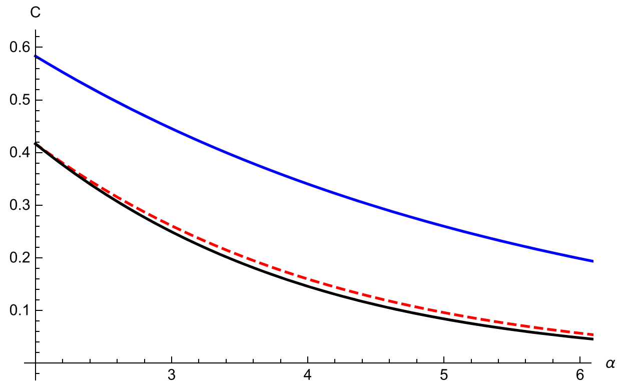

Fig. 1 The axis C represents the concurrence of $|\psi\rangle$, which is a function of α. The solid blue line represents the lower bound of concurrence of $|\psi\rangle$ in Example 1, the dashed red line represents the lower bound from our result, the solid black line represents lower bound from the result in [16]

3 Enhanced Monogamy Relations for EoF

In quantifying quantum entanglement, the entanglement of formation (EoF) [35, 36] is a well defined important measure of entanglement for bipartite systems. Let $\mathbb{H}_A$ and $\mathbb{H}_B$ be m and n dimensional ($m \leq n$) vector spaces, respectively. The EoF of a pure state $|\psi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$ is defined by

$$E(|\psi\rangle) = S(\rho_A), \tag{12}$$

where $\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$ and $S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$. For a bipartite mixed state $\rho_{AB} \in \mathbb{H}_A \otimes \mathbb{H}_B$, the entanglement of formation is given by,

$$E(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{13}$$

with the minimum taking over all possible pure state decompositions of ρ_{AB}.

Denote $f(x) = H\left(\frac{1+\sqrt{1-x}}{2}\right)$, where $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$. From (12) and (13), one has $E(|\psi\rangle) = f\left(C^2(|\psi\rangle)\right)$ for $2 \otimes m$ $(m \geq 2)$ pure state $|\psi\rangle$, and $E(\rho) = f\left(C^2(\rho)\right)$ for two-qubit mixed state ρ [30]. It is obvious that $f(x)$ is a monotonically increasing function for $0 \leq x \leq 1$. The function $f(x)$ satisfies the following relations:

$$f^{\sqrt{2}}(x^2+y^2) \geq f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2), \tag{14}$$

where $f^{\sqrt{2}}(x^2+y^2) = [f(x^2+y^2)]^{\sqrt{2}}$.

From [2] one sees that EoF does not satisfy the inequality $E_{A|BC} \geq E_{AB} + E_{AC}$. In [37] the authors showed that EoF is a monotonic function satisfying $E^2(C^2_{A|B_1B_2\cdots B_{N-1}}) \geq E^2(\sum_{i=1}^{N-1} C^2_{AB_i})$. For N-qubit systems, one has [17]

$$E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \geq E^{\alpha}_{AB_1} + E^{\alpha}_{AB_2} + \cdots + E^{\alpha}_{AB_{N-1}}, \tag{15}$$

where $E_{A|B_1B_2\cdots B_{N-1}}$ is the EoF of the state $\rho_{A|B_1\cdots B_{N-1}}$, E_{AB_i} is the EoF of the mixed state $\rho_{AB_i} = \mathrm{tr}_{B_1B_2\cdots B_{i-1},B_{i+1}\cdots B_{N-1}}(\rho)$, $i = 1, 2, \cdots, N-1$, $\alpha \geq \sqrt{2}$. In particular, we have following relations.

Lemma 2 *For any* $2 \otimes 2 \otimes 2^{N-2}$ *mixed state* $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B \otimes \mathbb{H}_C$*, if* $C_{AB} \geq C_{AC}$*, the following inequality holds for* $\alpha \geq \sqrt{2}$*,*

$$E^{\alpha}_{A|BC} \geq E^{\alpha}_{AB} + (2^t - 1)E^{\alpha}_{AC} + \frac{t}{2} E^{\sqrt{2}}_{AC}(E^{\alpha-\sqrt{2}}_{AB} - E^{\alpha-\sqrt{2}}_{AC}), \tag{16}$$

where $t = \frac{\alpha}{\sqrt{2}}$.

Proof The proof is similar to the proof of Lemma 1. □

Note that, for any N-qubit mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, $E^{\alpha}_{A|B_1B_2\cdots B_{N-1}}(\rho)$ no longer has similar relation like (6) in Proposition 1. However, the following proposition holds.

Proposition 3 *For any N-qubit mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for all $i = 1, 2, \cdots, N-2$ and $\alpha \geq \sqrt{2}$, we have*

$$E^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-2} h^{i-1}(E^{\alpha}_{AB_i} + R_{AB_i}) + h^{N-2}E^{\alpha}_{AB_{N-1}}, \tag{17}$$

where $h = 2^t - 1$, $t = \frac{\alpha}{\sqrt{2}}$, and $R_{AB_i} = \frac{t}{2}(E^{\sqrt{2}}_{AB_{i+1}} + \cdots + E^{\sqrt{2}}_{AB_{N-1}})(E^{\alpha-\sqrt{2}}_{AB_i} - E^{\alpha-\sqrt{2}}_{A|B_{i+1}\cdots B_{N-1}})$ with $i = 1, 2, \cdots, N-1$.

Proof For $\alpha \geq \sqrt{2}$, we have

$$\begin{aligned} & f^{\alpha}(x^2 + y^2) \\ = & \left(f^{\sqrt{2}}(x^2 + y^2)\right)^t \\ \geq & \left(f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2)\right)^t \\ \geq & f^{\alpha}(x^2) + (2^t - 1)f^{\alpha}(y^2) \\ & + \frac{t}{2} f^{\sqrt{2}}(y^2)[f^{\alpha-\sqrt{2}}(x^2) - f^{\alpha-\sqrt{2}}(y^2)], \end{aligned} \tag{18}$$

where the first inequality is due to the inequality (14), and without loss of generality, we assume $x^2 \geq y^2$, using the monotonicity of $f(x)$ and inequality (4), the second inequality is obtained.

Let $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_N-1}$ be the optimal decomposition of $E_{A|B_1B_2\cdots B_{N-1}}(\rho)$ for the N-qubit mixed state ρ, we have

$$\begin{aligned} & E_{A|B_1B_2\cdots B_{N-1}}(\rho) \\ & = \sum_i p_i E_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle) \\ & = \sum_i p_i f\left(C^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right) \\ & \geq f\left(\sum_i p_i C^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right) \\ & \geq f\left(\left[\sum_i p_i C_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right]^2\right) \\ & \geq f\left(C^2_{A|B_1B_2\cdots B_{N-1}}(\rho)\right), \end{aligned} \tag{19}$$

where the first inequality is due to that $f(x)$ is a convex function. The second inequality is due to the Cauchy-Schwarz inequality: $(\sum_i x_i^2)^{\frac{1}{2}}(\sum_i y_i^2)^{\frac{1}{2}} \geq \sum_i x_i y_i$, with $x_i = \sqrt{p_i}$

and $y_i = \sqrt{p_i} C_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)$. Due to the definition of concurrence and that $f(x)$ is a monotonically increasing function, we obtain the third inequality. Therefore, we have

$$\begin{aligned}
& E^{\alpha}_{A|B_1B_2\cdots B_{N-1}}(\rho) \\
\geq\ & f^{\alpha}(C^2_{AB_1} + C^2_{AB_2} + \cdots + C^2_{AB_{N-1}}) \\
\geq\ & f^{\alpha}(C^2_{AB_1}) + h \cdot f^{\alpha}(C^2_{AB_2} + \cdots + C^2_{AB_{N-1}}) \\
& + \frac{t}{2} f^{\sqrt{2}}(C^2_{AB_2} + \cdots + C^2_{AB_{N-1}})[f^{\alpha-\sqrt{2}}(C^2_{AB_1}) - f^{\alpha-\sqrt{2}}(C^2_{AB_2} + \cdots + C^2_{AB_{N-1}})] \\
\geq\ & f^{\alpha}(C^2_{AB_1}) + h \cdot f^{\alpha}(C^2_{AB_2} + \cdots + C^2_{AB_{N-1}}) \\
& + \frac{t}{2}[f^{\sqrt{2}}(C^2_{AB_2}) + \cdots + f^{\sqrt{2}}(C^2_{AB_{N-1}})] \cdot [f^{\alpha-\sqrt{2}}(C^2_{AB_1}) - f^{\alpha-\sqrt{2}}(C^2_{A|B_2\cdots B_{N-1}})] \\
\geq\ & f^{\alpha}(C^2_{AB_1}) + h \cdot f^{\alpha}(C^2_{AB_2}) + \cdots + h^{N-2} \cdot f^{\alpha}(C^2_{AB_{N-1}}) \\
& + h^{(N-3)} \cdot \frac{t}{2} f^{\sqrt{2}}(C^2_{AB_{N-1}})[f^{\alpha-\sqrt{2}}(C^2_{AB_{N-2}}) - f^{\alpha-\sqrt{2}}(C^2_{AB_{N-1}})] \\
& + \cdots + \frac{t}{2}[f^{\sqrt{2}}(C^2_{AB_2}) + \cdots + f^{\sqrt{2}}(C^2_{AB_{N-1}})] \cdot [f^{\alpha-\sqrt{2}}(C^2_{AB_1}) - f^{\alpha-\sqrt{2}}(C^2_{A|B_2\cdots B_{N-1}})] \\
\geq\ & E^{\alpha}_{AB_1} + hE^{\alpha}_{AB_2} + \cdots + h^{N-2}E^{\alpha}_{AB_N-1} + h^{N-3} \cdot \frac{t}{2}(E^{\sqrt{2}}_{AB_{N-1}})[E^{\alpha-\sqrt{2}}_{AB_{N-2}} - E^{\alpha-\sqrt{2}}_{AB_{N-1}}] + \cdots \\
& + \frac{t}{2}(E^{\sqrt{2}}_{AB_2} + \cdots + E^{\sqrt{2}}_{AB_{N-1}}) \cdot (E^{\alpha-\sqrt{2}}_{AB_1} - E^{\alpha-\sqrt{2}}_{A|B_2\cdots B_{N-1}}),
\end{aligned} \tag{20}$$

where we have used the monogamy inequality (15) to obtain the first inequality. By using the relation (14) and the monotonicity of the function $f^{\sqrt{2}}(x)$, we get the third and the fourth inequalities. Since for any $2\otimes 2$ quantum state ρ_{AB_i}, $E(\rho_{AB_i}) = f\left[C^2(\rho_{AB_i})\right]$, from (19) one gets the last inequality. □

Since $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$, $(i = 1, 2, \cdots, N-2)$, $R_{AB_i} \geq 0$ holds, and our results are tighter than (8) in [16].

Example 2 Let us consider the three-qubit state $|\psi\rangle$ in Example 1 again. Set $\lambda_0 = \lambda_2 = \frac{1}{2}$ and $\lambda_1 = \lambda_3 = \lambda_4 = \frac{\sqrt{6}}{6}$ in (11). One has $E^{\alpha}_{A|BC} = (0.674027)^{\alpha}$, $E^{\alpha}_{AB} + hE^{\alpha}_{AC} + \frac{\alpha}{2\sqrt{2}}(E^{\sqrt{2}}_{AC})(E^{\alpha-\sqrt{2}}_{AB} - E^{\alpha-\sqrt{2}}_{AC}) = (0.354579)^{\alpha} + h \cdot (0.258403)^{\alpha} + \frac{\alpha}{2\sqrt{2}} \cdot (0.258403)^{\sqrt{2}}[(0.354579)^{\alpha-\sqrt{2}} - (0.258403)^{\alpha-\sqrt{2}}]$. While the result in [16] gives $E^{\alpha}_{AB} + hE^{\alpha}_{AC} = (0.354579)^{\alpha} + h \cdot (0.258403)^{\alpha}$. We can verify that our result is better than the corresponding result in [16], see Fig. 2.

4 Enhanced Monogamy Relations for Negativity

Another well known quantifier of bipartite entanglement is the negativity, which is based on the positive partial transposition (PPT) criterion. Given a bipartite state ρ_{AB} in $\mathbb{H}_A \otimes \mathbb{H}_B$, the negativity is defined by [38] $N(\rho_{AB}) = (||\rho^{T_A}_{AB}||_1 - 1)/2$, where $\rho^{T_A}_{AB}$ is the partial transposed matrix of ρ_{AB} with respect to the subsystem A, $||\cdot||_1$ is the trace norm. The negativity is a convex function of ρ_{AB}. For convenience, we use $N(\rho_{AB}) = ||\rho^{T_A}_{AB}||_1 - 1$ [6]. For any bipartite pure state $|\psi\rangle_{AB}$, the negativity $N(\rho_{AB})$ is given by

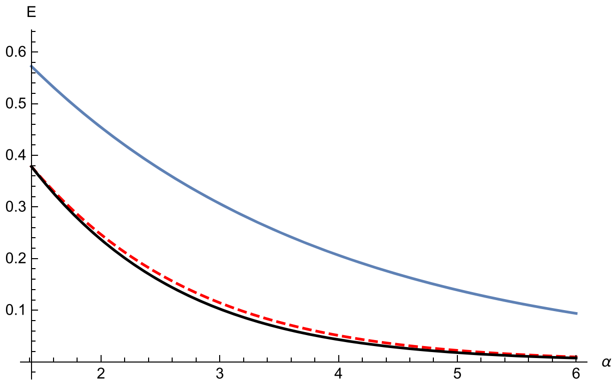

Fig. 2 The axis E represents the EoF of the state $|\psi\rangle$, which is a function of α. The solid blue line represents the lower bounds of EoF of the state $|\psi\rangle$ in Example 2, the dashed red line represents the lower bound from our result, and the solid black line represents the lower bound from the result in [16]

$N(|\psi\rangle_{AB}) = 2\sum_{i<j}\sqrt{\lambda_i\lambda_j} = (\mathrm{tr}\sqrt{\rho_A})^2 - 1$, where λ_i are the eigenvalues of the reduced density matrix of $|\psi\rangle_{AB}$. For a mixed state ρ_{AB}, the convex-roof extended negativity (CREN) is defined as

$$N_c(\rho_{AB}) = \min\sum_i p_i N(|\psi_i\rangle_{AB}), \tag{21}$$

where the minimum is taken over all possible pure state decompositions $\{p_i,\ |\psi_i\rangle_{AB}\}$ of ρ_{AB}. CREN gives a perfect discrimination of positive partial transposed bound entangled states and separable states in any bipartite quantum systems [39, 40].

Notice that there exists a relationship between CREN and concurrence. For any bipartite pure state $|\psi\rangle_{AB}$ in a $d\otimes d$ quantum system with Schmidt rank 2, $|\psi\rangle_{AB} = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$. One has $N(|\psi\rangle_{AB}) = \| \ |\psi\rangle\langle\psi|^{T_B} \ \|_1 - 1 = 2\sqrt{\lambda_0\lambda_1} = \sqrt{2(1-\mathrm{tr}\rho_A^2)} = C(|\psi\rangle_{AB})$. It follows that for any two-qubit mixed state $\rho_{AB} = \sum p_i|\psi_i\rangle_{AB}\langle\psi_i|$,

$$\begin{aligned} N_c(\rho_{AB}) &= \min\sum_i p_i N(|\psi_i\rangle_{AB}) \\ &= \min\sum_i p_i C(|\psi_i\rangle_{AB}) \\ &= C(\rho_{AB}). \end{aligned} \tag{22}$$

Here $N_{cAB} = N_c(\rho_{AB})$, then we have the following result.

Proposition 4 *For an N-qubit mixed state, if $N_{cAB_i} \geq N_{cAB_i\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $N_{cAB_j} \leq N_{cAB_j\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$ ($1 \leq m \leq N-3$, $N \geq 4$), the following holds for all $\alpha \geq 2$,*

$$N_{c\,A|B_1B_2\cdots B_{N-1}}^{\alpha} \geq \sum_{i=1}^{m} h^{i-1}(N_{c\,AB_i}^{\alpha} + Q_{AB_i}) + h^m \sum_{j=m+1}^{N-2} (hN_{c\,AB_j}^{\alpha} + \bar{Q}_{AB_j}) + h^m N_{c\,AB_{N-1}}^{\alpha}, \tag{23}$$

where $h = 2^{\frac{\alpha}{2}} - 1$, $Q_{AB_i} = \frac{\alpha}{4}N_{c\,A|B_{i+1}\cdots B_{N-1}}^{2}\left(N_{c\,AB_i}^{(\alpha-2)} - N_{c\,A|B_{i+1}\cdots B_{N-1}}^{(\alpha-2)}\right)$, $\bar{Q}_{AB_j} = \frac{\alpha}{4}N_{c\,AB_j}^{2}\left(N_{c\,A|B_{j+1}\cdots B_{N-1}}^{(\alpha-2)} - N_{c\,AB_j}^{(\alpha-2)}\right)$.

Proof From the inequality (5), we have

$$\begin{aligned}
&N_{c}{}^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \\
&\geq N_{c}{}^{\alpha}_{AB_1} + hN_{c}{}^{\alpha}_{A|B_2\cdots B_{N-1}} + \frac{\alpha}{4}N_{c}{}^{2}_{A|B_2\cdots B_{N-1}}(N_{c}{}^{(\alpha-2)}_{AB_1} - N_{c}{}^{(\alpha-2)}_{A|B_2\cdots B_{N-1}}) \\
&\geq N_{c}{}^{\alpha}_{AB_1} + h[N_{c}{}^{\alpha}_{AB_2} + hN_{c}{}^{\alpha}_{A|B_3\cdots B_{N-1}} + \frac{\alpha}{4}N_{c}{}^{2}_{A|B_3\cdots B_{N-1}}(N_{c}{}^{(\alpha-2)}_{AB_2} - N_{c}{}^{(\alpha-2)}_{A|B_3\cdots B_{N-1}})] \\
&\quad + \frac{\alpha}{4}N_{c}{}^{2}_{A|B_2\cdots B_{N-1}}(N_{c}{}^{(\alpha-2)}_{AB_1} - N_{c}{}^{(\alpha-2)}_{A|B_2\cdots B_{N-1}}) \\
&\geq \cdots \\
&\geq \sum_{i=1}^{m} h^{i-1}(N_{c}{}^{\alpha}_{AB_i} + Q_{AB_i}) + h^m N_{c}{}^{\alpha}_{A|B_{m+1}\cdots B_{N-1}}.
\end{aligned} \tag{24}$$

Similarly, as $N_{cAB_j} \leq N_{cA|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$, we get

$$\begin{aligned}
&N_{c}{}^{\alpha}_{A|B_{m+1}\cdots B_{N-1}} \\
&\geq N_{c}{}^{\alpha}_{A|B_{m+2}\cdots B_{N-1}} + hN_{c}{}^{\alpha}_{AB_{m+1}} + \frac{\alpha}{4}N_{c}{}^{2}_{AB_{m+1}}\left(N_{c}{}^{(\alpha-2)}_{A|B_{m+2}\cdots B_{N-1}} - N_{c}{}^{(\alpha-2)}_{AB_{m+1}}\right) \\
&\geq \sum_{j=m+1}^{N-2}(hN_{c}{}^{\alpha}_{AB_j} + \bar{Q}_{AB_j}) + N_{c}{}^{\alpha}_{AB_{N-1}}.
\end{aligned} \tag{25}$$

Combining (24) and (25), we complete the proof. □

In particular, if $N_{cAB_i} \geq N_{cA|B_{i+1}\cdots B_{N-1}}$ for all $i = 1, 2, \cdots, N-2$, we have the following proposition.

Proposition 5 *For any N-qubit state* $\rho \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, *if* $N_{cAB_i} \geq N_{cA|B_{i+1}\cdots B_{N-1}}$ *for all* $i = 1, 2, \cdots, N-2$, *we have*

$$N_{c\ A|B_1B_2\cdots B_{N-1}}^{\alpha} \geq \sum_{i=1}^{N-2} h^{i-1}(N_{c\ AB_i}^{\alpha} + Q_{AB_i}) + h^{N-2}N_{c\ AB_{N-1}}^{\alpha}, \tag{26}$$

for $\alpha \geq 2$, *where* $h = 2^{\frac{\alpha}{2}} - 1$, $Q_{AB_i} = \frac{\alpha}{4}N_{c}{}^{2}_{A|B_{i+1}\cdots B_{N-1}}(N_{c}{}^{\alpha-2}_{AB_i} - N_{c}{}^{\alpha-2}_{A|B_{i+1}\cdots B_{N-1}})$.

Example 3 Let us consider the three-qubit state $|\psi\rangle$ (11) again. From the definition of CREN, we have $N_{cA|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $N_{cAB} = 2\lambda_0\lambda_2$, and $N_{cAC} = 2\lambda_0\lambda_3$. Set $\lambda_0 = \lambda_2 = \frac{1}{2}$, $\lambda_1 = \lambda_3 = \lambda_4 = \frac{\sqrt{6}}{6}$, one has $N_{c}{}^{\alpha}_{A|BC} \geq N_{c}{}^{\alpha}_{AB} + hN_{c}{}^{\alpha}_{AC} + \frac{\alpha}{4}N_{c}{}^{2}_{AC}(N_{c}{}^{\alpha-2}_{AB} - N_{c}{}^{\alpha-2}_{AC}) = (\frac{1}{2})^{\alpha} + h\cdot(\frac{\sqrt{6}}{6})^{\alpha} + \frac{\alpha}{4}\cdot(\frac{\sqrt{6}}{6})^2[(\frac{1}{2})^{\alpha-2} - (\frac{\sqrt{6}}{6})^{\alpha-2}]$. While from [16] one has $N_{c}{}^{\alpha}_{AB} + hN_{c}{}^{\alpha}_{AC} = (\frac{1}{2})^{\alpha} + h\cdot(\frac{\sqrt{6}}{6})^{\alpha}$. One can see that our lower bound is tighter than the results in [16] for $\alpha \geq 2$, see Fig. 3.

5 Enhanced Monogamy Relations for Tsallis-*Q* Entanglement

The Tsallis entropy is a generalization of the standard Boltzmann-Gibbs entropy. The Tsallis-q entropy [41, 42] with respect to a non-negative number q, can be used to

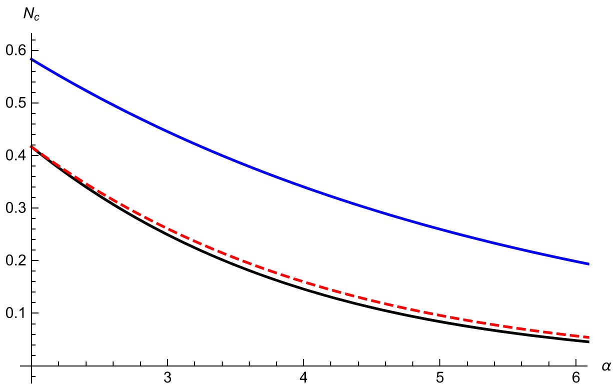

Fig. 3 The axis N_c stands for the negativity of $|\psi\rangle$, which is a function of α. The solid blue line represents the lower bound of negativity of $|\psi\rangle$ in Example 3, the dashed red line represents the lower bound from our result, the solid black line represents lower bound from the result in [16]

characterize classical statistical correlations inherent in quantum states [43]. For a bipartite pure state $|\psi\rangle_{AB}$, the Tsallis-q entanglement is defined by [20],

$$T_q(|\psi\rangle_{AB}) = S_q(\rho_A) = \frac{1}{q-1}(1 - tr(\rho_A^q)), \tag{27}$$

for any $q > 0$ and $q \neq 1$. If q tends to 1, $T_q(\rho)$ converges to the von Neumann entropy, i.e., $\lim_{q\to 1} T_q(\rho) = -tr(\rho \ln \rho)$. For a bipartite mixed state ρ_{AB}, the Tsallis-q entanglement is defined via the convex-roof extension, $T_q(\rho_{AB}) = \min \sum_i p_i T_q(|\psi_i\rangle_{AB})$, with the minimum taken over all possible pure state decompositions of ρ_{AB}.

In [44], the authors proved an analytic relationship between the Tsallis-q entanglement and the concurrence for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$,

$$T_q(|\psi\rangle_{AB}) = g_q(C^2(|\psi\rangle_{AB})), \tag{28}$$

where the function $g_q(x)$ is defined by

$$g_q(x) = \frac{1}{q-1}\left[1 - \left(\frac{1+\sqrt{1-x}}{2}\right)^q - \left(\frac{1-\sqrt{1-x}}{2}\right)^q\right]. \tag{29}$$

It has been shown that $T_q(|\psi\rangle) = g_q\left(C^2(|\psi\rangle)\right)$ for any $2 \otimes m$ ($m \geq 2$)-dimensional pure state $|\psi\rangle$, and $T_q(\rho) = g_q\left(C^2(\rho)\right)$ for two-qubit mixed state ρ [20]. Hence (28) holds for any q such that $g_q(x)$ in (29) is monotonically increasing and convex. In particular, $g_q(x)$ satisfies the following relations for $2 \leq q \leq 3$,

$$g_q(x^2 + y^2) \geq g_q(x^2) + g_q^2(y^2). \tag{30}$$

Lemma 3 *For any $2 \otimes 2 \otimes 2^{N-2}$ mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B \otimes \mathbb{H}_C$, if $C_{AB} \geq C_{AC}$, the following inequality holds for $\alpha \geq 1$,*

$$T_q{}^{\alpha}_{A|BC} \geq T_q{}^{\alpha}_{AB} + (2^{\alpha} - 1)T_q{}^{\alpha}_{AC} + \frac{\alpha}{2} T_{q\,AC}(T_q{}^{\alpha-1}_{AB} - T_q{}^{\alpha-1}_{AC}), \tag{31}$$

where $2 \leq q \leq 3$, N stands for the number of qubit systems, A and B are qubit systems, C is a 2^{N-2}-dimensional qudit system, consisting of $N-2$ qubit systems.

Proof The proof is similar to the proof of Lemma 1. □

The Tsallis-q entanglement satisfies $T_{q\,A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} T_{q\,AB_i}$ [20], where $i = 1, 2, \cdots N-1$, $2 \leq q \leq 3$. It is further proved that $T^2_{q\,A|B_1B_2\cdots B_{N-1}} \geq \sum_{i=1}^{N-1} T^2_{q\,AB_i}$ for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$ in [44].

Note that, for any N-qubit mixed state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_{N-1}}$, if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, m$, and $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \cdots, N-2$ $(1 \leq m \leq N-3,\ N \geq 4)$, $C(\rho)$ no longer satisfies the relation (6) in Proposition 1. Nevertheless, for the case that $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, N-2$, we have an enhanced monogamy relation for the Tsallis-q entanglement.

Proposition 6 *For an arbitrary N-qubit mixed state $\rho_{A|B_1\cdots B_{N-1}}$, if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \cdots, N-2 \quad (N \geq 3)$, the αth power of Tsallis-q entanglement satisfies the following monogamy relation,*

$$T_{q\,A|B_1B_2\cdots B_{N-1}}^{\ \alpha} \geq \sum_{i=1}^{N-2} h^{i-1}(T_{q\,AB_i}^{\ \alpha} + G_{AB_i}) + h^{N-2} T_{q\,AB_{N-1}}^{\ \alpha} \tag{32}$$

for $\alpha \geq 1$, where $h = 2^\alpha - 1$, and $G_{AB_i} = \frac{\alpha}{2}(T_{q\,AB_{i+1}} + \cdots + T_{q\,AB_{N-1}})(T_{q\,AB_i}^{\ \alpha-1} - T_{q\,A|B_{i+1}\cdots B_{N-1}}^{\ \alpha-1})$.

Proof For $\alpha \geq 1$, we have

$$\begin{aligned}
&g_q^\alpha(x^2+y^2) \\
&\geq \left(g_q(x^2)+g_q(y^2)\right)^\alpha \\
&\geq g_q{}^\alpha(x^2) + (2^\alpha-1)g_q{}^\alpha(y^2) + \frac{\alpha}{2} g_q(y^2)(g_q{}^{\alpha-1}(x^2) - g_q{}^{\alpha-1}(y^2)),
\end{aligned} \tag{33}$$

where the first inequality is due to the inequality (30), and the second inequality is obtained analogously from the proof of the second inequality in (5).

Let $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in \mathbb{H}_A \otimes \mathbb{H}_{B_1} \otimes \cdots \otimes \mathbb{H}_{B_N-1}$ be the optimal decomposition for the N-qubit mixed state ρ. We have

$$\begin{aligned}
&T_{q\,A|B_1B_2\cdots B_{N-1}}(\rho) = \sum_i p_i T_q(|\psi_i\rangle_{A|B_1B_2\cdots B_{N-1}}) \\
&= \sum_i p_i g_q\left[C^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right] \\
&\geq g_q\left[\sum_i p_i C^2_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right] \geq g_q\left(\left[\sum_i p_i C_{A|B_1B_2\cdots B_{N-1}}(|\psi_i\rangle)\right]^2\right) \\
&= g_q\left[C^2_{A|B_1B_2\cdots B_{N-1}}(\rho)\right],
\end{aligned} \tag{34}$$

where the first inequality is due to that $g_q(x)$ is a convex function. The second inequality is due to the Cauchy-Schwarz inequality: $(\sum_i x_i^2)^{\frac{1}{2}}(\sum_i y_i^2)^{\frac{1}{2}} \geq \sum_i x_i y_i$, with $x_i = \sqrt{p_i}$ and $y_i = \sqrt{p_i} C_{A|B_1 B_2 \cdots B_{N-1}}(|\psi_i\rangle)$. Due to the definition of the Tsallis-q entanglement and that $g_q(x)$ is a monotonically increasing function, we obtain the third inequality. Therefore, we have

$$
\begin{aligned}
& {T_q}^{\alpha}_{A|B_1 B_2 \cdots B_{N-1}}(\rho) \\
\geq\ & g_q^{\alpha}\left[\sum_i C^2(\rho_{AB_i})\right] \\
\geq\ & g_q^{\alpha}(C^2_{AB_1}) + h \cdot g_q^{\alpha}(C^2_{AB_2}) + \cdots + h^{(N-3)} \cdot g_q^{\alpha}(C^2_{AB_{N-2}}) + h^{(N-2)} \cdot {g_q}^{\alpha}(C^2_{AB_{N-1}}) \\
& + h^{(N-3)} \cdot \frac{\alpha}{2} {g_q}^{\alpha}(C^2_{AB_{N-1}})[{g_q}^{\alpha-1}(C^2_{AB_{N-2}}) - {g_q}^{\alpha-1}(C^2_{AB_{N-1}})] \\
& + \cdots + \frac{\alpha}{2}[{g_q}^{\alpha}(C^2_{AB_2}) + \cdots + {g_q}^{\alpha}(C^2_{AB_{N-1}})] \cdot [{g_q}^{\alpha-1}(C^2_{AB_1}) - {g_q}^{\alpha-1}(C^2_{A|B_2 \cdots B_{N-1}})] \\
\geq\ & {T_q}^{\alpha}_{AB_1} + h {T_q}^{\alpha}_{AB_2} + \cdots + h^{N-3} {T_q}^{\alpha}_{AB_{(N-2)}} + h^{N-2} {T_q}^{\alpha}_{AB_{(N-1)}} \\
& + h^{(N-3)} \cdot \frac{\alpha}{2} {T_q}^{\alpha}_{AB_{N-1}} ({T_q}^{\alpha-1}_{AB_{N-2}} - {T_q}^{\alpha-1}_{AB_{N-1}}) \\
& + \cdots + \frac{\alpha}{2}({T_q}^{\alpha}_{AB_2} + \cdots {T_q}^{\alpha}_{AB_{N-1}}) \cdot ({T_q}^{\alpha-1}_{AB_1} - {T_q}^{\alpha-1}_{A|B_2 \cdots B_{N-1}}),
\end{aligned}
$$

where we have used the monogamy inequality in (20) for N-qubit states ρ to obtain the first inequality. By using the fact that $g_q(x)$ is a monotonically increasing function and the inequality (4), we get the second inequality. Since for any $2 \otimes 2$ quantum state ρ_{AB_i}, $T_q(\rho_{AB_i}) = g_q\left[C^2(\rho_{AB_i})\right]$, from (34) one gets the last inequality. □

Example 4 Let us consider again the three-qubit state $|\psi\rangle$ (11). From the definition of Tsallis-q entanglement, we have $T_{q\,A|BC} = g_q[(2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2})^2]$, $T_{q\,AB} = g_q(4\lambda_0^2\lambda_2^2)$ and $T_{q\,AC} = g_q(4\lambda_0^2\lambda_3^2)$. Set $\lambda_0 = \lambda_2 = \frac{1}{2}$, $\lambda_1 = \lambda_3 = \lambda_4 = \frac{\sqrt{6}}{6}$ and $q = 2$, one has ${T_2}^{\alpha}_{A|BC} = (\frac{7}{24})^{\alpha} \geq {T_2}^{\alpha}_{AB} + (2^{\alpha} - 1){T_2}^{\alpha}_{AC} + \frac{\alpha}{2} T_{2\,AC}({T_2}^{\alpha-1}_{AB} - {T_2}^{\alpha-1}_{AC}) = (\frac{1}{8})^{\alpha} +$

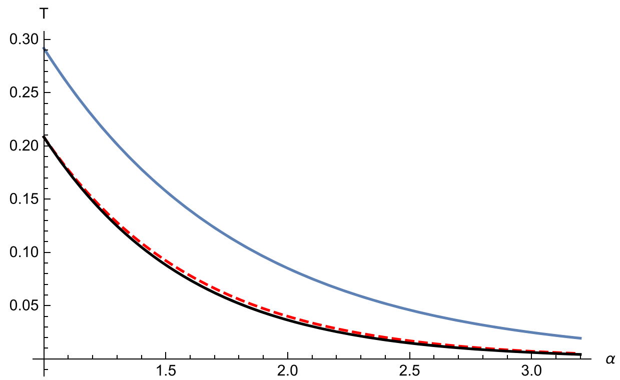

Fig. 4 The axis T represents the Tsallis-q of $|\psi\rangle$, which is a function of α. The solid blue line represents the lower bounds of Tsallis-q of $|\psi\rangle$ (q=2) in Example 4. The dashed red line represents the lower bound from our enhangced monogamy inequalities. The solid black line represents the lower bound from the result in [16]

$(2^\alpha - 1)(0.08333)^\alpha + \frac{0.08333\alpha}{2}[(\frac{1}{8})^{(\alpha-1)} - (0.08333)^{(\alpha-1)}]$. While the formula in [16] is $T2^\alpha_{AB} + (2^\alpha - 1)T2^\alpha_{AC} = (\frac{1}{8})^\alpha + (2^\alpha - 1)(0.08333)^\alpha$. One can see that our result is better than that in [16] for $\alpha \geq 1$, see Fig. 4.

6 Conclusion

Entanglement monogamy is a fundamental property of quantum multipartite states. The extension of the monogamy relation for multipartite entanglement is far more from trivial. We have explored the multipartite entanglement based on the monogamy of the αth-power of concurrence C^α ($\alpha \geq 2$), entanglement of formation E^α ($\alpha \geq \sqrt{2}$), negativity N_c^α ($\alpha \geq 2$) and Tsallis-q entanglement T_q^α ($\alpha \geq 1$). We have proposed a new class of monogamy relations of multipartite entanglement for arbitrary quantum states, and showed that these new monogamy relations have larger lower bounds and tighter than the existing monogamy relations presented in [21, 27, 28, 31]. These tighter monogamy relations give rise to finer characterization of the entanglement distributions among the subsystems of a multipartite system. Our approach may be also applied to the study of monogamy properties related to other quantum correlations.

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