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Representations of von Neumann Algebras and Ultraproducts

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Abstract
We will introduce the concept of ergodicity of states with respect to some group of transformations on a von Neumann algebra and its properties are studied. A connection between the ergodic states on the von Neumann algebra and the representations of von Neumann algebras associated to them will be described. We also study the properties of ultraproducts of von Neumann algebras with ergodic states and corresponding representations. Here we use ultraproducts of von Neumann algebras by Groh (J. Operator Theory **11**(2), 395–404 1984) and Raynaud (J. Operator Theory **48**(1), 41–68 2002). In particular, we will show that the ultraproduct of irreducibles representations isn't, generally speaking, irreducible.

Keywords Ergodic states · Representations · Ultraproducts

1 Introduction

The theory of ergodic states on $*$-algebras has been developed for invariant transformations. At the same time, for abelian algebras many results from the theory of quasi-invariant ergodic probability measures can be easily transferred. The concept of ergodicity of states can be considered for "quasiinvariant" states.

An ultraproduct does not preserve some "good" properties of its cofactors (see, for example, [8]). Therefore, the ultraproducts frequently lead to exotic results.

2 Ergodic States and Representations Irreducibles

Definition 1 Let $\mathcal{M}$ be a von Neumann algebra, φ and ψ be a normal states on $\mathcal{M}$. The states φ and ψ are said to be equivalent if $\varphi(x^*x) = 0 \iff \psi(x^*x) = 0$, $x \in \mathcal{M}$. The states φ and ψ are said to be singular if there is an operator $x \in \mathcal{M}$ such that $\varphi(x^*x) = 0$ and $\psi(1 - x^*x) = 0$.

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Note 1 1. The equivalence of states means the mutual absolute continuity of these states (see, for example, [3, 5]).
2. For von Neumann algebras it is possible to determine the equivalence and singularity of states only on projections.

We will write $\varphi \sim \psi$ for equivalent states and $\varphi \perp \psi$ for singular states.

Let φ and ψ be normal states on a von Neumann algebra $\mathcal{M}$. Denote $s(\varphi)$ and $s(\psi)$ the supports of states φ and ψ respectively. It is easy to see from definition that $\varphi \sim \psi$ if and only if $s(\varphi) = s(\psi)$ and $\varphi \perp \psi$ if and only if $s(\varphi)s(\psi) = 0$. It is clear also that any two faithful normal states φ and ψ are equivalent, since their supports $s(\varphi) = s(\psi) = 1$.

We will give a characterization of equivalence and orthogonality of states on a von Neumann algebra $\mathcal{M}$ by means of the cyclic representations associated with them.

Theorem 1 *Let $\mathcal{M}$ be a von Neumann algebra, φ and ψ be normal states on $\mathcal{M}$, let $(H_\varphi, \pi_\varphi, \xi_\varphi)$ and $(H_\psi, \pi_\psi, \xi_\psi)$ be the cyclic representations of $\mathcal{M}$ associated with φ and ψ respectively.*

(1) *$\varphi \sim \psi$ if and only if there is an $*$-isomorphism $\tau : \pi_\varphi(\mathcal{M}) \to \pi_\psi(\mathcal{M})$ such that $\tau(\pi_\varphi(x)) = \pi_\psi(x)$ for all $x \in \mathcal{M}$.*
(2) *$\varphi \perp \psi$ if and only if there is the projection $p \in \pi_\chi(\mathcal{M})'$, where $\chi = (\varphi + \psi)/2$, $\pi_\chi(\mathcal{M})'$ is the commutator of $\pi_\chi(\mathcal{M})$, such that*

$$\varphi(x) = (\xi_\chi, p\pi_\chi(x)\xi_\chi), \ \ \psi(x) = (\xi_\chi, (1-p)\pi_\chi(x)\xi_\chi).$$

Proof (1). We define $*$-morphism $\tau : \pi_\varphi(\mathcal{M}) \to \pi_\psi(\mathcal{M})$ by $\tau(\pi_\varphi(x)) = \pi_\psi(x)$ for all $x \in \mathcal{M}$. Let $\varphi(x^*x) = (\pi_\varphi(x)\xi_\varphi, \pi_\varphi(x)\xi_\varphi) = 0$. This condition holds if and only if $\psi(x^*x) = (\pi_\psi(x)\xi_\psi, \pi_\psi(x)\xi_\psi) = (\tau(\pi_\varphi(x))\xi_\varphi, \tau(\pi_\varphi(x))\xi_\varphi) = 0$. Means, $\pi_\varphi(x) = 0 \Leftrightarrow \tau(\pi_\varphi(x)) = 0$. Then τ is $*$-isomorphism of the algebras $\pi_\varphi(\mathcal{M})$ and $\pi_\psi(\mathcal{M})$.

We show that there is a positive operator h such that $h^{-1}\pi_\psi(x)h = \pi_\varphi(x)$. Put $h(\pi_\varphi(x))\xi_\varphi = \pi_\psi(x)\xi_\psi$. It follows from the argumentation above that the kernel of the operator h is trivial. Then h is a positive invertible operator implementing the isomorphism τ.

(2). We show that our definition of orthogonality of states is equivalent to the following: $\varphi \perp \psi$ if and only if the condition $\omega \le \varphi$, $\omega \le \psi$ implies $\omega = 0$. Indeed, let the last condition hold true. Then $s(\omega) \le s(\varphi)$, $s(\omega) \le s(\psi)$. If $s(\omega) \ne 0$, then the states φ and ψ cannot be orthogonal. The opposite is obvious. Then the equivalence of our conditions follows from the Lemma 4.1.19 in [2].

□

Let G be a group acting in a von Neumann algebra $\mathcal{M}$. Denote by $Aut(\mathcal{M})$ the group of all $*$-automorphisms of $\mathcal{M}$, and let $\tau : G \to Aut(\mathcal{M})$ be a homomorphism. For every element $g \in G$ we denote by τ_g the $*$-automorphism of algebra $\mathcal{M}$ corresponding to g (for homomorphism τ). We introduce the transformation of state φ using $*$-automorphism τ: $\varphi_{\tau_g}(p) = \varphi(\tau_{g^{-1}}(p))$, $g \in G$, p is a projection.

Definition 2 Let $\mathcal{M}$ be a σ-finite von Neumann algebra and let φ be a normal state on $\mathcal{M}$. We denote $Aut_\varphi(\mathcal{M}) = \{\tau_g \in Aut(\mathcal{M}) : \varphi \sim \varphi_{\tau_g}, g \in G\}$. If $\varphi \sim \varphi_{\tau_g}$ for all $\tau_g \in Aut_\varphi(\mathcal{M})$ then the state φ is called quasiinvariant with respect to the action of the group $Aut_\varphi(\mathcal{M})$.

Theorem 2 *Let φ be a normal state on von Neumann algebra $\mathcal{M}$, ϱ be a $*$-automorphism of $\mathcal{M}$ such that $\varphi_\varrho \sim \varphi$, $(H_\varphi, \pi_\varphi, \xi_\varphi)$ be the cyclic representation of $\mathcal{M}$ associated with state φ. Then there is a unique invertible positive operator h_φ affiliated to $\pi_\varphi(\mathcal{M})$ such that*

$$\pi_\varphi(\varrho(x)) = h_\varphi \pi_\varphi(x) h_\varphi^{-1},\ x \in \mathcal{M}.$$

Proof It follows from Theorem 1 for the cyclic representation $(H_\varphi, \pi_\varphi \circ \varrho, \xi_\varphi)$. □

Definition 3 The quasiinvariant state φ is said to be ergodic with respect to the group $Aut_\varphi(\mathcal{M})$ if for any projection p the condition $\tau_g(p) = p$ for every $\tau_g \in Aut_\varphi(\mathcal{M})$ implies $\varphi(p) = 0$ or $\varphi(1 - p) = 0$.

In other words, the state φ is ergodic with respect to the group $Aut_\varphi(\mathcal{M})$ if for any projection $p \in \mathcal{M}$, such that $0 < \varphi(p) < 1$ there exists $\tau_g \in Aut_\varphi(\mathcal{M})$ such that $\tau_g(p) \neq p$. Note also that if a state φ is ergodic with respect to $Aut_\varphi(\mathcal{M})$, then it is also ergodic with respect to $Aut(\mathcal{M})$. This definition generalizes the definition of ergodicity for a group of invariant transformations on an algebra (see, for example, [11]), and in the case of an faithful state coincides with it. Further we will consider the one-parameter group, that is, put $G = \mathbb{R}$.

We consider some properties of ergodic states on von Neumann algebra.

Theorem 3 (dichotomy theorem) *Let φ and ψ are normal states on the σ-finite von Neumann algebra $\mathcal{M}$ such that the supports $s(\varphi)$ and $s(\psi)$ commute. Let φ be ergodic state with respect to the $Aut_\varphi(\mathcal{M})$ and ψ is ergodic with respect to the $Aut_\psi(\mathcal{M})$. If $Aut_\varphi(\mathcal{M}) = Aut_\psi(\mathcal{M})$ then the states φ and ψ are either equivalent, or singular.*

Proof Since the states φ and ψ are quasiinvariant with respect to the $Aut_\varphi(\mathcal{M})$ for any $\tau_t \in Aut_\varphi(\mathcal{M})$ we have $\tau_t(s(\varphi)) = s(\varphi)$, $\tau_t(s(\psi)) = s(\psi)$.

Suppose that the states φ and ψ are not equivalent and are nonsingular. Denote $r = s(\varphi)s(\psi)$. Then $r \neq s(\varphi), r \neq s(\psi), r \neq \mathbf{0}, r \neq \mathbf{1}$ and $0 < \varphi(r) < 1$. At the same time, since the supports $s(\varphi)$ and $s(\psi)$ commute, $\tau_t(r) = \tau_t(s(\varphi)s(\psi)) = \tau_t(s(\varphi))\tau_t(s(\psi)) = s(\varphi)s(\psi) = r$ for all $\tau_t \in Aut_\varphi(\mathcal{M})$. This contradicts the ergodicity of the state φ. □

Corollary 1 *If a normal state φ is ergodic with respect to $Aut_\varphi(\mathcal{M})$, then for any $\tau_t \in Aut(\mathcal{M})$ the states φ and φ_{τ_t} are either equivalent or orthogonal.*

Proof The proof follows from the fact that the state φ_{τ_t} is also ergodic and the supports of the φ and φ_{τ_t} commute. □

Definition 4 [2] Given a representation (H, π) of a von Neumann algebra $\mathcal{M}$, a subspace L of H is called an invariant subspace of (H, π) if $\pi(x)L \subseteq L$ for every $x \in \mathcal{M}$. If (H, π) has no invariant subspace other than H and $\{0\}$, then it is said to be irreducible.

Theorem 4 *Let φ be a normal faithful state on the σ-finite von Neumann algebra $\mathcal{M}$ and (H_φ, π_φ) its the cyclic representation. Consider the following conditions:*

i) *The state φ is ergodic with respect to the group $Aut_\varphi(\mathcal{M})$;*
ii) *The representation (H_φ, π_φ) is irreducible.*

Then condition ii) follows from condition i). If the algebra $\mathcal{M}$ is abelian, then conditions i) and ii) are equivalent.

Proof Let the state φ be ergodic with respect to the group $Aut_\varphi(\mathcal{M})$. Suppose that the representation (H_φ, π_φ) is not irreducible. This means that there is a nontrivial invariant subspace $H_1 \subseteq H_\varphi$. Take the orthogonal projection p with the range on the subspace H_1. Since the subspace H_1 is invariant under $\pi_\varphi(\mathcal{M})$, the projector p is contained in the commutator $\pi_\varphi(\mathcal{M})'$. So, for any positive invertible operator h affiliated to the $\pi_\varphi(\mathcal{M})$ we have $hp = ph$. Hence, $hph^{-1} = p$. Then it follows from Theorem 2 that $\pi_\varphi^{-1}(p)$ is invariant with respect to the action of the group $Aut_\varphi(\mathcal{M})$. This contradicts the ergodicity of the state φ.

Opposite, let the representation (H_φ, π_φ) be irreducible and the state φ not be ergodic. Then there is an non-trivial projection $p \in \mathcal{M}$ invariant with respect to the action of the group $Aut_\varphi(\mathcal{M})$. Therefore, for all $\tau_t \in Aut_\varphi(\mathcal{M})$ $\tau_t(p) = p$. Note that $\pi_\varphi(p)$ is a projection on H_φ and $\pi_\varphi(\tau_t(p)) = \pi_\varphi(p)$. We denote by $L = \pi_\varphi(p)H_\varphi$ the image of the projection $\pi_\varphi(p)$. Take $\xi \in L$. Consider $\pi_\varphi(q)\xi = \pi_\varphi(q)\pi_\varphi(p)\xi = \pi_\varphi(pq)\xi \in L$, where q is a projection on $\mathcal{M}$. Thus L is a non-trivial invariant subspace of H_φ. This contradicts the irreducibility of (H_φ, π_φ). □

3 Ultraproducts and Representations

Everywhere further $\mathscr{U}$ is a nontrivial ultrafilter in the set $\mathbb{N}$ of natural numbers.

Definition 5 [7] Consider a sequence $(H_n, \|\cdot\|)_{n\in\mathbb{N}}$ of Banach spaces. The ultraproduct $(H_n)_\mathscr{U}$ is the quotient $l^\infty(\mathbb{N}, H_n)/\mathcal{N}_\mathscr{U}$, where

$$l^\infty(\mathbb{N}, H_n) = \{(h_n), h_n \in H_n : \sup_n \|h_n\| < \infty\},$$

$$\mathcal{N}_\mathscr{U} = \{(h_n) \in l^\infty(\mathbb{N}, H_n) : \lim_\mathscr{U} \|h_n\| = 0\}.$$

Here $\mathcal{N}_\mathscr{U}$ is the closed subspace of $l^\infty(\mathbb{N}, H_n)$. We denote an element of $(H_n)_\mathscr{U}$ by $(h_n)_\mathscr{U}$. Then the relation

$$\|(h_n)_\mathscr{U}\| = \lim_\mathscr{U} \|h_n\|$$

defines the norm on ultraproduct $(H_n)_\mathscr{U}$. In this case $((H_n)_\mathscr{U}, \|\cdot\|)$ is a Banach space.

Now let (H_n) be a sequence of Hilbert spaces with the norm $\|\cdot\| = (\cdot,\cdot)$, and $(H_n)_\mathscr{U}$ be an ultraproduct of (H_n) with a scalar product

$$((h_n^1)_\mathscr{U}, (h_n^2)_\mathscr{U}) = \lim_\mathscr{U}(h_n^1, h_n^2), \quad h_n^1, h_n^2 \in H_n.$$

Then $(H_n)_\mathscr{U}$ is a Hilbert space.

Let x_n be a linear and bounded operator on H_n, $n \in \mathbb{N}$, with the condition $\sup_n \|x_n\| < \infty$. We define on $(H_n)_\mathscr{U}$ the ultraproduct of the sequence (x_n):

$$(x_n)_\mathscr{U}((h_n)_\mathscr{U}) = (x_n(h_n))_\mathscr{U}.$$

Then the operator $(x_n)_\mathscr{U}$ is linear and bounded and at the same time

$$\|(x_n)_\mathscr{U}\| = \lim_\mathscr{U} \|x_n\|.$$

Let $(\mathbb{B}(H_n))$ be a sequence of algebras of linear and bounded operators on H_n, and $(\mathbb{B}(H_n))_{\mathcal{U}}$ be an ultraproduct of sequence $(\mathbb{B}(H_n))$. Define $\varrho_{\mathcal{U}} : (\mathbb{B}(H_n))_{\mathcal{U}} \to \mathbb{B}((H_n)_{\mathcal{U}})$ by

$$\varrho_{\mathcal{U}}((x_n)_{\mathcal{U}})(h_n)_{\mathcal{U}} := (x_n(h_n))_{\mathcal{U}}, \quad (x_n) \in l^{\infty}(\mathbb{N}, \mathbb{B}(H_n)), \quad h_n \in H_n.$$

It is easy to check that

$$\|\varrho_{\mathcal{U}}((x_n)_{\mathcal{U}})(h_n)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\| = \|(x_n)_{\mathcal{U}}\|.$$

It is known, [1], that $\varrho_{\mathcal{U}}$ is injective and the image $\varrho_{\mathcal{U}}((\mathbb{B}(H_n))_{\mathcal{U}})$ is strongly dense in $\mathbb{B}((H_n)_{\mathcal{U}})$, but $\varrho_{\mathcal{U}}$ is not surjective (if H is a separable infinite-dimensional Hilbert space).

Definition 6 [4, 10] Let $M_n \subseteq \mathbb{B}(H_n)$ be a fixed faithful representation of von Neumann algebra $\mathcal{M}_n$ on a Hilbert space H_n. The abstract ultraproduct of the sequence (M_n, H_n) is defined as the strong operator closure of $\varrho_{\mathcal{U}}((M_n)_{\mathcal{U}})$ in $\mathbb{B}((H_n)_{\mathcal{U}})$. The Groh-Raynaud ultraproduct of sequence $(\mathcal{M}_n)$ is defined as the abstract ultraproduct of the sequence (M_n, H_n), where we choose the standard (cyclic) representation of $\mathcal{M}_n$.

Note 2 Let $(\mathcal{M}_n)$ be a sequence of σ-finite von Neumann algebras and let a normal faithful state φ_n on $\mathcal{M}_n$ be given for each $n \in \mathbb{N}$. Then each $\mathcal{M}_n$ acts standardly on $H_n = L^2(\mathcal{M}_n, \varphi_n)$.

Definition 7 [9] Let $(\mathcal{M}_n)$ be a sequence of σ-finite von Neumann algebras, and let φ_n be a normal faithful state on $\mathcal{M}_n$ for each $n \in \mathbb{N}$. Put

$$l^{\infty}(\mathbb{N}, \mathcal{M}_n) = \{(x_n), x_n \in \mathcal{M}_n : \sup_n \|x_n\| < \infty\},$$

$$\mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n) = \{(x_n) \in l^{\infty}(\mathbb{N}, \mathcal{M}_n) : \lim_{\mathcal{U}} \varphi_n(x_n^* x_n + x_n x_n^*)^{\frac{1}{2}} = 0\}.$$

$$\mathcal{M}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n) = \{(x_n) \in l^{\infty}(\mathbb{N}, \mathcal{M}_n) :$$
$$(x_n)\mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n) \subset \mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n), \mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n)(x_n) \subset \mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n)\}.$$

We then define the ultraproduct for the sequence of von Neumann algebras with normal faithful states

$$(\mathcal{M}_n, \varphi_n)_{\mathcal{U}} = \mathcal{M}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n)/\mathcal{N}_{\mathcal{U}}(\mathcal{M}_n, \varphi_n).$$

At last, we define a state on $(\mathcal{M}_n, \varphi_n)_{\mathcal{U}}$:

$$\varphi_{\mathcal{U}}((x_n)_{\mathcal{U}}) = \lim_{\mathcal{U}} \varphi_n(x_n).$$

It is known, [1], that $(\mathcal{M}_n, \varphi_n)_{\mathcal{U}}$ is a von Neumann algebra with the normal faithful state $\varphi_{\mathcal{U}}$.

Let $(\mathcal{M}_n)$ be a sequence of σ-finite von Neumann algebras, and let φ_n be a normal faithful state on $\mathcal{M}_n$ for each $n \in \mathbb{N}$, $(H_{\varphi_n}, \pi_{\varphi_n})$ be the cyclic representation $(\mathcal{M}_n)$ associated with state φ_n.

It is known, [1], that the representation $((H_{\varphi_n})_{\mathcal{U}}, (\pi_{\varphi_n})_{\mathcal{U}})$ of the Ocneanu ultraproduct $(\mathcal{M}_n, \varphi_n)_{\mathcal{U}}$ coincides with the ultraproduct $(H_{\varphi_n}, \pi_{\varphi_n})_{\mathcal{U}}$ of the representations $(H_{\varphi_n}, \pi_{\varphi_n})$ up to isometry.

It is well known that the ultraproduct does not preserve some properties of its cofactors. It is easy to prove that the ultraproduct of the sequence of finite algebras is not even a σ-finite algebra.

Let $\mathcal{M}_n$ be a σ-finite von Neumann algebra and let φ_n be a normal state on $\mathcal{M}_n$, $n \in \mathbb{N}$. We denote $Aut_{\varphi_n}(\mathcal{M}_n) = \{\tau_{t_n} \in Aut(\mathcal{M}_n) : \varphi_n \sim (\varphi_n)_{\tau_{t_n}}, t \in \mathbb{R}\}$. Consider the standard ultraproduct $(Aut_{\varphi_n}(\mathcal{M}_n))_{\mathcal{U}}$ of the sequence $(Aut_{\varphi_n}(\mathcal{M}_n))$.

Definition 8 [6] Let $(\mathcal{M}_n)$ be a sequence of σ-finite von Neumann algebras, let φ_n and ψ_n be a normal states on $\mathcal{M}_n$, $n \in \mathbb{N}$. The sequence (φ_n) is said to be *contigual* with respect to the sequence (ψ_n) if

$$\psi_n(x_n^* x_n) \to 0 \quad \text{implies} \quad \varphi_n(x_n^* x_n) \to 0, \; x_n \in \mathcal{M}_n, \quad (n \to \infty);$$

If the sequence (φ_n) is contigual with respect to the sequence (ψ_n) and the sequence (ψ_n) is contigual with respect to the sequence (φ_n) then the sequences (φ_n) and (ψ_n) are said to be *mutually contigual*;

We will write for any mutually contigual sequences (φ_n) and (ψ_n): $(\varphi_n) \trianglelefteq \trianglerighteq (\psi_n)$. These notions generalize the concepts of equivalence of states.

Theorem 5 1) *In general,* $(Aut_{\varphi_n}(\mathcal{M}_n))_{\mathcal{U}} \neq Aut_{(\varphi_n)_{\mathcal{U}}}(\mathcal{M}_n)_{\mathcal{U}}$*;*
2) *The state* $(\varphi_n)_{\mathcal{U}}$ *is quasiinvariant with respect to the group*

$$Aut_{(\varphi_n)_{\mathcal{U}}}(\mathcal{M}_n)_{\mathcal{U}} = \{(\tau_{t_n})_{\mathcal{U}} : \tau_{t_n} \in Aut_{\varphi_n}(\mathcal{M}_n), \; (\varphi_n) \trianglelefteq \trianglerighteq ((\varphi_n)_{\tau_{t_n}})\}.$$

Proof The proof follows from Theorem 1 and the Example 2 on [6]. □

Theorem 6 *In general, the ultraproduct of a sequence of irreducible representations of σ-finite von Neumann algebras with faithful normal states is not irreducible.*

Proof The proof follows from the Theorem 4 and the following example. □

Example 1 Let $\Omega_n = \mathbb{R}^n$, μ_n be Gaussian measure $\mathcal{N}(0, I_n)$, G_n be the group of shifts on the elements of $\mathbb{R}^n$, $n \in \mathbb{N}$. It is known, (see [8]), that the ultraproduct $\mu_{\mathcal{U}}$ is quasi-invariant with respect to action on the group $G = \{x = (x_n)_{\mathcal{U}} : \sup_n \|x_n\|_{\ell_2} < \infty\}$.

Put $\mathcal{A}_n = L^\infty(\Omega_n, \mu_n)$, $\varphi_n(f_n) = \int f_n(x) d\mu_n$. It is clear that the state φ_n on the algebra $\mathcal{A}_n$ is normal and faithful. We will consider the action τ_n of the group G_n on the algebra $\mathcal{A}_n$ as above. It is obvious that the action τ_n is non-singular and free with respect to the state φ_n.

Now we will show that the measure $\mu_{\mathcal{U}}$ isn't ergodic with respect to action $\tau_{\mathcal{U}}$ of the group G. It is shown, [8], that for any element $x \in G$ we have $\mu_{\mathcal{U}}(B \Delta (B - x)) = 0$, where $B = (B_n)_{\mathcal{U}}$, B_n is the ball on the Ω_n of the radius $\sqrt{n}$ and with the center at zero, $A \Delta B$ is the symmetric difference of sets A and B. It is clear that measure $\mu_{\mathcal{U}}(B) = 1/2$. Therefore the $\mu_{\mathcal{U}}$ is not ergodic.

Further, we will put $p = I_B$. Then $\tau_x(p) = p$ for all $x \in G$, at the same time we have $\varphi_{\mathcal{U}}(p) \neq 0$ and $\varphi_{\mathcal{U}}(p) \neq 1$. Therefore the state $\varphi_{\mathcal{U}}$ is not ergodic. Follows from Theorem 4 that the representation associated with the state $\varphi_{\mathcal{U}}$ is not irreducible.

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References

1. Ando, H., Haagerup, U.: Ultraproducts of von Neumann algebras. J. Funct. Anal. **266**(12), 6842–6913 (2014)

2. Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics. Springer, Berlin/Heidelberg (1987)
3. Chetcutti, E., Hamhalter, J.: Vitali-Hahn-Saks Theorem for vector measures on operator algebras. Q. J. Math. **57**, 479–493 (2006)
4. Groh, U.: Uniform ergodic theorems for identity preserving Schwarz maps on W^*-algebras. J. Operator Theory **11**(2), 395–404 (1984)
5. Gudder, S.P.: A Radon-Nikodym theorem for $*$-algebras. Pac. J. Math. **80**(1), 141–149 (1979)
6. Haliullin, S.: Contiguity and entire separability of States on von Neumann algebras. Int. J. Theor. Phys. **56**(2), 3889–3894 (2017)
7. Heinrich, S.: Ultraproducts in Banach space theory. J. für die reine und angewandte Math. **313**, 72–104 (1980)
8. Mushtari, D.H., Haliullin, S.G.: Linear spaces with a probability measure, ultraproducts and contiguity. Lobachevskii J. Math. **35**(2), 138–146 (2014)
9. Ocneanu, A.: Actions of Discrete Amenable Groups on von Neumann Algebras, Lect Notes in Math., vol. 1138. Springer, New York/Berlin (1985)
10. Raynaud, Y.: On ultrapowers of noncommutative L_p-spaces. J. Operator Theory **48**(1), 41–68 (2002)
11. Takesaki, M.: Theory of Operator Algebras III. Springer, Berlin (2003)