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The Discrete Center-of-Mass Tomogram

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Abstract
A new tomographic function is introduced for the description of qudit states. We utilize the discrete space phase formalism and an analogy with continuous variables systems. It is proved that the new function is nonnegative, normalized and that it unambiguously determines the state. The new function is called the discrete center-of-mass tomogram as it has a counterpart used to determine the states of continuous variables systems.

Keywords Wigner function · Discrete phase space · Tomographic probability representation of quantum mechanics · Center-of-mass tomogram

1 Introduction

There are various quantum state representations that are used in many applications of quantum information theory, quantum optics, etc.

In the traditional formalism, a Hilbert space $\mathcal{H}$ is associated with a considered quantum system. A pure state is described by the wave function $|\psi\rangle$, which is an element of this Hilbert space. In the case of statistical ensembles of quantum systems, a state is described by the density operator $\hat{\rho}$ [1, 2], that is a Hermitian self-adjoint, positive-semidefinite operator defined on the Hilbert space $\mathcal{H}$.

Also, the states of a quantum system can be described by functions defined on the phase space, just as in the case of the description of classical systems. As an example we can

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mention the quasi-probability distribution functions [3–7] or tomographic functions [8, 9]. The latter functions are true probability distributions.

There are many fields where the Wigner function is applied [15, 16]. It was introduced in [3] and was the first object of the kind of quasiprobabilities. The Wigner function is defined on the phase space and normalized as the classical probability density. However, unlike its classical counterpart, the Wigner function can take negative values. This fact can be considered as an effect of the uncertainty relations [10–12] and can be used in attempts to understand the quantum-classical correspondence [13, 14]. Also, it was proved that the Wigner function took negative values for every pure state that was not the gaussian one [17]. This fact is known as Hudson's theorem. It was generalized for multiparticle quantum systems in [18].

In [8] the tomographic probability representation of quantum states was proposed. In the framework, the state of a continuous variables quantum system is described by the symplectic tomogram, which is the product of the Radon transformation [19] of the Wigner function [3]. Also, the symplectic tomogram is a generalization of the optical tomogram, that is measured employing the homodyne detector [20–22] for the determination of a photonic state.

In the cases of the multimode electromagnetic field or multiparticle systems, there are different possibilities to introduce tomographic probability representations. The state of these systems can be described by conditional probability distributions of 1, 2,..., or N random variables, where the parameter N denotes the number of modes of the electromagnetic field. These possibilities of the quantum state representation were discussed in [23–25]. The quantum state can be described by the conditional probability distribution of one variable having the physical meaning of the center of mass of all the particles. The corresponding tomographic function is called the center-of-mass tomogram. The Hilbert space of the N-mode system is a tensor product of the Hilbert spaces of every mode. We can combine some of these subspaces in one. Therefore some modes (or particles) are described by a single Hilbert space, these modes or particles constitute a cluster. In [25], it was shown that for every partition of the Hilbert space of multimode (multiparticle) quantum system it was possible to describe its state by the conditional probability distributions of n random variables, where n was the number of elements in the partition of the initial Hilbert space or the number of clusters. Each variable was the center of mass of particles in the cluster. This tomographic function was called the cluster tomogram.

In the case of discrete variables systems (qudits), we can use the phase space formalism and describe the qudit states by an analog of Wigner function that is called the discrete Wigner function. There has been a great development in the area, see [26–29]. The general approach in building Wigner functions from Lie groups was considered in [30].

Like its counterpart for continuous variables systems, the discrete Wigner function can take negative values. There is also known an analog of Hudson theorem [31]. It states that the Wigner function of the stabilizer state is a nonnegative one. Also, it is known that quantum computational circuits consisting only of the Clifford gates produce the stabilizer states and can be effectively simulated by classical computers [32, 33]. Hence, there is the connection between the quantum computational speed-up and negativity of the discrete Wigner function, which was discussed in [34–36].

Since it is possible to construct the phase space for a discrete-variable quantum system, we can develop this formalism further by introducing discrete tomographic functions. In the present paper, we build the tomographic function for describing the state of a composite qudit system by being based on the notion of the center-of-mass tomogram. On the one hand, such a system can be considered as one d-level qudit, with d is a composite number (in our notation pure states of a d-level qudit are elements of a Hilbert space $\mathcal{H}_d$ of dimension d). On the other hand, it is possible to consider any qudit as a composite one if d is not a prime. Moreover, by introducing additional degrees of freedom we can present the quantum d-level system as a composite one for arbitrary d, see [37–40]. However, the involvement of the last technic goes beyond the scope of the present research. Here, we treat the d-level quantum system as a composite one only for the cases of composite d. Finally, our interest is in the development of the tomographic representations for this kind of quantum systems. We mentioned that the state of continuous variables composite system could have been described by the center-of-mass tomogram or by the cluster tomogram. An analogical representation we introduce for qudit states.

The tomographic functions are true probability distributions that in general can be obtained in POVM measurements. In other words, for some measurement schemes called IC-POVM (informational complete POVM) the state of a quantum system is determined by the probabilities of the possible outcomes. In the case of a continuous variables system, the state is determined by the probability density. Also, the tomographic functions can be considered as a set of conditional probability distributions. For instance the symplectic tomogram is a conditional probability density distribution $\mathcal{W}(X \mid \mu,\ \nu)$ of the quadrature values $X(\mu, \nu) = \mu q + \nu p$ (here q is the generalized coordinate and p is the generalized momentum). Parameters μ and ν determines the measuring observable. By introducing a distribution $P(\mu,\ \nu)$ we can construct POVM with the probability density of its outcomes of the form $P(X) = \mathcal{W}(X \mid \mu,\ \nu)P(\mu,\ \nu)$. The general discussion about representations of the quantum states by the conditional probabilities can be found in [41–43].

As it was mentioned, there were different ways to define the discrete Wigner function. We find it useful to follow the approach proposed in [26] and developed in [44, 45]. In the representation, d-level quantum system, where d is not a prime, is considered as a composite one. At the same time, for a prime d the phase space of a d-level quantum system can be constructed straightforwardly as a finite plane $\mathbb{Z}_d \times \mathbb{Z}_d$. The sum of values of the discrete Wigner function along the line (this object is additionally defined for a finite space) is nonnegative and represents the probability of the outcome when measuring an observable whose eigenvectors form one of the $d + 1$ bases of the full set of MUB (mutual unbiased bases) [46, 47]. The phase space of a composite qudit (i.e. when d is not a prime) is a Cartesian product of phase spaces of the qudit subsystems. A corresponding discrete Wigner function is constructed in consideration of these subsystems as individual particles.

The existence of the full set of MUB is proven only for prime or power of prime d [48]. Presumably, for other d the set of MUB of maximal size contains lesser than $d + 1$ bases. Note, that the minimum number of orthogonal measurements that are needed for state determination of a d-level quantum system is $d + 1$. In the case, the state can be described by $d + 1$ d-dimensional probability distributions. Thus, the full set of MUB, if it exists, produces a probability representation of qudit states. Consideration within the framework of a generic star-product scheme can be found in [49]. Moreover, the usage of the discrete phase space formalism allows us to speak about an analogy between the MUB probability representation and the symplectic tomography of continuous-variable systems states. We use this idea to extend the center-of-mass tomography formalism for the representation of the states of composite discrete variables quantum systems.

The presented MUB scheme possesses some redundant information. In the case, the qudit state is determined by $d^2 + d$ real parameters, while we need only $d^2 - 1$ for the determination of the density operator. Also, there are POVM schemes that produce non-redundant representations of quantum states, see for instance the SIC POVM approach that was studied and widely used in quantum Bayesianism reformulation of quantum mechanics (QBism) [50, 51]. An interesting connection between SIC POVM and MUB in the framework of finite geometry was discussed in [52].

We can use different partitions of the Hilbert space of composite discrete variables quantum system to define the analogies of the aforementioned cluster and center-of-mass tomograms. In this work, we are aimed to build the qudit representation where its state is described by the analog of the center-of-mass-tomogram. This object can be constructed straightforwardly from the formula of the continuous center-of-mass tomogram by replacing the integral by the sum and delta function by Kronecker symbol, and by using modular arithmetic. However, it should be proven that the map to the discrete center-of-mass tomogram (as we call this analog of continuous center-of-mass tomogram) is invertible. In order to build the complete representation, it is also important to find the inverse transform of this map. The discrete center-of-mass tomogram can be defined if the Hilbert space dimension d is not a prime. In our framework d must be a power of some $p \in \mathbb{N} \setminus \{1\}$. It was soluted to analyze the cases of prime p due to the relative simplicity in calculations.

The paper is organized as follows. In Sections 2 and 3 the information about the main aspects of the phase space formalism and the tomographic probabilities is given. In particular, in Section 2 we consider the representations of the states of the continuous variables quantum systems. In the next section, the review of the discrete Wigner function approach is presented. Then, in Section 4 we introduce the new tomographic function to describe the states of qudit systems. The construction is made utilizing the analogy between the continuous and discrete variables quantum systems. We also prove that the new function is nonnegative, normalized and it is possible to restore the density operator from its values. In Section 5, for demonstration of the new approach to describe qudit systems we consider the ququart case. The concluding remarks are presented in Section. 6.

2 Tomographic probability representation of quantum states

The state of a classical system is described by the probability density function defined on a phase space, i.e. by the function $f(q,\ p)$ of the canonical variables q and p, where q is the generalized coordinate and p is the generalized momentum. The function $f(q,\ p)$ is normalized and positively semi-definite.

The state of a quantum system is described by the density operator $\hat{\rho}$ defined on a Hilbert space $\mathcal{H}$ of pure states. The density operator is a Hermitian self-adjoint, positively semi-definite, normalized (i.e. $\mathrm{Tr}\,\hat{\rho} = 1$).

Due to the uncertainty relations [10, 11, 53], it is impossible to construct the joint probability distributions of the canonical variables q and p. Nevertheless, the state of the quantum system can be described by the function defined on the phase space (here and below we put $\hbar = 1$)

$$W(q,\ p) = \frac{1}{2\pi} \int du\ \rho\left(q + \frac{u}{2},\ q - \frac{u}{2}\right) e^{-ipu}, \tag{1}$$

where

$$\rho\left(q + \frac{u}{2},\ q - \frac{u}{2}\right) = \left\langle q + \frac{u}{2}\right| \hat{\rho} \left|q - \frac{u}{2}\right\rangle \tag{2}$$

is the matrix element of the density operator written in the q-representation.

The function $W(q,\ p)$ is called the Wigner function. It possesses the property of normalization, i.e. $\int dq\ dp\ W(q,\ p) = 1$. However, it is not the true probability distribution as it can take negative values. Actually, a pure state is given by the nonnegative Wigner function, only if the corresponding wave function is of the form of the Gaussian function [17].

Let us discuss the properties of the quasiprobability distribution (1) by using the star-product formalism [54–56]. The Wigner function is obtained by the linear transformation of the density operator. For every point $(q,\ p)$ in the phase space we determine an operator $\hat{\mathcal{U}}(q,\ p)$ called dequantizer. Then, the Wigner transform of the density operator can be presented in the form

$$W(q,\ p) = \text{Tr}\,\left(\hat{\rho}\cdot\hat{\mathcal{U}}(q,\ p)\right). \tag{3}$$

In the case, for dequantizers we have the following expressions

$$\hat{\mathcal{U}}(q,\ p) = 2\hat{D}\left(\sqrt{2}(q+ip)\right)\hat{P}, \tag{4}$$

where $\hat{D}(\frac{\alpha}{\sqrt{2}}) = \exp\left(i\text{Im}\alpha\cdot\hat{q} - i\text{Re}\alpha\cdot\hat{p}\right)$ is the displacement operator and $\hat{P}$ is operator of reflection (parity operator), i.e. $\hat{P}|q\rangle = |-q\rangle$.

The map (3) is an invertible one. The corresponding transform is known as the Weyl transform or the Weyl quantization [57] and can be presented in the form

$$\hat{\rho} = \int dq\ dp\ W(q,\ p)\hat{\mathcal{D}}(q,\ p). \tag{5}$$

Here, for every point in the phase space we determine operators $\hat{\mathcal{D}}$ called quantizers. In the case of the Wigner-Weyl transforms the quantizers and dequantizers coincide up to coefficient

$$\hat{\mathcal{D}}(q,\ p) = 2\pi\,\hat{\mathcal{U}}(q,\ p). \tag{6}$$

The star-product formalism can be applied not only for the density operator. By introducing quantizers and dequantizers we can map an arbitrary operator defined on the Hilbert space into c-valued function and vise versa

$$\hat{A}\to f_A(x) = \text{Tr}\,\left(\hat{\rho}\cdot\hat{\mathcal{U}}(x)\right), \quad f_A(x)\to\hat{A} = \sum_x f_A(x)\hat{\mathcal{D}}(x). \tag{7}$$

Here, the parameter x denotes all variables of function f_A and $\sum_x$ means the summation in the case of discrete values of x and the integration in the case of continuous values of x. The function f_A is called the symbol of operator $\hat{A}$. In the case of the Wigner-Weyl transforms, it is also common to call the Wigner function as the Weyl symbol of the density operator.

The schemes, where quantizers and dequantizers of one value x coincides up to a coefficient, are called self-dual and were studied in [56]. The Wigner-Weyl transforms are a special case of such schemes.

Although the Wigner function is not a true probability distribution, we can restore all statistical characteristics of an arbitrary observable, for instance

$$\langle A\rangle = \text{Tr}\,\left(\hat{\rho}\hat{A}\right) = \int dq\ dp\ A(q,\ p)W(q,\ p), \tag{8}$$

where $\hat{A}$ is the operator of the observable and $A(q,\ p)$ is proportional to its Weyl symbol

$$A(q,\ p) = \int dq\ dp\ \left\langle q+\frac{u}{2}\right|\hat{A}\left|q-\frac{u}{2}\right\rangle e^{-ipu}. \tag{9}$$

In particular, by integrating over the variable p, we obtain a density function of the probability distribution of outcomes of the measurement of the observable $\hat{q}$. Similarly, it is obtained a probability distribution function for an arbitrary observable of the form $\hat{X} = \mu\hat{q} + \nu\hat{p}$. Due to the properties of dequantizers of the Wigner-Weyl transformation, we always obtain nonnegative values. Indeed, the integration of the dequantizers along the line $\mu q + \nu p = X$ gives us the projector on the eigenstate of the corresponding operator $\hat{X}$. Hence, for obtaining the probability distribution of the observable $\hat{X}$, we need to integrate the Wigner function along the line $\mu q + \nu p = X$

$$\mathcal{W}(X \mid \mu,\ \nu) = \int dq\ dp\ W(q,\ p)\delta(X - \mu q - \nu p). \tag{10}$$

We can notice that $\int dX\ \mathcal{W}(X \mid \mu,\ \nu) = 1$. Thus, the function $\mathcal{W}(X \mid \mu,\ \nu)$ is the conditional probability distribution density of the random variable X given the values of the real parameters μ and ν. The map (10) is known as the Radon transform and it is an invertible one. Hence, the state of the quantum system can be described by the conditional probability distribution density $\mathcal{W}(X \mid \mu,\ \nu)$, which is called the symplectic tomogram. The connection of function $\mathcal{W}(X \mid \mu,\ \nu)$ with the density operator can be presented in the following way

$$\mathcal{W}(X \mid \mu,\ \nu) = \mathrm{Tr}\ \left(\hat{\rho}\delta(X - \mu\hat{q} - \nu\hat{p})\right) = \langle X;\ \mu,\ \nu|\hat{\rho}|X;\ \mu,\ \nu\rangle, \tag{11}$$

where $|X;\ \mu,\ \nu\rangle$ is eigenstate of the operator $\hat{X} = \mu\hat{q} + \nu\hat{p}$.

The function $\mathcal{W}(X \mid \mu,\ \nu)$ can be considered as the tomographic symbol of the density operator, where the corresponding dequantizers have the form

$$\hat{\mathcal{U}}(X,\ \mu,\ \nu) = \delta\left(X - \mu\hat{q} - \nu\hat{p}\right). \tag{12}$$

The density operator can be restored by using the inverse map $\hat{\rho} = \int dX\ d\mu\ d\nu \mathcal{W}(X,\ \mid \mu,\ \nu)\hat{\mathcal{D}}(X,\ \mu,\ \nu)$, where for quantizers we have

$$\hat{\mathcal{D}}(X,\ \mu,\ \nu) = \frac{1}{2\pi}\ e^{i(X-\mu\hat{q}-\nu\hat{p})}. \tag{13}$$

Let us proceed to the consideration of multimode or multiparticle states. For clarity, assume that there are N particles and everyone can be described by two parameters q_i and p_i. The Hilbert space for N-mode system can be presented as a tensor product of single-particle Hilbert spaces, i.e. $\mathcal{H} = \bigotimes_{k=1}^{N} \mathcal{H}_k$. In this case, the dimension of the corresponding phase space is $2N$ and the Wigner function has the following form

$$W(\mathbf{q},\ \mathbf{p}) = \frac{1}{(2\pi)^N} \int d^N u\ \rho\left(\mathbf{q} + \frac{\mathbf{u}}{2},\ \mathbf{q} - \frac{\mathbf{u}}{2}\right) e^{-i\mathbf{p}\cdot\mathbf{u}}, \tag{14}$$

where $\mathbf{q} = \left[q_1\ ...\ q_N\right]^T$ and $\mathbf{p} = \left[p_1\ ...\ p_N\right]^T$.

The joint probability distribution of the observables $\hat{q}_i$ and $\hat{p}_j$ exists if $i \neq j$. The function $W(\mathbf{q},\ \mathbf{p})$ is not the true probability density, but we can obtain by means of corresponding integral transformations the probability distributions of all observables $\hat{q}_i$ and $\hat{p}_i$ $\forall i = 1, ...,\ N$ and some of their compositions. In general, we can obtain the joint probability distributions of any set of commuting observables.

The tomographic symbol of the density operator of the state of a single-particle subsystem can be obtain by linear transform $\hat{\rho}_k \to \mathrm{Tr}\ \left(\hat{\rho}_k \cdot \hat{\mathcal{U}}(X_k,\ \mu_k,\ \nu_k)\right)$, where index k denotes the k-th particle. For the state $\hat{\rho}$ of the whole system, we can construct quantizers and dequantizers that are tensor products of quantizers and dequantizers for single-particle subsystems, i.e.

$$\hat{\mathcal{U}}(\mathbf{X},\ \boldsymbol{\mu},\ \boldsymbol{\nu}) = \prod_{k=1}^{N} \delta\left(X_k - \mu_k \hat{q}_k - \nu_k \hat{p}_k\right), \tag{15}$$

$$\hat{\mathcal{D}}(\mathbf{X},\ \boldsymbol{\mu},\ \boldsymbol{\nu}) = \frac{1}{(2\pi)^N} \prod_{k=1}^{N} e^{i(X_k - \mu_k \hat{q}_k - \nu_k \hat{p}_k)}, \tag{16}$$

where $\mathbf{X}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ consist of parameters X_k, μ_k and ν_k relatively. By using (15, 16), we can introduce the multidimensional symplectic tomogram that can be expressed as a transformation of the multidimensional Wigner function

$$\mathcal{W}(\mathbf{X} \mid \boldsymbol{\mu},\ \boldsymbol{\nu}) = \int d^N q\ d^N p\ W(\mathbf{q},\ \mathbf{p}) \prod_{k=1}^{N} \delta(X_k - \mu_k q_k - \nu_k p_k) \tag{17}$$

where q_k and p_k are coordinate and momentum of k-th particle. In other words, $2N$-dimensional phase space is split in the Cartesian product of phase spaces, then, in every phase space we integrate the Wigner function along the line $\mu_k q_k + \nu_k p_k = X_k$.

If instead of performing the elementwise multiplication in the expression (17) we decide to calculate the scalar product of vectors $\boldsymbol{\mu}$ and $\mathbf{q}$ ($\boldsymbol{\nu}$ and $\mathbf{p}$), we obtain the function that has the fewer number of variables and has the form

$$\mathcal{W}_{cm}(X \mid \boldsymbol{\mu},\ \boldsymbol{\nu}) = \int d^K q\ d^K p\ \delta(X - \boldsymbol{\mu} \cdot \mathbf{q} - \boldsymbol{\nu} \cdot \mathbf{p}) W(\mathbf{q},\ \mathbf{p}). \tag{18}$$

The function (18) is known as the center-of-mass tomogram [23]. Its values are nonnegative and it is normalized, i.e. $\int dX\ \mathcal{W}_{cm}(X \mid \boldsymbol{\mu},\ \boldsymbol{\nu}) = 1$. So, the center-of-mass tomogram is true probability distribution density. Let us note that the transformation (18) is an invertible one. In the star-product formalism the dequatizers and quantizers of the center-of-mass tomogram have the form

$$\hat{\mathcal{U}}(X,\ \boldsymbol{\mu},\ \boldsymbol{\nu}) = \delta\left(X - \sum_{k=1}^{N} \mu_k \hat{q}_k - \sum_{k=1}^{N} \nu_k \hat{p}_k\right), \tag{19}$$

$$\hat{\mathcal{D}}(X,\ \boldsymbol{\mu},\ \boldsymbol{\nu}) = \frac{1}{(2\pi)^N}\ \exp\left(X - \sum_{k=1}^{N} \mu_k \hat{q}_k - \sum_{k=1}^{N} \nu_k \hat{p}_k\right). \tag{20}$$

Thus, we describe the state of the quantum multimode system by the conditioned probability distribution of one random variable X. In other words, the center-of-mass tomogram is the function of $2N + 1$ parameters. The variable X can be presented as the sum of variables $X_k = \mu_k q_k + \nu_k p_k$, where every variable X_k relates to k-th particle. Let us suppose that $\nu_k = 0\ \forall k = 1, ...,\ N$. In the case variable X is the weighted sum of coordinates of particles, $X = \sum_{k=1}^{N} \mu_k q_k$. If we assume that $\mu_k = \frac{m_k}{\sum_{l=1}^{N} m_l}$, then we see that variable X has the meaning of center of mass of the multiparticle system.

3 The discrete Wigner function

Let us consider the qudit systems. In this case, the density matrix $\hat{\rho}$ of the state acts on a Hilbert space $\mathcal{H}_d$ of finite dimension d and can be represented as a square matrix of size

$d \times d$. As the density matrix is a Hermitian positively semidefinite normalized operator, the state of the qudit is given by $d^2 - 1$ real parameters.

As it was discussed in Section 2, the state of continuous-variable quantum system can be described by a function defined on the phase space ($(q,\ p)$ - space), e.g. by the Wigner function $W(q,\ p)$.

By analoguous way, the phase space is constructed in the case of qudit systems. In [26–29] the formalism of quasi-distributions for finite-level systems was introduced and developed. Various approaches to the construction of the function, which is analogous to the Wigner function for qudits [58–60, 70] as well as approaches of introduction of other discrete quasiprobabilities [61–63], have been proposed. Also, this approach was considered in the context of the star product formalism [64–66]. The problem of the quantum state reconstruction in terms of the Wigner function and the discrete phase space was discussed in [29]. There are various papers where the formalism of the phase space was applied, i.e. [67–69, 71].

Following the approach [26], we consider the two cases

1. d is a prime number. The correspondig qudit systems can not be composite. Also, the formalism of the phase space in the case is closely related to the notion of mutually unbiased bases (MUB) [72–74]. Two bases $\{|e_i\rangle\}|_{i=1}^{d}$ and $\{|f_i\rangle\}|_{i=1}^{d}$ are mutually unbiased if

$$|\langle f_i|e_j\rangle| = \frac{1}{\sqrt{d}}, \quad \forall i, j = 1, ..., \ d. \tag{21}$$

Let us consider the d-level system such that it is possible to find $d+1$ different MUBs. Measurements of the ensemble of qudits in these bases give us $d^2 - 1$ independent probabilistic parameters. That allows us to reconstruct the density matrix of the state. However, the existence of $d+1$ mutually unbiased bases is proved for the cases when d is a prime or power of a prime number [48]. Also, for arbitrarily d the number of MUB cannot exceed $d + 1$ [47]. It is assumed that it is not possible to construct a complete MUB set for all d that are not powers of a prime number. General information on the MUB can be found in the review [75].

2. d is not a prime number. The corresponding quantum systems are considered as composite ones. In the continuous case, the Wigner function of the state of a multimode quantum system has the form (14), and its dequantizers and quantizers are the products of the dequantizers and quantizers for single-particle subsystems, see (4). The same approach can be used to define the discrete Wigner function of the state of the composite discrete-variable system. We remind that in the paper we consider any d-level quantum system, where d is not a prime, as a composite one.

3.1 *d* is a prime number

Let us choose the basis $\{|n\rangle\}|_{n=0}^{d-1}$. In the considered case it is possible to construct the complete set of MUB, i.e. for the given basis $\{|n\rangle\}|_{n=0}^{d-1}$ there are d bases $\{|b,\ n\rangle\}|_{n=0}^{d-1}$, $b = 1, ...,\ d$, such that

$$|\langle b,\ n|b',\ n'\rangle| = \frac{1 - \delta_{b,b'}}{\sqrt{d}} + \delta_{b,b'}\delta_{n,n'}. \tag{22}$$

Here, we use designation $|n\rangle = |0,\ n\rangle$.

Obviously, the Fourier images of the states $|n\rangle$ form an element of MUB set. We put that this basis corresponds to the number $b = 1$

$$|1, n\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d} \cdot nk} |k\rangle. \tag{23}$$

The phase space of the described physical systems is the space $\mathbb{Z}_d \times \mathbb{Z}_d$, where $\mathbb{Z}_d$ is the field of residues modulo the prime number d. The corresponding coordinates a_1 and a_2 are associated with the states of the bases $\{|n\rangle\}|_{n=0}^{d-1}$ and $\{|1,\ n\rangle\}|_{n=0}^{d-1}$ relatively. In other words, we encode the basis states by the elements of the finite field $\mathbb{Z}_d$.

Let us introduce operators $\hat{Z}$ and $\hat{X}$ of the form

$$\hat{Z} = \sum_{a_1 \in \mathbb{Z}_d} \omega^{a_1} |a_1\rangle\langle a_1|, \quad \hat{X} = \sum_{a_1 \in \mathbb{Z}_d} \omega^{a_2} |1,\ a_2\rangle\langle 1,\ a_2|. \tag{24}$$

Here we use encoding via the finite field elements and notation $\omega = e^{\frac{2\pi i}{d}}$. The operator $\hat{Z}$ is diagonal in the basis $\{|n\rangle\}|_{n=0}^{d-1}$ and $\hat{X}$ is diagonal in the $\{|1,\ n\rangle\}|_{n=0}^{d-1}$.

The discrete Wigner function is the map $W :\ \mathbb{Z}_d \times \mathbb{Z}_d \to \mathbb{R}$, where

$$W(a_1,\ a_2) = \mathrm{Tr}\left(\hat{\mathcal{U}}(a_1,\ a_2) \cdot \hat{\rho}\right),\quad a_1,\ a_2\ \in\ \mathbb{Z}_d, \tag{25}$$

and the operators $\hat{\mathcal{U}}(a_1,\ a_2)$ are called dequantizers and for the case $d \neq 2$ according to [26, 73] they have the form

$$\hat{\mathcal{U}}(a_1,\ a_2) = \frac{1}{d^2} \sum_{k \in \mathbb{Z}_d} \sum_{l \in \mathbb{Z}_d} \omega^{a_2 l - a_1 k} \omega^{-2^{-1} kl} \hat{Z}^k \hat{X}^l. \tag{26}$$

From the expressions for operators $\hat{Z}$ and $\hat{X}$ (24) we obtain

$$\hat{\mathcal{U}}(a_1,\ a_2) = \frac{1}{d} \sum_{k \in \mathbb{Z}_d} \sum_{l \in \mathbb{Z}_d} |k\rangle\langle l| \delta_{2a_1, k+l} \omega^{a_2 \cdot (k-l)}, \tag{27}$$

Hence, we obtain an expression for a discrete Wigner function, which has a similar form to the Wigner function of a system with continuous variables

$$W(a_1,\ a_2) = \frac{1}{d} \sum_{n \in \mathbb{Z}_d} \langle a_1 - n|\hat{\rho}|a_1 + n\rangle e^{\left(\frac{4\pi i}{d}\right) n a_2}. \tag{28}$$

The function $W(a_1,\ a_2)$ has the following properties

1. It is normalized, i.e.

$$\sum_{a_1=0}^{d-1} \sum_{a_2=0}^{d-1} W(a_1,\ a_2) = 1 \tag{29}$$

2. Summation by the variable a_2 gives us the probability distribution $P_0(n)$ of outcomes of measurement in the basis $\{|n\rangle\}|_{n=0}^{d-1}$

$$\sum_{a_2=0}^{d-1} W(a_1,\ a_2) = \langle a_1|\hat{\rho}|a_1\rangle = P_0(a_1). \tag{30}$$

Here, we use the following identity $\frac{1}{N}\sum_{k=0}^{N-1}\exp\left(\frac{2\pi i}{N}\cdot nk\right) = \delta_{n,0}$. Analogously, when summing by the variable a_1, we obtain the distribution of outcomes of measurement in the basis $\{|1,\ n\rangle\}|_{n=0}^{d-1}$

$$\sum_{a_1=0}^{d-1} W(a_1,\ a_2) = \frac{1}{d}\sum_{m=0}^{d-1}\sum_{n=0}^{d-1}\langle m|\hat{\rho}|n\rangle e^{\frac{2\pi i}{d}\cdot(n-m)a_2} = \langle 1,\ a_2|\hat{\rho}|1,\ a_2\rangle = P_1(a_2). \quad (31)$$

3. For two functions $W_1(a_1,\ a_2)$ and $W_2(a_1,\ a_2)$

$$d\sum_{a_1=0}^{d-1}\sum_{a_2=0}^{d-1} W_1(a_1,\ a_2)W_2(a_1,\ a_2) = \mathrm{Tr}\ \left(\hat{\rho}_1\cdot\hat{\rho}_2\right), \quad (32)$$

where $\hat{\rho}_1$ and $\hat{\rho}_2$ are corresponding to W_1 and W_2 density operators.

The density operator can be reconstructed from the values of the discrete Wigner function by the transformation

$$\hat{\rho} = \sum_{a_1=0}^{d-1}\sum_{a_2=0}^{d-1} W(a_1,\ a_2)\hat{\mathcal{D}}(a_1,\ a_2), \quad (33)$$

where the operators $\hat{\mathcal{D}}(a_1,\ a_2)$ are known as quantizers, and they coincide with the corresponding dequantizers up to a coefficient

$$\hat{\mathcal{D}}(a_1,\ a_2) = d\cdot\hat{\mathcal{U}}(a_1,\ a_2). \quad (34)$$

Let us discuss some properties of the discrete Wigner function related to geometrical objects in the discrete phase space. The line in $\mathbb{Z}_d\times\mathbb{Z}_d$ is the set of points $(a_1,\ a_2)$ satisfying the equation

$$\mu a_1 + \nu a_2 = \kappa,\ \ \mu,\ \nu,\ \kappa\ \in\ \mathbb{Z}_d. \quad (35)$$

In a phase space of size $d\times d$ there are only d^2+d of such objects. Two lines are parallel, that is, they have no intersections if the free terms κ differ while the coefficients μ and ν are the same for the equations of the lines. Let us denote the lines in a finite phase space by specifying the triple $(\mu,\ \nu,\ \kappa)$. Note that dividing by a nonzero coefficient leads to an equation describing the same line. In particular, this means that it is sufficient for us to consider the following variants of values of coefficients

– $(\mu,\ 1,\ \kappa)$, where $\mu,\ \kappa\in\mathbb{Z}_d$,
– $(1,\ 0,\ \kappa)$, where $\kappa\ \in\ \mathbb{Z}_d$.

Each one of $d+1$ given pairs $(\mu,\ \nu)$ specifies a set of parallel lines (every set contains d lines) that called striations. The case of qutrit striations is presented in Fig. 1. The corresponding phase space consists of $3\times 3 = 9$ points. In the figure, the column number is counted from the left to the right and denotes the variable a_1. The row number is counted from the bottom to the top and corresponds to the variable a_2.

Using the concept of lines we can generalize the property 2 of the discrete Wigner functions: the sum of the values of the Wigner function at all points $(a_1,\ a_2)$ of the line $(\mu,\ 1,\ \kappa)$ is a probability to find the system in the MUB state $|\mu+1,\ \kappa\rangle$

$$\sum_{a_1,\ a_2:\ \mu a_1+a_2=\kappa} W(a_1,\ a_2) = \langle\mu+1,\ \kappa|\hat{\rho}|\mu+1,\ \kappa\rangle. \quad (36)$$

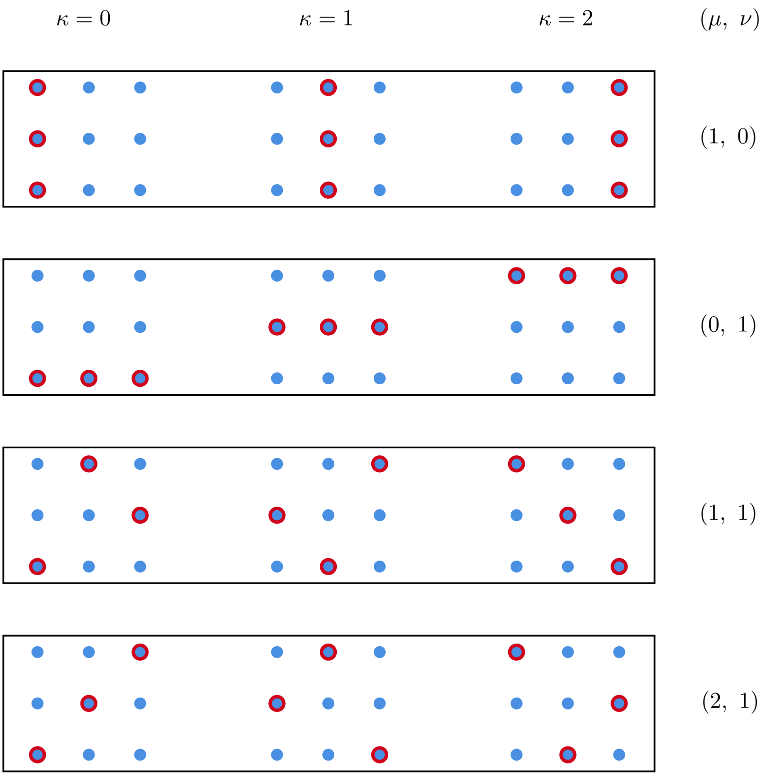

Fig. 1 The striations of the phase space of qutrit. The lines of one striation are placed in one rectangular box

Indeed, for the sum of dequantizers we have

$$\sum_{a_1=0}^{d-1} \hat{\mathcal{U}}(a_1,\ \kappa - \mu a_1) = |\mu + 1,\ \kappa\rangle\langle\mu + 1,\ \kappa|, \tag{37}$$

where

$$|\mu + 1,\ \kappa\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{\kappa k - 2^{-1}\mu k^2} |k\rangle. \tag{38}$$

The sum of the values along the line $(1,\ 0,\ \kappa)$, as already shown, gives the probability to detect the system in the state of the basis $|\kappa\rangle = |0,\ \kappa\rangle$.

Note that the expressions of dequantizers (27) and the definition of the Wigner function (28) are not suitable for descriptions of states of qubit systems $d = 2$. At least we can see that $2a_1 = 0$. In the case of $d = 2$, we have $\omega = (-1)$, $\hat{Z} = \hat{\sigma}_z$, $\hat{X} = \hat{\sigma}_x$, where $\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are Pauli matrices. The corresponding dequantizers have the form

$$\hat{\mathcal{U}}(a_1,\ a_2) = \frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} (-1)^{a_2 l - a_1 k} i^{-kl} \hat{\sigma_z^k} \hat{\sigma_x^l}. \tag{39}$$

Finally, we obtain the expression

$$\hat{\mathcal{U}}(a_1,\ a_2) = \frac{1}{4}\left(\hat{I} + \mathbf{u}(a_1,\ a_2)\cdot\hat{\boldsymbol{\sigma}}\right), \tag{40}$$

where

$$\mathbf{u}(a_1,\ a_2) = \left[(-1)^{a_2}\ \ (-1)^{a_1+a_2}\ \ (-1)^{a_1}\right]^T, \tag{41}$$

and for all a_1 and a_2 $\hat{\mathcal{D}}(a_1,\ a_2) = 2\,\hat{\mathcal{U}}(a_1,\ a_2)$.

The qubit density operator can be presented in the form

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \mathbf{r}\cdot\hat{\boldsymbol{\sigma}}\right), \tag{42}$$

where $\mathbf{r} = \left[x\ \ y\ \ z\right]^T$ is the vector of real parameters and $\hat{\boldsymbol{\sigma}} = \left[\hat{\sigma}_x\ \ \hat{\sigma}_y\ \ \hat{\sigma}_z\right]$ are Pauli matrices. Therefore, for the Wigner function of qubit state we obtain

$$W(a_1,\ a_2) = \frac{1}{4}\left(1 + (-1)^{a_2}x + (-1)^{a_1+a_2}y + (-1)^{a_1}z\right). \tag{43}$$

3.2 *d* is not a prime number

The Hilbert space can be presented in the form

$$\mathcal{H}_d = \mathcal{H}_{p_1}\otimes\ldots\otimes\mathcal{H}_{p_K}, \tag{44}$$

where $d = p_1 \ldots p_K$ is the decomposition of the number d into prime factors. The corresponding phase space can be presented as a Cartesian product $\mathbb{Z}_{p_1}^2 \times \ldots \times \mathbb{Z}_{p_K}^2$. A continuous analog of this construction is the phase space of a multimode system of continuous variables. By using (26) for each subspace $\mathcal{H}_{p_k}$, we obtain the operators $\hat{\mathcal{U}}_k(a_1^{(k)},\ a_2^{(k)})$, where $a_1^{(k)},\ a_2^{(k)} \in \mathbb{Z}_{p_k}$. Then, the Wigner function of the whole system $\mathcal{H}_d$ is constructed through dequantizers of the form

$$\hat{\mathcal{U}}(\mathbf{a}_1,\ \mathbf{a}_2) = \bigotimes_{k=1}^{K}\hat{\mathcal{U}}_k(a_1^{(k)},\ a_2^{(k)}). \tag{45}$$

Here $\mathbf{a}_1 = \left[a_1^{(1)}\ \ldots\ a_1^{(K)}\right]^T$, $\mathbf{a}_2 = \left[a_2^{(1)}\ \ldots\ a_2^{(K)}\right]^T$.

4 The discrete center-of-mass tomogram

In Section 2 the product of the Radon transformation of the Wigner function was associated with symplectic and optical tomographic functions. In the case of existence of the full set of MUB (and it is true when d is a prime number), we can describe the state by means of a tomographic function of the form

$$w_{MUB}(n,\ b) = \mathrm{Tr}\ \left(|b,\ n\rangle\langle b,\ n|\cdot\hat{\rho}\right), \quad b = 0, \ldots,\ d,\ n \in \mathbb{Z}_d, \tag{46}$$

and in this case dequantizers $\hat{\mathcal{U}} = |b,\ n\rangle\langle b,\ n|$ are projectors on the MUB states $|\mu,\ \kappa\rangle$. Similarly, in the continuous case, the symplectic tomogram dequantizer $\hat{\mathcal{U}}(X,\ \mu,\ \nu) = \delta\left(X - \mu\hat{q} - \nu\hat{p}\right)$ is the projector on the corresponding to the eigenvalue X eigenstate of the operator $\hat{X} = \mu\hat{q} + \nu\hat{p}$. In the paper [49], the quantizers of the scheme for the transition from the w_{MUB} function to the density matrix were obtained

$$\hat{\mathcal{D}}(\kappa,\ \mu) = |\mu,\ \kappa\rangle\langle\mu,\ \kappa| - \frac{1}{d+1}\hat{I}. \tag{47}$$

Let us suppose that d is not a prime number. By analogy with the continuous case, it is possible to introduce a tomographic function of the form

$$w(\boldsymbol{\kappa} \mid \boldsymbol{\mu},\ \boldsymbol{\nu}) = \sum_{\mathbf{a}_1 \in \bigotimes_{k=1}^{N} \mathbb{Z}_{p_k}} \sum_{\mathbf{a}_2 \in \bigotimes_{k=1}^{N} \mathbb{Z}_{p_k}} W(\mathbf{a}_1,\ \mathbf{a}_2) \prod_{k=1}^{K} \delta_{\mu_k a_1^{(k)} + \nu_k a_2^{(k)},\ \kappa_k}, \tag{48}$$

where $d = p_1 ... p_N$ is a prime factors decomposition of the composite d and

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_K \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} \nu_1 \\ \vdots \\ \mu_K \end{bmatrix}, \quad \boldsymbol{\kappa} = \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_K \end{bmatrix}, \tag{49}$$

where $\mu_k,\ \nu_k,\ \kappa_k \in \mathbb{Z}_{p_k}$. Thus, we see that the case of composite d is analog of the case of the multiparticle system of continuous variables. Reconstruction of the discrete Wigner function from the values of the tomographic function $w(\boldsymbol{\kappa}, \boldsymbol{\mu},\ \boldsymbol{\nu})$ is possible by means of the iterative procedure described in [76].

In this section we propose to continue the construction of analogies between discrete and continuous cases by introducing the discrete counterpart of the center-of-mass tomogram

$$w_{cm}(\kappa \mid \boldsymbol{\mu},\ \boldsymbol{\nu}) = \sum_{\mathbf{a}_1 \in \mathbb{Z}_p^N} \sum_{\mathbf{a}_2 \in \mathbb{Z}_p^N} W(\mathbf{a}_1,\ \mathbf{a}_2) \delta_{\kappa,\ \boldsymbol{\mu}\cdot\mathbf{a}_1 + \boldsymbol{\nu}\cdot\mathbf{a}_2}. \tag{50}$$

Note, that the proposed definition is appropriate for cases where d is a power of a prime number. Otherwise, it is not clear by what rules elements of different fields would be added. If d is a power of a composite number b, then the parameters μ_k, ν_k, κ_k are elements of $\mathbb{Z}_b$, and $\mathbb{Z}_b$ is a ring, not a field. This circumstance complicates the analysis.

So, we consider a quantum system of dimension $d = p^N$, where p is a prime number. The components of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ and the variable κ belong to the field of residues modulo p, i.e. $\mathbb{Z}_p$. The phase space is the space $\mathbb{Z}_p^{2N}$. We will also use the notation

$$\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{bmatrix}. \tag{51}$$

In the space $\mathbb{Z}_p^{2N}$ the equation $\boldsymbol{\gamma} \cdot \mathbf{x} = \kappa$ specifies a finite set that we call a hyperplane of $2N - 1$ dimension. The sum of the values of the discrete Wigner function over the points belonging to the hyperplane defined by the parameters $\boldsymbol{\gamma}$ and κ is the value of the discrete center-of-mass tomogram. Note, that the values of the function $w_{cm}(\kappa \mid \boldsymbol{\gamma})$ are nonnegative. Indeed, it is not difficult to obtain a connection of the discrete center-of-mass tomogram with a tomographic function $w(\boldsymbol{\kappa} \mid \boldsymbol{\gamma})$ defined in (48)

$$w_{cm}(\kappa \mid \boldsymbol{\gamma}) = \sum_{\boldsymbol{\upsilon} \in \mathbb{Z}_p^N} w(\boldsymbol{\upsilon} \mid \boldsymbol{\gamma}) \delta_{\kappa,\ \sum_{n=1}^{N} \upsilon_n}. \tag{52}$$

The function $w(\boldsymbol{\upsilon} \mid \boldsymbol{\gamma})$ is nonnegative, therefore, each term in the last expression is nonnegative, hence the values of the function $w_{cm}(\kappa \mid \boldsymbol{\gamma})$ are also nonnegative. Furthermore, it follows from (52) that

$$\sum_{\kappa=0}^{p-1} w_{cm}(\kappa \mid \boldsymbol{\gamma}) = 1. \tag{53}$$

It is also necessary to prove that the function $w_{cm}(\kappa \mid \boldsymbol{\gamma})$ uniquely determines the state of the quantum system. To do this, it is sufficient to show that the discrete Wigner function is uniquely restored by the values of the discrete center-of-mass tomogram.

First, we denote the pair of vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ as one vector $\boldsymbol{\gamma} = (\boldsymbol{\mu}, \ \boldsymbol{\nu})$, and pair of vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ as $\boldsymbol{\alpha} = (\mathbf{a}_1, \ \mathbf{a}_2)$. Then, for (50) we have

$$w_{cm}(\kappa \mid \boldsymbol{\gamma}) = \sum_{\boldsymbol{\alpha}} A(\kappa, \ \boldsymbol{\gamma}; \ \boldsymbol{\alpha}) W(\boldsymbol{\alpha}), \tag{54}$$

where $A(\kappa, \ \boldsymbol{\gamma}; \ \boldsymbol{\alpha}) = \delta_{\kappa, \ \boldsymbol{\gamma}\cdot\boldsymbol{\alpha}}$. As we consider the case of $d = p^N$, where p is a prime number, so $\boldsymbol{\alpha}, \ \boldsymbol{\gamma} \ \in \mathbb{Z}_p^{2N}$. The functions $w_{cm}(\kappa \mid \boldsymbol{\gamma})$ and $W(\boldsymbol{\alpha})$ can be considered as p^{2N+1}-dimensional and p^{2N}-dimensional vectors. Therefore, the function values $A(\kappa, \ \boldsymbol{\gamma}; \ \boldsymbol{\alpha})$ form a matrix $\hat{A}$ of size $p^{2N+1} \times p^{2N}$, where $(\kappa, \ \boldsymbol{\gamma})$ decodes the row index of $\hat{A}$ and $\boldsymbol{\alpha}$ decodes the column index.

The linear transform is invertible if and only if a square matrix $\hat{B} = \hat{A}^T \hat{A}$ of size $p^{2N} \times p^{2N}$ is invertible. In the case, the transform of discrete center-of-mass tomogram to discrete Wigner function is determined by the matrix $\hat{B}^{-1} \cdot \hat{A}^T$. We can see that

$$B(\boldsymbol{\alpha}, \ \boldsymbol{\beta}) = \sum_{\boldsymbol{\gamma}\in\mathbb{Z}_p^{2N}} \delta_{\boldsymbol{\gamma}\cdot\boldsymbol{\alpha}, \ \boldsymbol{\gamma}\cdot\boldsymbol{\beta}}. \tag{55}$$

Let us consider the equation $\boldsymbol{\gamma} \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta}) = \mathbf{0}$. If $\boldsymbol{\alpha} = \boldsymbol{\beta}$ arbitrary $\boldsymbol{\gamma} \in \mathbb{Z}_p^{2N}$ is a solution. Therefore, all the diagonal elements of the matrix $\hat{B}$ are equal to p^{2N}. In the case of $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ the equation $\boldsymbol{\gamma} \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta}) = \mathbf{0}$ has p^{2N-1} solutions. Thus, the elements of the matrix $\hat{B}$ can be presented in the form

$$B(\boldsymbol{\alpha}, \ \boldsymbol{\beta}) = p^{2N-1} \cdot C(\boldsymbol{\alpha}, \ \boldsymbol{\beta}), \tag{56}$$

where

$$C(\boldsymbol{\alpha}, \ \boldsymbol{\beta}) = \begin{cases} p, & \boldsymbol{\alpha} = \boldsymbol{\beta} \\ 1, & \boldsymbol{\alpha} \neq \boldsymbol{\beta} \end{cases}. \tag{57}$$

The value of function $C(\boldsymbol{\alpha}, \ \boldsymbol{\beta})$ form the invertable $\hat{C}$, and for elements $\hat{C}^{-1}$ we have

$$C^{-1}(\boldsymbol{\alpha}, \ \boldsymbol{\beta}) = \frac{1}{p^{2N+1} - p^{2N} + p^2 - 2p + 1} \begin{cases} p^{2N} + p - 2, & \boldsymbol{\alpha} = \boldsymbol{\beta} \\ -1, & \boldsymbol{\alpha} \neq \boldsymbol{\beta} \end{cases}. \tag{58}$$

Thus, from (56) we have $\hat{B}^{-1} = p^{1-2N} \cdot \hat{C}^{-1}$. Finally, the inverse map for (54) has the following form

$$\mathrm{W}(\boldsymbol{\alpha}) = \frac{1}{p^{2N-1}} \sum_{\boldsymbol{\beta}\in\mathbb{Z}_p^{2N}} \ \sum_{(\kappa, \ \boldsymbol{\gamma})\in\mathbb{Z}_p^{2N+1}} C^{-1}(\boldsymbol{\alpha}, \ \boldsymbol{\beta}) \, \delta_{\kappa, \ \boldsymbol{\gamma}\cdot\boldsymbol{\beta}} \, w_{cm}(\kappa \mid \boldsymbol{\gamma}) \tag{59}$$

5 The ququart case

To demonstrate the representations of qudit states given in the paper, we construct the discrete center-of-mass tomogram for the simplest case in which it is possible, i.e. for the case of ququart.

The density operator $\hat{\rho}$ of the ququart state is the operator defined on the Hilbert space $\mathcal{H}_4$. The construction of the corresponding Wigner function is carried out according to the scheme proposed in Section 3. We present the original Hilbert space as the tensor product of the Hilbert spaces of pure qubit states $\mathcal{H}_4 = \mathcal{H}_2^{(A)} \otimes \mathcal{H}_2^{(B)}$. Thus, the original ququart is considered as a composite system of two qubits denoted as A and B.

The phase space of each qubit is a finite plane $\mathbb{Z}_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. The corresponding coordinates are denoted as a_1^A, a_2^A, a_1^B, a_2^B. A point in the ququart phase space is defined by the values of these four parameters. In other words, the ququart phase space is a Cartesian product of the qubit phase spaces A and B. The points of the resulting phase space $\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$ can be arranged on a plane, see Fig. 2.

The Wigner function of the state of ququart is determined for the constructed phase space $\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$, the corresponding dequantizers have the form

$$\hat{\mathcal{U}}(a_1^A,\ a_1^B,\ a_2^A,\ a_2^B) = \hat{\mathcal{U}}(a_1^A,\ a_2^A) \otimes \hat{\mathcal{U}}(a_2^B,\ a_2^B). \tag{60}$$

In this case the qubit operators $\hat{\mathcal{U}}(a_1,\ a_2)$ are defined by the expression (39).

For the values of the discrete Wigner function of ququart state we have

$$\sum_{a_1^A,\ a_2^A\ \in\ \Lambda_A}\ \sum_{a_1^B,\ a_2^B\ \in\ \Lambda_B} W(a_1^A,\ a_1^B,\ a_2^A,\ a_2^B) \geq 0, \tag{61}$$

where Λ_A and Λ_B are lines in the phase spaces of qubits A and B correspondingly.

$$\Lambda_A = \left\{a_1^A,\ a_2^A\ :\ \mu^A a_1^A + \nu^A a_1^A = \kappa^A,\ \ \mu^A,\ \nu^A,\ \kappa^A \in \mathbb{Z}_2\right\}, \tag{62}$$

$$\Lambda_B = \left\{a_1^B,\ a_2^B\ :\ \mu^B a_1^B + \nu^B a_1^B = \kappa^B,\ \ \mu^B,\ \nu^B,\ \kappa^B \in \mathbb{Z}_2\right\}. \tag{63}$$

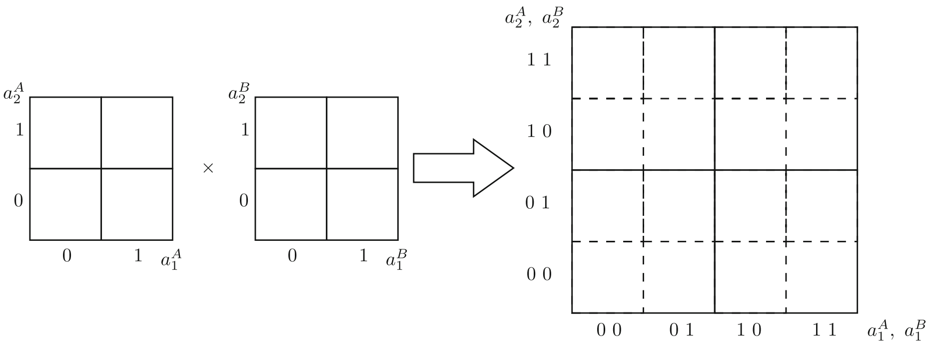

Fig. 2 Representation of the ququart phase space as a Cartesian product of the phase spaces of two qubits. A point in the qubit phase space $A(B)$ is given by the pair $(a_1^{A(B)},\ a_2^{A(B)})$, with $a_1^{A(B)},\ a_2^{A(B)} \in \mathbb{Z}_2$

The center-of-mass tomogram of the ququart state can be presented in the form

$$w_{cm}(\kappa \mid \mu^A,\ \mu^B,\ \nu^A,\ \nu^B) = \sum_{a_1^A,\ a_1^B,\ a_2^A,\ a_2^B \in \Omega} W(a_1^A,\ a_1^B,\ a_2^A,\ a_2^B), \tag{64}$$

where the set Ω is a finite hyperplane of dimension 3 and consists of 8 points

$$\Omega = \{a_1^A,\ a_1^A,\ a_2^A,\ a_2^B\ :\ \mu^A a_1^A + \mu^B a_2^B + \nu^A a_2^A + \nu^B a_2^B = \kappa,\\ \mu^A,\ \mu^B,\ \nu^A,\ \nu^B,\ \kappa\ \in\ \mathbb{Z}_2\}. \tag{65}$$

Thus, the center-of-mass tomogram of the ququart state is the sum of the values of the Wigner function over points belonging to the finite hyperplane Ω. In other words, the center-of-mass tomogram is defined on the set of all finite three-dimensional spaces $\Omega \in \mathbb{Z}_2^2 \times \mathbb{Z}_2^2$. For each value κ, there are 15 different Ω, and since $\kappa \in \mathbb{Z}_2$, there are only 30 three-dimensional finite subspaces in the space $\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$. Note that subspaces Ω given by equations with the same coefficients $\mu^{A(B)}$, $\nu^{A/B}$ and differing free terms κ have no common points. Thus, it is possible to divide the initial phase space into a pair of disjoint subspaces of dimension 3. In Fig. 3 we present all the possible divisions.

Based on the property of normalization of the Wigner function, we can conclude

$$w_{cm}(0 \mid \mu^A,\ \mu^B,\ \nu^A,\ \nu^B) + w_{cm}(1 \mid \mu^A,\ \mu^B,\ \nu^A,\ \nu^B) = 1. \tag{66}$$

This means that for determination of the center-of-mass tomogram, it is sufficient for us to determine 15 of its values. Let us find a linear map of the elements of the density matrix to the elements of the probability vector composed of the values $w_{cm}(0 \mid \mu^A,\ \mu^B,\ \nu^A,\ \nu^B)$. To do this, let us vectorize the density matrix, i.e., let us walk through its elements row by row and write them in one column, which is denoted as $\boldsymbol{\rho}$ and which consists of 16 elements. Also, let's make a column vector $\boldsymbol{\mathcal{P}}$ from the values $w_{cm}(0 \mid \mu^A,\ \mu^B,\ \nu^A,\ \nu^B)$ ordered lexicographically by the values of the four $(\mu^A,\ \mu^B,\ \nu^A,\ \nu^B)$. The final size of the vector is equal to 15. For convenience of calculations we will add the element $w_{cm}(1 \mid 1,\ 1,\ 1,\ 1)$. Then, the mapping from $\boldsymbol{\rho}$ to $\boldsymbol{\mathcal{P}}$ is given by the square matrix $\hat{\mathcal{U}}$ of the size 16×16

$$\boldsymbol{\mathcal{P}} = \hat{\mathcal{U}} \cdot \boldsymbol{\rho}, \tag{67}$$

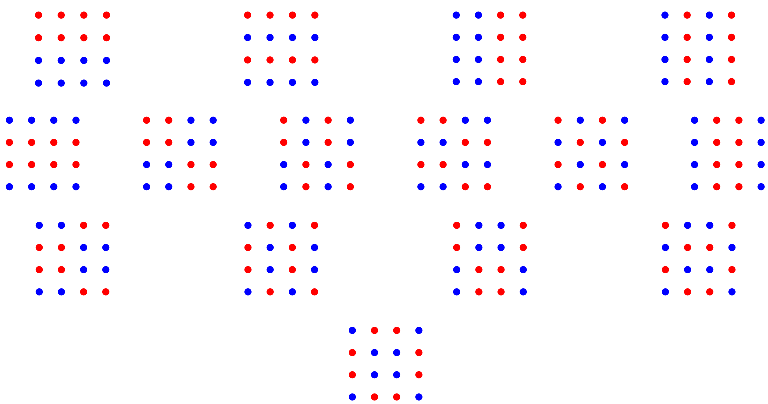

Fig. 3 Divisions of the ququart phase space into pairs of disjoint finite subspaces of dimension 3. Blue color indicates the points related to the case $\kappa = 0$, red color relates to the case $\kappa = 1$

Using the definition of the ququart center-of-mass tomogram (64) and making the corresponding calculations, we obtain

$$\hat{\mathcal{U}} = \frac{1}{2}\cdot\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1\\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1\\
2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & i & 0 & 0 & 1 & 0 & i & -i & 0 & 1 & 0 & 0 & -i & 0 & 1\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1\\
1 & 0 & 0 & i & 0 & 1 & i & 0 & 0 & -i & 1 & 0 & -i & 0 & 0 & 1\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1\\
1 & i & 0 & 0 & -i & 1 & 0 & 0 & 0 & 0 & 1 & i & 0 & 0 & -i & 1\\
1 & 0 & 0 & i & 0 & 1 & -i & 0 & 0 & i & 1 & 0 & -i & 0 & 0 & 1\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\\
1 & 0 & i & 0 & 0 & 1 & 0 & -i & -i & 0 & 1 & 0 & 0 & i & 0 & 1\\
1 & i & 0 & 0 & -i & 1 & 0 & 0 & 0 & 0 & 1 & -i & 0 & 0 & i & 1\\
1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1\\
1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix} \tag{68}$$

In the previous section, we proved the possibility of reconstructing the values of the discrete Wigner function from the values of the discrete center-of-mass tomogram. This also means that we can reconstruct the density matrix. In particular, it is easy to see that the mapping (67) is reversible, i.e. there exists $\hat{\mathcal{D}} = \hat{\mathcal{U}}^{-1}$ and

$$\boldsymbol{\rho} = \hat{\mathcal{D}} \cdot \boldsymbol{\mathcal{P}}. \tag{69}$$

Finally, the calculations give us the expression of the inverse matrix

$$\hat{\mathcal{D}} = \frac{1}{2}\cdot\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & -i & -1+i & -1+i\\
1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -i & 0 & -1+i & -1+i\\
0 & 0 & 1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & -i & 0 & 0 & 0 & -1+i & i\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & i & -1-i & -1-i\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1\\
0 & 0 & 1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & -1\\
1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & -1 & 0 & 0 & 0 & i & 0 & 0 & 0\\
1 & 0 & 0 & 0 & i & 0 & 0 & 0 & 1 & 0 & 0 & 0 & i & 0 & -1-i & -1-i\\
0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1\\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -i & 0 & 0 & 0 & i & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & i & 0 & 0 & 0 & i & 0 & 0 & 0 & -1-i & -i\\
1 & 0 & 0 & 0 & i & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -i & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & i & 0 & 0 & 0 & -i & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix} \tag{70}$$

6 Summary

In conclusion, we summarize the main results of our work. It was well known that it was possible to construct the phase space for qudit systems and to describe their states by the discrete Wigner function. Also, there is an analogy in the construction of the tomographic functions of qudits and the continuous variables system. For example, the qudit tomographic function associated with MUB measurements corresponds to the symplectic tomogram.

In the paper, we exploited this idea to construct a new tomographic function that described the qudit states. This function $w_{cm}(\kappa \mid \boldsymbol{\mu}, \ \boldsymbol{\nu})$ is an analog of the center-of-mass tomogram used to describe the multiparticle continuous variables system. The definition of our function is presented as expression (50) that connects it with the discrete Wigner function. The new function was called the discrete center-of-mass tomogram. We showed that it was nonnegative one and the map from the discrete Wigner function to the discrete center-of-mass tomogram was invertible. The latter property means that we can restore the discrete Wigner function as well as the density operator from the values of the discrete center-of-mass tomogram. We also paid attention to the simplest case where it was possible to describe the quantum states by the new tomographic function, i.e. we studied the case of ququart system. Here, we found the expressions for linear operators of the direct and inverse maps from density operator to the discrete center-of-mass tomogram.

Thus, we propose a new approach to describe the qudit states. The considered representation can be used in the case of a d-level quantum system with d is the power of a prime number.

Finally, we should note that the value of the discrete center-of-mass tomogram is the sum of the discrete Wigner function values over a finite hyperplane of dimension $2N - 1$, where $d = p^N$ and p is a prime number. However, we can develop the analogy between continuous and discrete quantum systems even further by using the finite space geometric objects, and we can construct the function that can be called the discrete cluster tomogram. This function can be obtained by summing the discrete Wigner function values over finite hyperplanes of dimension lower than $2N - 1$. This function should be studied in future research.

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