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The Three Coefficient Matrices can Completely Determine Six Algebraically Independent Local Invariants of Three-qubit States

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Abstract
The coefficient matrix and local unitary (LU) transformation invariant of three-qubit states are studied in this paper. There is general relation between the coefficient matrix and independent LU transformation invariant existing which is shown. Especially, the three coefficient matrix can completely determine six algebraically independent local invariants for three-qubit pure states.

Keywords Coefficient matrix · LU transformation · Three-qubit pure states

1 Introduction

Entanglement is known to be a promising resource which enables us to execute various quantum tasks such as teleportationquantum computing, quantum cryptography, etc. [1, 2]. With experimental technology development, multi-mode entanglement states based on noise correlation and squeezing of parametrically amplified multi-wave mixing literatures in atomic ensemble are achieved [3–5]. It is imminent of application of quantum entanglement, however, an important question for quantum information theory is how to determine if a given state is entangled. A common feature of entanglement for multipartite quantum systems is that the non-local properties do not change under local transformations, i.e. the unitary operations act independently on each of the subsystems. Hence the entanglement can all be characterized by the invariants under LU transformations. Two quantum states are locally equivalent under LU transformations if and only if all these invariants have equal values [6]. Numerous researchers have investigated the equivalent classes of three-qubit

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states specified by LU transformation invariants [7–9]. Sudbery studied invariants of three-qubit states under LU transformations with polynomials. It has been shown that three-qubit pure states have six independent invariants [10]. With the six independent invariants, we can determine whether two three-qubit pure states are LU-equivalent or not.

In 2015, Li et al. have proposed a simpler and more efficient approach to the stochastic local operations and classical communication (SLOCC) classification of general n-qubit pure states in [11]. For two n-qubit pure states connected by SLOCC, Li proved that the ranks of the coefficient matrices are equal whether or not the permutation of qubits is fulfilled on both states. This theorem provides a way of partitioning all the n-qubit states into different families. Zhang et al. [12] has studied the invariants of arbitrary dimensional multipartite quantum states under local unitary operations. They give a set of invariants in terms of singular values of coefficient matrices of multipartite pure states and multipartite mixed states.

Recently, Zha et al. proposed the relation of SLOCC invariants and the character polynomial of the square matrix for three and four qubit states [13].

In this paper we give a set of invariants in terms of coefficient matrices of three pure states. Furthermore, the general relation between the coefficient matrix and independent LU transformation invariant is discussed.

2 The Definition of Coefficient Matrices for Three-qubit Pure States

An arbitrary pure state $|\psi\rangle$ of the three qubits A, B and C is expressed in the form of

$$\begin{aligned}|\psi\rangle_{ABC} &= a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle \\ &+ a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle\end{aligned} \tag{1}$$

For three qubit pure state, the coefficient matrix can be defined

$$M_{A(BC)} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 & a_7 \end{pmatrix}, M_{B(AC)} = \begin{pmatrix} a_0 & a_1 & a_4 & a_5 \\ a_2 & a_3 & a_6 & a_7 \end{pmatrix}, M_{C(AB)} = \begin{pmatrix} a_0 & a_2 & a_4 & a_6 \\ a_1 & a_3 & a_5 & a_7 \end{pmatrix} \tag{2}$$

3 The Six Algebraically Independent Local Invariants of Three Qubit Pure State

It is known [14–16] that there are six algebraically independent local invariants of three qubit pure state.

There is one independent invariant of degree 2,

$$I_1 = \langle\psi|\psi\rangle \tag{3}$$

The three linearly independent quartic invariants are

$$I_2 = tr\left(\rho_A^2\right), \ I_3 = tr\left(\rho_B^2\right), \ I_4 = tr\left(\rho_C^2\right) \tag{4}$$

There is one independent invariant of degree 6

$$I_5 = 3tr[(\rho_A \otimes \rho_B)\,\rho_{AB}] - tr\left(\rho_A^3\right) - tr\left(\rho_B^3\right) \tag{5}$$

There is one independent invariant of degree 8, A convenient, and physically significant, choice is the 3-tangle identified by Coffman, Kundu and Wootters [9] :

$$I_6 = \tau_{ABC}^2 \tag{6}$$

We can show that $\tau_{ABC} = |C_{AB}| = |C_{AC}| = |C_{BC}|$

where $C_{AB} = \langle \psi^* \left| \hat{\sigma}_{Ay}\hat{\sigma}_{By}\hat{\sigma}_{Cx} \right| \psi\rangle^2 + \langle \psi^* \left| \hat{\sigma}_{Ay}\hat{\sigma}_{By}\hat{\sigma}_{Cz} \right| \psi\rangle^2 - \langle \psi^* \left| \hat{\sigma}_{Ay}\hat{\sigma}_{By} \right| \psi\rangle^2$

From (1), one can obtain

$$C_{AB} = 4[(a_0a_7 + a_1a_6 - a_2a_5 - a_3a_4)^2 + 4\,(a_0a_6 - a_2a_4)\,(a_3a_5 - a_1a_7)]$$

Similarly, it is easy to show

$$\begin{aligned} C_{AC} &= 4[(a_0a_7 - a_1a_6 + a_2a_5 - a_3a_4)^2 + 4\,(a_0a_5 - a_1a_4)\,(a_3a_6 - a_2a_7)] \\ C_{AB} &= 4[(a_0a_7 - a_1a_6 - a_2a_5 + a_3a_4)^2 + 4\,(a_0a_3 - a_1a_2)\,(a_5a_6 - a_4a_7)] \end{aligned} \tag{7}$$

4 The Relation of Local Invariants with Coefficient Matrices of Three Qubit Pure State

From (1)–(4), we have

$$I_1 = tr\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right) \tag{8}$$

It is easy to show that

$$2\,(1 - I_2) = 4Det\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right);$$

and $2\,(1 - I_3) = 4Det\left(M_{B(AC)}M^{\dagger}_{B(AC)}\right)$.

With

$$2\,(1 - I_4) = 4Det\left(M_{C(AB)}M^{\dagger}_{C(AB)}\right). \tag{9}$$

$$\begin{aligned} I_5 \;=\; & 3tr\left[\left(M_{A(BC)}M^{\dagger}_{A(BC)} \otimes M_{B(AC)}M^{\dagger}_{B(AC)}\right)\tilde{M}_{C(AB)}M^{*}_{C(AB)}\right] \\ & - tr\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right)^3 - tr\left(M_{B(AC)}M^{\dagger}_{B(AC)}\right)^3 \end{aligned} \tag{10}$$

From [13],

$$F^{n}_{1,2,\cdots i}(\psi) = (\nu)^{\otimes(i)}C^{n}_{1,2,\cdots i}(\psi)(\nu)^{\otimes(n-i)}\left[C^{n}_{1,2,\cdots i}(\psi)\right]^T \tag{11}$$

where $\nu = i\sigma_y$ and σ_y are the Pauli operators.

we have

$$\begin{aligned} F^{(3)}_1 &= \nu M_{A(BC)}(\nu \otimes \nu)M^{\sim}_{A(BC)} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 & a_7 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_0 & a_4 \\ a_1 & a_5 \\ a_2 & a_6 \\ a_3 & a_7 \end{pmatrix} \\ &= \begin{pmatrix} a_0a_7 - a_1a_6 - a_2a_5 + a_3a_4 & 2(a_4a_7 - a_5a_6) \\ -2(a_0a_3 - a_1a_2) & -(a_0a_7 - a_1a_6 - a_2a_5 + a_3a_4) \end{pmatrix} \end{aligned} \tag{12}$$

where $\nu = i\sigma_y$ and σ_y are the Pauli operators.

It is also easy to show that

$$\tau_{ABC} = \left|Det F^{(3)}_1\right|; \tag{13}$$

Finally, we give the values of these invariants for some special states

For a factorized state

$$|\varphi\rangle_{ABC} = b|100\rangle + a|111\rangle, |a|^2 + |b|^2 = 1 \tag{14}$$

Then, we have

$$M_{A(BC)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & a \end{pmatrix}, M_{B(AC)} = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, M_{C(AB)} = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

Therefore, $M_{A(BC)}M^{\dagger}_{A(BC)} = \begin{pmatrix} 0 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix}$; $M_{B(AC)}M^{\dagger}_{B(AC)} = \begin{pmatrix} |b|^2 & 0 \\ 0 & |a|^2 \end{pmatrix}$;

$$M_{C(AB)}M^{\dagger}_{C(AB)} = \begin{pmatrix} |b|^2 & 0 \\ 0 & |a|^2 \end{pmatrix}$$

$$I_1 = tr\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right) = |a|^2 + |b|^2;$$

$$Det\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right) = 0;$$

$$Det\left(M_{B(AC)}M^{\dagger}_{B(AC)}\right) = |a|^2|b|^2;$$

$$Det\left(M_{C(AB)}M^{\dagger}_{C(AB)}\right) = |a|^2|b|^2;$$

$$I_2 = 1 - 2Det\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right) = 1;$$

$$I_3 = 1 - 2Det\left(M_{B(AC)}M^{\dagger}_{B(AC)}\right) = |a|^4 + |b|^4;$$

$$I_4 = 1 - 2Det\left(M_{C(AB)}M^{\dagger}_{C(AB)}\right) = |a|^4 + |b|^4;$$

$$\begin{aligned} I_5 &= 3tr\left[\left(M_{A(BC)}M^{\dagger}_{A(BC)} \otimes M_{B(AC)}M^{\dagger}_{B(AC)}\right)\tilde{M}_{C(AB)}M^{*}_{C(AB)}\right] \\ &- tr\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right)^3 - tr\left(M_{B(AC)}M^{\dagger}_{B(AC)}\right)^3 \\ &= 3tr\left[\left(\begin{pmatrix} 0 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix} \otimes \begin{pmatrix} |b|^2 & 0 \\ 0 & |a|^2 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & |b|^2 & 0 \\ 0 & 0 & 0 & |a|^2 \end{pmatrix}\right] - tr\begin{pmatrix} 0 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix}^3 - tr\begin{pmatrix} |b|^2 & 0 \\ 0 & |a|^2 \end{pmatrix}^3 \\ &= |a|^6 + |b|^6 \end{aligned}$$

$$\begin{aligned} F_1^{(3)} &= \nu M_{A(BC)}(\nu \otimes \nu)M^{\sim}_{A(BC)} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & a \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \\ 0 & 0 \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2ab \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$\tau_{ABC} = \left|Det F_1^{(3)}\right| = 0$$

For a generalized GHZ state

$$|\psi\rangle_{ABC} = p|000\rangle + q|111\rangle \tag{15}$$

Then, we have

$$M_{A(BC)} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, M_{B(AC)} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, M_{C(AB)} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

$$I_1 = tr\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right) = |p|^2 + |q|^2;$$

$$Det\left(M_{A(BC}M^{\dagger}_{A(BC)}\right) = |p|^2|q|^2;$$

$$Det\left(M_{B(AC)}M^{\dagger}_{B(AC)}\right) = |p|^2|q|^2;$$

$$Det\left(M_{C(AB)}M^{\dagger}_{C(AB)}\right) = |p|^2|q|^2;$$

$$I_2 = 1 - 2Det\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right) = |p|^4 + |q|^4;$$

$$I_3 = 1 - 2Det\left(M_{B(AC)}M^{\dagger}_{B(AC)}\right) = |p|^4 + |q|^4;$$

$$I_4 = 1 - 2Det\left(M_{C(AB)}M^{\dagger}_{C(AB)}\right) = |p|^4 + |q|^4;$$

$$\begin{aligned} I_5 &= 3tr\left[\left(M_{A(BC)}M^{\dagger}_{A(BC)} \otimes M_{B(AC)}M^{\dagger}_{B(AC)}\right)\hat{M}_{C(AB)}M^{*}_{C(AB)}\right] \\ &- tr\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right)^3 - tr\left(M_{B(AC)}M^{\dagger}_{B(AC)}\right)^3 \\ &= |p|^6 + |q|^6 \end{aligned}$$

$$\begin{aligned} F_1^{(3)} &= \nu M_{A(BC)}(\nu \otimes \nu)M^{\sim}_{A(BC)} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & q \end{pmatrix} \\ &= \begin{pmatrix} pq & 0 \\ 0 & pq \end{pmatrix} \end{aligned}$$

Therefore, we have

$$\tau_{A(BC)} = \left|Det F_1^{(3)}\right| = 4p^2q^2$$

For the W state

$$|\varphi\rangle_{ABC} = \lambda_1|001\rangle + \lambda_2|010\rangle + \lambda_4|100\rangle \tag{16}$$

Using (2), we can also obtain

$$M_{A(BC)} = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & 0 \\ \lambda_4 & 0 & 0 & 0 \end{pmatrix}, M_{B(AC)} = \begin{pmatrix} 0 & \lambda_1 & \lambda_4 & 0 \\ \lambda_2 & 0 & 0 & 0 \end{pmatrix}, M_{A(BC)} = \begin{pmatrix} 0 & \lambda_2 & \lambda_4 & 0 \\ \lambda_1 & 0 & 0 & 0 \end{pmatrix}$$

$$I_1 = tr\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right) = |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_4|^2$$

$$Det\left(M_{A(BC)}M^{\dagger}_{A(BC)}\right) = |\lambda_1|^2 + |\lambda_2|^2|\lambda_4|^2$$

$$Det\left(M_{B(AC)}M_{B(AC)}^{\dagger}\right) = |\lambda_1|^2 + |\lambda_4|^2|\lambda_2|^2$$

$$Det\left(M_{C(AB)}M_{C(AB)}^{\dagger}\right) = |\lambda_2|^2 + |\lambda_4|^2|\lambda_1|^2$$

$$\begin{aligned} I_5 &= 3tr\left[\left(M_{A(BC)}M_{A(BC)}^{\dagger} \otimes M_{B(AC)}M_{B(AC)}^{\dagger}\right)\tilde{M}_{C(AB)}M_{C(AB)}^{*}\right] \\ &- tr\left(M_{A(BC)}M_{A(BC)}^{\dagger}\right)^3 - tr\left(M_{B(AC)}M_{B(AC)}^{\dagger}\right)^3 \\ &= |\lambda_1|^6 + |\lambda_2|^6 + |\lambda_4|^6 + 3|\lambda_1|^2|\lambda_2|^2|\lambda_4|^2 \end{aligned}$$

$$\begin{aligned} F_1^{(3)} &= \nu M_{A(BC)}(\nu \otimes \nu)M_{A(BC)} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \lambda_1 & \lambda_2 & 0 \\ \lambda_4 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \lambda_4 \\ \lambda_1 & 0 \\ \lambda_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -2\lambda_1\lambda_2 & 0 \end{pmatrix} \end{aligned}$$

$$\tau_{ABC} = \left|Det F_1^{(3)}\right| = 0$$

It is well know that the entanglement measure of qubits A, with the pair (BC) is $\tau_{A(BC)} = 2(1 - I_2) = 4Det\left(M_{A(BC)}M_{A(BC)}^{\dagger}\right)$

Similarly,

$$\tau_{B(AC)} = 2(1 - I_3) = 4Det\left(M_{B(AC)}M_{B(AC)}^{\dagger}\right)$$

$$\tau_{C(AB)} = 2(1 - I_4) = 4Det\left(M_{C(AB)}M_{C(AB)}^{\dagger}\right)$$

Therefore, we can see those entanglement measure $\tau_{A(BC)}$, $\tau_{B(AC)}$, $\tau_{C(AB)}$, τ_{ABC}, can be easy to obtain by coefficient matrices.

5 Conclusion

In the present paper, we have proposed three coefficient matrix for three qubit pure states, and the mathematical connection between independent LU transformation invariant and the coefficient matrices was established, which indicates that entanglement classification and quantification are closely linked to the coefficient matrices. Examples were discussed to show that the coefficient matrices is capable of dealing with three qubit pure states. We give six algebraically independent local invariants in terms of coefficient matrices. For instance, by means of the coefficient matrix, it can be easily to obtain entanglement measure of 3-tangle τ_{ABC} . As these LU invariants can be explicitly calculated, our approach gives a simple way in verifying the LU equivalence of given three qubit pure states. We expect that our work could come up with further theoretical and experimental results.

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